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CAVITY QED WITH DEGENERATE ATOMIC LEVELS AND POLARIZATION-DEGENERATE FIELD MODE

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The Jaynes – Cummings model with degenerate atomic levels and polarization-degenerate field mode is considered. The general expression for the system evolution operator is derived. The analytical expressions for such operators in the case of low values ($J \leq 3/2$) of atomic angular momentum are obtained. The polarization properties of the photon emitted into the cavity by an excited atom are studied with an account of relaxation processes for arbitrary angular momenta of atomic levels.

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1. INTRODUCTION

The original Jaynes – Cummings model [1] and its numerous descendants [2] constitute basic theoretical tools in quantum optics (see, for example, the textbooks [3–5] and the reviews [6–9]). In the present article one more extension of Jaynes – Cummings model, taking into account the degeneracy of atomic levels and polarization degeneracy of the field mode, is considered. The reason for such extension is that the levels of isolated atoms in the micro-cavity are degenerate in the projections of the angular momentum on the quantization axis (unless this degeneracy is somehow lifted), while the single frequency mode of the quantized field in the cavity may obtain two independent polarizations in the plane perpendicular to the cavity axis (unless the symmetry in the cavity is somehow broken). For example, in the experiments [10–14] the field mode was on resonance with the transition between Rydberg states of ^{85}Rb atoms with the angular momenta of atomic levels $J = 3/2, 5/2$, while in the experiments [15–22] the field mode was on resonance with the transitions between the hyperfine-structure components with the angular momenta $F = 1, 2, 3$ of electronic levels $5S_{1/2}$ and $5P_{3/2}$ of ^{85}Rb and ^{87}Rb atoms. On the other hand the degeneracy of atomic levels may be employed to control the interaction of the atom with the field mode by means of atomic preparation in some special states, while the control of the field polarization in the cavity may provide tools for handling the photon-polarization qubits. The Jaynes – Cummings model with degenerate atomic levels but with the fixed polarization of the field mode was considered in [23–25] with an application to the theory of one-atom maser. However for the applications of atom-cavity systems as universal nodes of quantum networks [20] the two polarization states of the photon in the cavity should be taken into account. The various aspects of the Jaynes – Cummings model with degenerate atomic levels and polarization-degenerate field mode were considered previously in [26–30]. In the present article the general solution of such a model, based on the conservation laws of energy and projection of the angular momentum on the quantization axis, is proposed. In the first section the basics of the model are described and the decomposition of the space of system states into the subspaces invariant under the action the system Hamiltonian is performed. It is shown that the problem of diagonalization of the Hamiltonian is reduced to the diagonalization of Hermitian matrix of the dimension equal to the degree of degeneracy of atomic levels. In the second section the evolution operator of the system is considered and an example of the dynamics of an atom with the resonant transition $J_0 = 3 \to J_1 = 4$ in the unpolarized thermal field is studied numerically. In the third section the analytical expressions for the eigenvalues and eigenvectors of the system Hamiltonian are presented for the low values ($J_0, J_1 \leq 3/2$) of atomic angular momen-

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tum. In the fourth section the polarization dynamics of the photon stored in the cavity is discussed. In the fifth section the polarization properties of the photon emitted into the cavity by an excited atom is studied with an account of relaxation processes.

2. THE MODEL BASICS

The model system consists of a single two-level atom with degenerate levels and a single polarization-degenerate mode of the quantized electromagnetic field, the basis states of the system space being the product of atomic and field states

$$|J_a, m_a\rangle|n_1, n_2\rangle.$$

The atomic states

$$|J_a, m_a\rangle \tag{1}$$

are characterized by the number $a = 0, 1$ of the atomic energy level E_a, $a = 0$ refers to the ground state, $a = 1$ refers to the excited state. The frequency of the atomic transition: $\omega_0 = (E_1 - E_0)/\hbar$, $E_1 = \hbar\omega_0/2$, $E_0 = -\hbar\omega_0/2$, J_a is the total angular momentum of atomic level E_a, m_a is the projection of this angular momentum on the quantization axis z. The quantization axis z is directed along the micro-cavity axis. The states of the field mode

$$|n_1, n_2\rangle \tag{2}$$

are characterized by the number of photons n_k with the polarization $\mathbf{s}_k$, $k = 1, 2$, where $\mathbf{s}_k$ are the two orthonormal vectors in the xy plane: $\mathbf{s}_i\mathbf{s}_j^* = \delta_{ij}$, $i, j = 1, 2$, $n = n_1 + n_2$ being the total photon number.

The Hamiltonian of the model system

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{3}$$

is the sum of the free system Hamiltonian

$$\hat{H}_0 = \hbar\omega\hat{n} + \frac{1}{2}\hbar\omega_0\left(\hat{P}_A^{(1)} - \hat{P}_A^{(0)}\right), \tag{4}$$

and the interaction operator

$$\hat{V} = -\left(\hat{\mathbf{d}} + \hat{\mathbf{d}}^\dagger\right)\hat{\mathbf{E}}. \tag{5}$$

Here

$$\hat{P}_A^{(a)} = \sum_{m_a=-J_a}^{J_a} |J_a, m_a\rangle\langle J_a, m_a|, \tag{6}$$

is the projection operator on the subspace of atomic states referring to the level a,

$$\hat{\mathbf{d}} = \sum_{m_0,m_1} (\mathbf{d})_{m_0m_1}^{01} |J_0, m_0\rangle\langle J_1, m_1|, \tag{7}$$

is the operator of the electric dipole moment transition $J_1 \to J_0$, $(\mathbf{d})_{m_0m_1}^{01}$ being the matrix elements of this operator, ω is the frequency of the field mode, the electric field operator

$$\hat{\mathbf{E}} = \hat{\mathbf{e}} + \hat{\mathbf{e}}^\dagger, \quad \hat{\mathbf{e}} = ie_0(\mathbf{s}_1\hat{a}_1 + \mathbf{s}_2\hat{a}_2), \tag{8}$$

is expressed through the annihilation $\hat{a}_k$ and creation $\hat{a}_k^\dagger$ operators of the photon with the polarization $\mathbf{s}_k$, $\hat{n}_k = \hat{a}_k^\dagger\hat{a}_k$ being the number operator of such photons, $\hat{n} = \hat{n}_1 + \hat{n}_2$,

$$e_0 = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}, \tag{9}$$

V is the mode volume, the atom is placed at the center of the cavity $z = 0$.

In the rotating wave approximation the Hamiltonian (3) may be written as follows:

$$\hat{H} = \hbar\omega\hat{h}_0 - \frac{\hbar}{2}\hat{\Omega}, \quad \hat{h}_0 = \hat{n} + \frac{1}{2}\left(\hat{P}_A^{(1)} - \hat{P}_A^{(0)}\right), \tag{10}$$

$$\hat{\Omega} = \Delta\left(\hat{P}_A^{(0)} - \hat{P}_A^{(1)}\right) + \theta\left(\hat{G} + \hat{G}^\dagger\right), \tag{11}$$

$$\hat{G} = -i\left(\hat{G}_1 + \hat{G}_2\right), \quad \hat{G}_k = \hat{g}_k\hat{a}_k^\dagger \quad \hat{g}_k = \hat{\mathbf{g}}\mathbf{s}_k^*, \tag{12}$$

$$\Delta = \omega_0 - \omega, \quad \theta = 2de_0/\hbar, \quad \hat{\mathbf{g}} = \hat{\mathbf{d}}/d, \tag{13}$$

d being the reduced matrix element of the electric dipole moment operator of the transition $J_1 \to J_0$.

Let us choose the two orthonormal polarization vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ to stand for σ^+ and σ^- polarizations correspondingly, the circular components of these vectors being

$$s_{1q} = \delta_{q,-1}, \quad s_{2q} = \delta_{q,+1}. \tag{14}$$

With such a choice the field states (2) may be defined by the total photon number n and by the projection σ of the photon angular momentum on the quantization axis z:

$$|n, \sigma\rangle, \tag{15}$$

where $n = n_1 + n_2$, $\sigma = n_1 - n_2$, n_1 is the number of the photons with the σ^+ polarization $\mathbf{s}_1$, n_2 is the number of the photons with the σ^- polarization $\mathbf{s}_2$. At the given photon number n the projection σ attains the following $(n + 1)$ values:

$$\sigma = -n + 2k, \quad k = 0, 1, 2, ..., n. \tag{16}$$

Then the action of operators $\hat{G}_1$ and $\hat{G}_2$ (12) on the system states is as follows:

$$\hat{G}_1^\dagger|J_0, m\rangle|n, \sigma\rangle = G_{1,mn\sigma}|J_1, m+1\rangle|n-1, \sigma-1\rangle, \tag{17}$$

$$\hat{G}_2^\dagger|J_0, m\rangle|n, \sigma\rangle = G_{2,mn\sigma}|J_1, m-1\rangle|n-1, \sigma+1\rangle. \tag{18}$$

The matrix elements $G_{1,mn\sigma}$ and $G_{2,mn\sigma}$ of these operators are expressed through Wigner $3j$-symbols:

$$G_{1,mn\sigma} = f_1(m)\sqrt{\frac{1}{2}(n+\sigma)}, \tag{19}$$

$$G_{2,mn\sigma} = f_2(m)\sqrt{\frac{1}{2}(n-\sigma)}, \tag{20}$$

$$f_1(m) = (-1)^{J_0-m}\begin{pmatrix} J_0 & 1 & J_1 \\ -m & -1 & m+1 \end{pmatrix}, \tag{21}$$

$$f_2(m) = (-1)^{J_0-m}\begin{pmatrix} J_0 & 1 & J_1 \\ -m & 1 & m-1 \end{pmatrix}. \tag{22}$$

As it follows from (17), (18) the interaction operator $\hat{\Omega}$ (11) does not change the excitation number $n = p + a$, where p is the photon number and $a = 0, 1$ is the number of the atomic level, and it does not change the projection $l = m + \sigma$ of the system total angular momentum on the quantization axis z. Consequently the space V of the system states may be decomposed into a set of subspaces $V^{(nl)}$ with given numbers n and l which are invariant under the action of the interaction operator $\hat{\Omega}$:

$$V = \bigcup_{n=0}^{\infty}\bigcup_{l=-L}^{L} V^{(nl)}, \quad \hat{\Omega}V^{(nl)} = V^{(nl)}, \tag{23}$$

where $L = J_0 + n$ is the maximum value of the system total angular momentum projection on the quantization axis at given excitation number n. The interaction operator $\hat{\Omega}$ may be expanded into the sum of operators acting invariantly in subspaces $V^{(nl)}$:

$$\hat{\Omega} = \sum_{n=0}^{\infty}\sum_{l=-L}^{L}\hat{\Omega}^{(nl)}, \quad \hat{\Omega}^{(nl)} = \hat{P}^{(nl)}\hat{\Omega}\hat{P}^{(nl)}, \tag{24}$$

where $\hat{P}^{(nl)}$ is the projection operator on the subspace $V^{(nl)}$. The subspaces $V^{(nl)}$ may be further decomposed into two subspaces:

$$V^{(nl)} = V^{(0,n,l)}\bigcup V^{(1,n-1,l)}, \tag{25}$$

where $V^{(anl)}$ is the subspace of states with the atom at level $a = 0, 1$, photon number n and $m_a + \sigma = l$. The orthonormal basis of the subspace $V^{(anl)}$ consists of the states:

$$|w_k^{(anl)}\rangle = |J_a, m_k\rangle|n, l-m_k\rangle,\ m_k = m_{min}^{(anl)} + 2k, \tag{26}$$

$k = 0, 1, ..., N^{(anl)} - 1$, where $m_{min}^{(anl)}$ is the minimum value of m_a at given n and l, $N^{(anl)}$ is the dimension of the subspace $V^{(anl)}$. The subspace $V^{(anl)}$ is invariant under the action of operator $\hat{D}_a$, where $\hat{D}_0 = \hat{G}\hat{G}^{\dagger}$, $\hat{D}_1 = \hat{G}^{\dagger}\hat{G}$, while $\hat{G}$ is defined by (12). Instead of the orthonormal basis (26) in the subspace $V^{(anl)}$ let us consider the orthonormal basis consisting of the eigenvectors of the Hermitian operator $\hat{D}_a$ acting in this subspace. Next, the subspace $V^{(anl)}$ may be decomposed into two subspaces:

$$V^{(anl)} = V_d^{(anl)}\bigcup V_c^{(anl)}, \tag{27}$$

where the orthonormal basis $|d_k^{(anl)}\rangle$, $k = 0, 1, ..., N_d^{(anl)} - 1$, of the the subspace $V_d^{(anl)}$ consists of the eigenvectors of the operator $\hat{D}_a$ with zero eigenvalues, while the orthonormal basis $|v_k^{(anl)}\rangle$, $k = 0, 1, ..., N_c^{(anl)} - 1$ ($N_d^{(anl)} + N_c^{(anl)} = N^{(anl)}$), of the subspace $V_c^{(anl)}$ consists of the eigenvectors of the operator $\hat{D}_a$ with real positive non-zero eigenvalues $[\xi_k^{(anl)}]^2$:

$$\hat{D}_a|v_k^{(anl)}\rangle = [\xi_k^{(anl)}]^2|v_k^{(anl)}\rangle. \tag{28}$$

If the states $|v_k^{(0,n,l)}\rangle$ in the subspace $V_c^{(0,n,l)}$ are the orthonormal eigenvectors of the operator $\hat{D}_0$ with non-zero eigenvalues:

$$[\xi_k^{(0,n,l)}]^2 = [\xi_k^{(nl)}]^2, \tag{29}$$

then the states

$$|v_k^{(1,n-1,l)}\rangle = \frac{1}{\xi_k^{(nl)}}\hat{G}^{\dagger}|v_k^{(0,n,l)}\rangle \tag{30}$$

are the orthonormal eigenvectors of the operator $\hat{D}_1$ in the subspace $V_c^{(1,n-1,l)}$ with non-zero eigenvalues:

$$[\xi_k^{(1,n-1,l)}]^2 = [\xi_k^{(0,n,l)}]^2 = [\xi_k^{(nl)}]^2. \tag{31}$$

So, the subspace

$$V_c^{(nl)} = V_c^{(0,n,l)}\bigcup V_c^{(1,n-1,l)} \tag{32}$$

may be decomposed into $N_c^{(nl)} = N_c^{(0,n,l)} = N_c^{(1,n-1,l)}$ two-dimensional subspaces

$$V_{ck}^{(nl)} = V_{ck}^{(0,n,l)}\bigcup V_{ck}^{(1,n-1,l)}, \tag{33}$$

($k = 0, 1, ..., N_c^{(nl)} - 1$) which are invariant under the action of interaction operator $\hat{\Omega}$. The orthonormal basis in the subspace $V_{ck}^{(nl)}$ consists of two states $|v_k^{(0,n,l)}\rangle$ and $|v_k^{(1,n-1,l)}\rangle$. In this subspace:

$$\hat{G}^{\dagger}|v_k^{(0,n,l)}\rangle = \xi_k^{(nl)}|v_k^{(1,n-1,l)}\rangle, \tag{34}$$

$$\hat{G}|v_k^{(1,n-1,l)}\rangle = \xi_k^{(nl)}|v_k^{(0,n,l)}\rangle. \tag{35}$$

The eigenvalues and eigenvectors of the interaction operator $\hat{\Omega}$ in the subspace $V_{ck}^{(nl)}$ are as follows:

$$\hat{\Omega}|u_k^{(nl,\pm)}\rangle = \pm\Omega_k^{(nl)}|u_k^{(nl,\pm)}\rangle, \tag{36}$$

$$|u_k^{(nl,\pm)}\rangle = c_k^{(nl,\pm)}|v_k^{(0,n,l)}\rangle \pm c_k^{(nl,\mp)}|v_k^{(1,n-1,l)}\rangle, \tag{37}$$

$$\Omega_k^{(nl)} = \sqrt{\Delta^2 + \theta^2\xi_k^{(nl)2}}, \tag{38}$$

$$c_k^{(nl,\pm)} = \sqrt{\frac{1}{2}\left(1 \pm \frac{\Delta}{\Omega_k^{(nl)}}\right)}. \tag{39}$$

Consequently the interaction operator $\hat{\Omega}^{(nl)}$ (24) in the subspace $V^{(nl)}$ may be presented in such a way:

$$\hat{\Omega}^{(nl)} = \hat{\Omega}_d^{(nl)} + \hat{\Omega}_c^{(nl)}, \tag{40}$$

$$\hat{\Omega}_d^{(nl)} = \Delta\left(\hat{P}_d^{(0,n,l)} - \hat{P}_d^{(1,n-1,l)}\right), \tag{41}$$

$$\hat{\Omega}_c^{(nl)} = \sum_{k=0}^{N_c^{(nl)}-1} \Omega_k^{(nl)}\left(\hat{P}_{ck}^{(nl,+)} - \hat{P}_{ck}^{(nl,-)}\right), \tag{42}$$

where

$$\hat{P}_d^{(anl)} = \sum_{k=0}^{N_d^{(anl)}-1} |d_k^{(anl)}\rangle\langle d_k^{(anl)}| \tag{43}$$

and

$$\hat{P}_{ck}^{(nl,\pm)} = |u_k^{(nl,\pm)}\rangle\langle u_k^{(nl,\pm)}| \tag{44}$$

are the projection operators on the subspaces $V_d^{(anl)}$ and $V_{ck}^{(nl,\pm)}$ correspondingly.

So the problem of diagonalization of the interaction operator $\hat{\Omega}^{(nl)}$ is reduced to the problem of diagonalization of the $N^{(anl)} \times N^{(anl)}$ Hermitian matrix

$$D_{k'k}^{(anl)} = \langle w_{k'}^{(anl)}|\hat{D}_a|w_k^{(anl)}\rangle, \tag{45}$$

of the operator $\hat{D}_a$ in the subspace $V^{(anl)}$. The non-zero elements of the matrix $D_{k'k}^{(anl)}$ are expressed through the functions $G_{k,mn\sigma}$ defined by the formulae (19)–(22).

The minimum $m_{min}^{(anl)}$ and maximum $m_{max}^{(anl)}$ values of the projection m_a of the atom angular momentum on the quantization axis at given n and l, and the dimension $N^{(anl)} = 1+(m_{max}-m_{min})/2$ of the subspace $V^{(anl)}$ may be found from the conditions:

$$|m_k| \le J_a,\ |l - m_k| \le n,\ m_k = m_{min}^{(anl)} + 2k, \tag{46}$$

where $k = 0, 1, ..., N^{(anl)} - 1$. Since the projection of the photon angular momentum σ varies with the step 2 (16) the two sets of subspaces $V_p^{(anl)}$ $(p = 0, 1)$ may be distinguished with

$$l = -J_a - n + p + 2s,\ s = 0, 1, ..., L_p^{(an)}, \tag{47}$$

where

$$L_p^{(an)} = J_p^{(a)} + n, \tag{48}$$

$$J_p^{(a)} = \left\{ \begin{array}{c} J_a - p,\ \text{for integer } J_a \\ J_a - \frac{1}{2},\ \text{for half-integer } J_a \end{array} \right\}. \tag{49}$$

Then we obtain from (46), (47):

$$m_{min}^{(anlp)} = \left\{ \begin{array}{c} -J_a + p,\ 0 \le s \le n \\ -J_a + p + 2(s-n), n < s \le L_p^{(an)} \end{array} \right\}. \tag{50}$$

$$N_p^{(anl)} = \left\{ \begin{array}{c} s + 1,\ 0 \le s \le s_{min}^{(anp)} \\ s_{min}^{(anp)} + 1,\ s_{min}^{(anp)} < s \le s_{max}^{(anp)} \\ L_p^{(an)} + 1 - s,\ s_{max}^{(anp)} < s \le L_p^{(an)} \end{array} \right\}, \tag{51}$$

where

$$s_{min}^{(anp)} = \min\{n, J_p^{(a)}\},\ s_{max}^{(anp)} = \max\{n, J_p^{(a)}\}. \tag{52}$$

The maximum dimension $N_m^{(anp)}$ of the subspaces $V_p^{(anl)}$ is defined by the formula:

$$N_m^{(anp)} = s_{min}^{(anp)} + 1 \le J_p^{(a)} + 1. \tag{53}$$

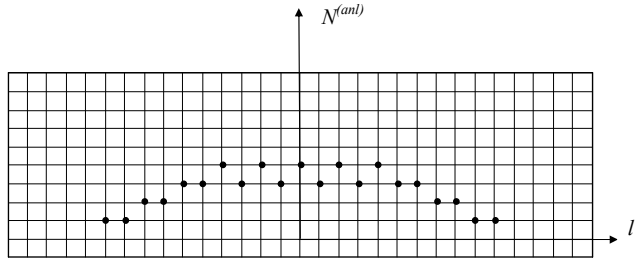

Fig. 1. The dependence of the subspace dimension $N^{(anl)}$ on l for $J_a = 3$, $n = 7$

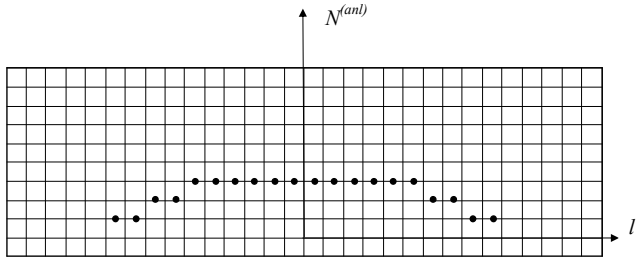

Fig. 2. The dependence of the subspace dimension $N^{(anl)}$ on l for $J_a = 5/2$, $n = 7$

The typical dependencies of the dimensions $N^{(anl)}$ of the subspaces $V^{(anl)}$ on the projection l of the system angular momentum on the quantization axis z are presented in the Figs. 1 and 2 in the cases of integer

$J_a = 3$ and half-integer $J_a = 5/2$ values of atomic angular momenta correspondingly and the photon number $n = 7 > J_p^{(a)}$.

3. THE SYSTEM DYNAMICS

The system dynamics is described by the equation for its density matrix $\hat{\sigma}$:

$$\frac{d\hat{\rho}}{dt} = \frac{i}{2}\left[\hat{\Omega}, \hat{\rho}\right], \quad \hat{\rho} = \hat{u}^{\dagger}\hat{\sigma}\hat{u}, \quad \hat{u} = e^{-i\omega t\hat{h}_0}, \tag{54}$$

where $\hat{\rho}$ is the slowly-varying system density matrix. The solution of the equation (54) is expressed through the evolution operator $\hat{S}(t)$:

$$\hat{\rho}(t) = \hat{S}(t)\hat{\rho}(0)\hat{S}^{\dagger}(t), \quad \hat{S}(t) = \exp\left\{\frac{i}{2}\hat{\Omega}t\right\}, \tag{55}$$

$$\hat{S}(t) = \sum_{n=0}^{\infty}\sum_{l=-L}^{L}\hat{S}^{(nl)}(t), \tag{56}$$

$$\hat{S}^{(nl)}(t) = \hat{P}^{(nl)}\exp\left\{\frac{i}{2}\hat{\Omega}^{(nl)}t\right\}. \tag{57}$$

From the expressions (40)–(44) for the interaction operator we obtain the following expressions for the evolution operator in terms of eigenvalues and eigenvectors of operators $\hat{D}_a$ defined in the equations (28)–(39):

$$\hat{S}^{(nl)}(t) = \hat{S}_d^{(nl)}(t) + \sum_{a,b=0}^{1}\hat{S}_{ab}^{(nl)}(t), \tag{58}$$

$$\hat{S}_d^{(nl)}(t) = e^{i\Delta t/2}\hat{P}_d^{(0,n,l)} + e^{-i\Delta t/2}\hat{P}_d^{(1,n-1,l)}, \tag{59}$$

$$\hat{S}_{00}^{(nl)}(t) = \sum_{k=0}^{N_c^{(nl)}-1} C_k^{(nl)}(t)|v_k^{(0nl)}\rangle\langle v_k^{(0nl)}|, \tag{60}$$

$$\hat{S}_{11}^{(nl)}(t) = \sum_{k=0}^{N_c^{(nl)}-1} [C_k^{(nl)}(t)]^*|v_k^{(1,n-1,l)}\rangle\langle v_k^{(1,n-1,l)}|, \tag{61}$$

$$\hat{S}_{01}^{(nl)}(t) = \sum_{k=0}^{N_c^{(nl)}-1} S_k^{(nl)}(t)|v_k^{(0nl)}\rangle\langle v_k^{(1,n-1,l)}|, \tag{62}$$

$$\hat{S}_{10}^{(nl)}(t) = \sum_{k=0}^{N_c^{(nl)}-1} S_k^{(nl)}(t)|v_k^{(1,n-1,l)}\rangle\langle v_k^{(0nl)}|, \tag{63}$$

$$C_k^{(nl)}(t) = \cos\left(\frac{\Omega_k^{(nl)}t}{2}\right) + i\frac{\Delta}{\Omega_k^{(nl)}}\sin\left(\frac{\Omega_k^{(nl)}t}{2}\right), \tag{64}$$

$$S_k^{(nl)}(t) = i\frac{\theta\xi_k^{(nl)}}{\Omega_k^{(nl)}}\sin\left(\frac{\Omega_k^{(nl)}t}{2}\right). \tag{65}$$

As an example let us consider the dynamics of population of the atomic excited level

$$n_1(t) = tr\left\{\hat{P}_A^{(1)}\hat{\rho}(t)\right\} \tag{66}$$

in the case when the atom is initially in the equilibrium ground state and the field is in the equilibrium thermal state, so that the initial system density matrix is as follows:

$$\hat{\rho}(0) = \frac{\hat{P}_A^{(0)}}{2J_0+1}\sum_{n=0}^{\infty}\frac{p_n}{n+1}\sum_{\sigma=-n}^{n}|n,\sigma\rangle\langle n,\sigma|, \tag{67}$$

$$p_n = \frac{n_c^n}{(1+n_c)^{n+1}}, \tag{68}$$

where n_c is an average photon number. In this case we obtain from (55)–(68):

$$n_1(t) = \frac{1}{2J_0+1}\sum_{n=0}^{\infty}\frac{p_n}{n+1}\sum_{l=-(J_0+n)}^{J_0+n}F_{nl}(t), \tag{69}$$

$$F_{nl}(t) = \sum_{k=0}^{N_c^{(nl)}-1}\frac{\theta^2\xi_k^{(nl)2}}{\Omega_k^{(nl)2}}\sin^2\left(\frac{\Omega_k^{(nl)}t}{2}\right). \tag{70}$$

The dependence of the population n_1 of the excited atomic level on the dimensionless time $\tau = \theta t$ is presented in the Fig. 3 for the transition $J_0 = 3 \to J_1 = 4$, $\Delta = 0.1\theta$ and $n_c = 3$. The solid line refers to the case of the equilibrium initial state of the system (67), while

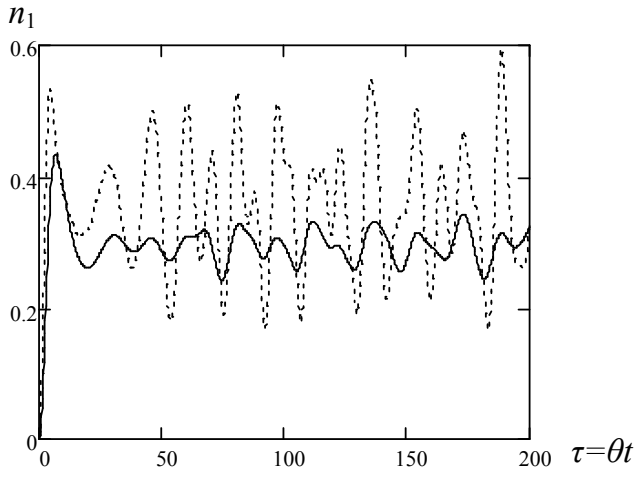

Fig. 3. The dependence of the population n_1 of the excited atomic level on the dimensionless time $\tau = \theta t$ for the transition $J_0 = 3 \to J_1 = 4$, $\Delta = 0.1\theta$ and $n_c = 3$. The solid line refers to the thermally equilibrium initial state of the system, while the dotted line refers to the case, when the atom is initially in a "stretched" ground state $|J_0, J_0\rangle$ and only σ^+ mode of the thermal field is present

the dotted line refers to the case of the system initial density matrix

$$\hat{\rho}(0) = |J_0, J_0\rangle\langle J_0, J_0| \sum_{n=0}^{\infty} p_n |n, n\rangle\langle n, n|, \tag{71}$$

when the atom is initially in a "stretched" ground state $|J_0, J_0\rangle$ and only σ^+ mode of the thermal field (68) is present. In this case the original Jaynes – Cummings model with two-level atom and single-mode field is realised:

$$n_1(t) = \sum_{n=0}^{\infty} p_n \frac{\theta^2 \xi_n^2}{\Omega_n^2} \sin^2\left(\frac{\Omega_n t}{2}\right), \tag{72}$$

$$\Omega_n = \sqrt{\Delta^2 + \theta^2 \xi_n^2},\ \xi_n = \sqrt{\frac{n}{2J_0+3}}\delta_{J_1, J_0+1}. \tag{73}$$

4. THE TRANSITIONS WITH LOW VALUES OF ATOMIC ANGULAR MOMENTUM

The matrix elements of the evolution operator (58) are expressed through the eigenvalues and eigenvectors of operators $\hat{D}_a$ in the subspaces $V^{(anl)}$ (59)–(65). Since the dimensions of these subspaces do not exceed the value J_a+1 (53) these eigenvalues and eigenvectors may be easily obtained analytically in the case of low values of atomic angular momentum $J_a \le 3/2$. The explicit expressions of these eigenvalues and eigenvectors for such transitions are presented in this section.

4.1. Transitions $J_0 = 0 \to J_1 = 1$

For the subspaces of the set $p = 0$ (47) with

$$l = -n + 2s,\ s = 0, 1, ..., n, \tag{74}$$

at the values $s = 0$ and $s = n$ ($l = \mp n$):

$$N^{(0,n,l)} = N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,\pm n)}\rangle = |0, 0\rangle|n, \pm n\rangle,\ \xi^{(nl)} = \sqrt{n/3}, \tag{75}$$

$$|v^{(1,n-1,\pm n)}\rangle = i|1, \pm 1\rangle|n-1, \pm(n-1)\rangle, \tag{76}$$

while at the values $s = 1, ..., n-1$:

$$N^{(0,n,l)} = N_c^{(nl)} = 1,$$

$$N^{(1,n-1,l)} = 2, \quad N_d^{(1,n-1,l)} = 1,$$

$$|v^{(0,n,l)}\rangle = |0, 0\rangle|n, l\rangle,\ \xi^{(nl)} = \sqrt{n/3}, \tag{77}$$

$$|v^{(1,n-1,l)}\rangle = c_0|w_0^{(1,n-1,l)}\rangle + c_1|w_1^{(1,n-1,l)}\rangle, \tag{78}$$

$$|d^{(1,n-1,l)}\rangle = c_1|w_0^{(1,n-1,l)}\rangle - c_0|w_1^{(1,n-1,l)}\rangle, \tag{79}$$

where

$$c_0 = i\sqrt{\frac{n-l}{2n}},\ c_1 = i\sqrt{\frac{n+l}{2n}}, \tag{80}$$

$$|w_k^{(1,n-1,l)}\rangle = |1, -1+2k\rangle|n-1, l+1-2k\rangle, \tag{81}$$

$$k = 0,\ 1.$$

For the subspaces of the set $p = 1$ (47) with

$$l = -n + 1 + 2s,\ s = 0, 1, ..., n-1, \tag{82}$$

$$N^{(0,n,l)} = 0, \quad N^{(1,n-1,l)} = N_d^{(1,n-1,l)} = 1,$$

$$|d^{(1,n-1,l)}\rangle = i|1, 0\rangle|n-1, l\rangle. \tag{83}$$

4.2. Transitions $J_0 = 1 \to J_1 = 0$

For the subspaces of the set $p = 0$ (47) with

$$l = -(n+1) + 2s,\ s = 0, 1, ..., n, \tag{84}$$

at the values $s = 0$ and $s = n+1$ ($l = \mp(n+1)$):

$$N^{(0,n,l)} = N_d^{(0,n,l)} = 1, \quad N^{(1,n-1,l)} = 0,$$

$$|d^{(0,n,\pm(n+1))}\rangle = |1, \pm 1\rangle|n, \pm n\rangle, \tag{85}$$

while at the values $s = 1, ..., n$:

$$N^{(0,n,l)} = 2,\ N_d^{(0,n,l)} = 1,\ N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,l)}\rangle = c_0|w_0^{(0,n,l)}\rangle + c_1|w_1^{(0,n,l)}\rangle, \tag{86}$$

$$|d^{(0,n,l)}\rangle = c_1|w_0^{(0,n,l)}\rangle - c_0|w_1^{(0,n,l)}\rangle, \tag{87}$$

$$|v^{(1,n-1,l)}\rangle = i|0, 0\rangle|n-1, l\rangle,\ \xi^{(nl)} = \sqrt{(n+1)/3}, \tag{88}$$

where

$$c_0 = \sqrt{\frac{n+1+l}{2(n+1)}},\ c_1 = \sqrt{\frac{n+1-l}{2(n+1)}}, \tag{89}$$

$$|w_k^{(0,n,l)}\rangle = |1, -1+2k\rangle|n, l+1-2k\rangle,\ k = 0, 1. \tag{90}$$

For the subspaces of the set $p = 1$ (47) with

$$l = -n + 2s,\ s = 0, 1, ..., n, \tag{91}$$

$$N^{(0,n,l)} = N_d^{(0,n,l)} = 1, \quad N^{(1,n-1,l)} = 0,$$

$$|d^{(0,n,l)}\rangle = |1, 0\rangle|n, l\rangle. \tag{92}$$

4.3. Transitions $J_0 = 1 \to J_1 = 1$

For the subspaces of the set $p = 0$ (47) with

$$l = -(n+1) + 2s,\ s = 0, 1, ..., n+1, \tag{93}$$

at the values $s = 0$ and $s = n+1$ ($l = \mp(n+1)$):

$$N^{(0,n,l)} = N_d^{(0,n,l)} = 1, \quad N^{(1,n-1,l)} = 0,$$

the states $|d^{(0,n,\pm(n+1))}\rangle$ are defined by the equation (85), while at the values $s = 1, ..., n$:

$$N^{(0,n,l)} = 2,\ N_d^{(0,n,l)} = 1,\ N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,l)}\rangle = -c_0|w_0^{(0,n,l)}\rangle + c_1|w_1^{(0,n,l)}\rangle, \tag{94}$$

$$|d^{(0,n,l)}\rangle = c_1|w_0^{(0,n,l)}\rangle + c_0|w_1^{(0,n,l)}\rangle, \tag{95}$$

$$|v^{(1,n-1,l)}\rangle = i|1,0\rangle|n-1,l\rangle,\ \xi^{(nl)} = \sqrt{(n+1)/6}, \tag{96}$$

where c_0, c_1 and the states $|w_k^{(0,n,l)}\rangle$ are defined by the equations (89), (90).

For the subspaces of the set $p = 1$ (47) with

$$l = -n + 2s,\ s = 0, 1, ..., n, \tag{97}$$

at the values $s = 0$ and $s = n$ ($l = \mp n$):

$$N^{(0,n,l)} = N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,\pm n)}\rangle = |1,0\rangle|n,\pm n\rangle,\ \xi^{(nl)} = \sqrt{n/6}, \tag{98}$$

$$|v^{(1,n-1,\pm n)}\rangle = i|1,\pm 1\rangle|n-1,\pm(n-1)\rangle, \tag{99}$$

while at the values $s = 1, ..., n-1$:

$$N^{(0,n,l)} = N_c^{(nl)} = 1,\ N^{(1,n-1,l)} = 2,\ N_d^{(1,n-1,l)} = 1,$$

$$|v^{(0,n,l)}\rangle = |1,0\rangle|n,l\rangle,\ \xi^{(nl)} = \sqrt{n/6}, \tag{100}$$

$$|v^{(1,n-1,l)}\rangle = c_0|w_0^{(1,n-1,l)}\rangle - c_1|w_1^{(1,n-1,l)}\rangle, \tag{101}$$

$$|d^{(1,n-1,l)}\rangle = c_1|w_0^{(1,n-1,l)}\rangle + c_0|w_1^{(1,n-1,l)}\rangle, \tag{102}$$

where c_0, c_1 and the states $|w_k^{(0,n,l)}\rangle$ are defined by the equations (80), (81).

4.4. Transitions $J_0 = 1/2 \to J_1 = 1/2$

For the subspaces of the set $p = 0$ (47) with

$$l = -(n+1/2) + 2s,\ s = 0, 1, ..., n, \tag{103}$$

at the value $s = 0$ ($l = -(n+1/2)$):

$$N^{(0,n,l)} = N_d^{(0,n,l)} = 1,\ N^{(1,n-1,l)} = 0,$$

$$|d^{(0,n,-(n+1/2))}\rangle = |1/2,-1/2\rangle|n,-n\rangle, \tag{104}$$

while at the values $s = 1, ..., n$:

$$N^{(0,n,l)} = N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,l)}\rangle = |1/2,-1/2\rangle|n,l+1/2\rangle, \tag{105}$$

$$|v^{(1,n-1,l)}\rangle = i|1/2,1/2\rangle|n-1,l-1/2\rangle, \tag{106}$$

$$\xi^{(nl)} = \sqrt{(n+l+1/2)/6}. \tag{107}$$

For the subspaces of the set $p = 1$ (47) with

$$l = -(n-1/2) + 2s,\ s = 0, 1, ..., n, \tag{108}$$

at the value $s = n$ ($l = (n+1/2)$):

$$N^{(0,n,l)} = N_d^{(0,n,l)} = 1,\ N^{(1,n-1,l)} = 0,$$

$$|d^{(0,n,(n+1/2))}\rangle = |1/2,1/2\rangle|n,n\rangle, \tag{109}$$

while at the values $s = 0, ..., n-1$:

$$N^{(0,n,l)} = N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,l)}\rangle = |1/2,1/2\rangle|n,l-1/2\rangle, \tag{110}$$

$$|v^{(1,n-1,l)}\rangle = -i|1/2,-1/2\rangle|n-1,l+1/2\rangle, \tag{111}$$

$$\xi^{(nl)} = \sqrt{(n-l+1/2)/6}. \tag{112}$$

4.5. Transitions $J_0 = 1/2 \to J_1 = 3/2$

For the subspaces of the set $p = 0$ (47) with

$$l = -(n+1/2) + 2s,\ s = 0, 1, ..., n, \tag{113}$$

$$\xi^{(nl)} = \sqrt{(2n-l-1/2)/12}, \tag{114}$$

for all l (113), at the value $s = 0$ ($l = -(n+1/2)$):

$$N^{(0,n,l)} = N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,l)}\rangle = |1/2,-1/2\rangle|n,-n\rangle, \tag{115}$$

$$|v^{(1,n-1,l)}\rangle = i|3/2,-3/2\rangle|n-1,-n+1\rangle, \tag{116}$$

at the value $s = n$ ($l = n-1/2$):

$$N^{(0,n,l)} = N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,l)}\rangle = |1/2,-1/2\rangle|n,n\rangle, \tag{117}$$

$$|v^{(1,n-1,l)}\rangle = i|3/2,1/2\rangle|n-1,n-1\rangle, \tag{118}$$

while at the values $s = 1, ..., n-1$: $N^{(0,n,l)} = N_c^{(nl)} = 1$,

$$N^{(1,n-1,l)} = 2,\ N_d^{(1,n-1,l)} = 1,$$

$$|v^{(0,n,l)}\rangle = |1/2,-1/2\rangle|n,l+1/2\rangle, \tag{119}$$

$$|v^{(1,n-1,l)}\rangle = c_0|w_0^{(1,n-1,l)}\rangle + c_1|w_1^{(1,n-1,l)}\rangle, \tag{120}$$

$$|d^{(1,n-1,l)}\rangle = c_1|w_0^{(1,n-1,l)}\rangle - c_0|w_1^{(1,n-1,l)}\rangle, \tag{121}$$

where

$$c_0 = \sqrt{\frac{3(n-l-1/2)}{2(2n-l-1/2)}},\ c_1 = \sqrt{\frac{(n+l+1/2)}{2(2n-l-1/2)}}, \tag{122}$$

$$|w_k^{(1,n-1,l)}\rangle = i|3/2,-3/2+2k\rangle|n-1,l+3/2-2k\rangle. \tag{123}$$

For the subspaces of the set $p=1$ (47) with

$$l = -(n-1/2)+2s,\ s=0,1,...,n, \tag{124}$$

$$\xi^{(nl)} = \sqrt{(2n+l-1/2)/12}, \tag{125}$$

for all l (124), at the value $s=0$ ($l=-(n-1/2)$):

$$N^{(0,n,l)} = N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,l))}\rangle = |1/2,1/2\rangle|n,-n\rangle, \tag{126}$$

$$|v^{(1,n-1,l)}\rangle = i|3/2,-1/2\rangle|n-1,-n+1\rangle, \tag{127}$$

at the value $s=n$ ($l=n+1/2$):

$$N^{(0,n,l)} = N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(0,n,l))}\rangle = |1/2,1/2\rangle|n,n\rangle, \tag{128}$$

$$|v^{(1,n-1,l)}\rangle = i|3/2,3/2\rangle|n-1,n-1\rangle, \tag{129}$$

while at the values $s=1,...,n-1$:

$$N^{(0,n,l)} = N_c^{(nl)} = 1,\ N^{(1,n-1,l)} = 2,\ N_d^{(1,n-1,l)} = 1,$$

$$|v^{(0,n,l)}\rangle = |1/2,1/2\rangle|n,l-1/2\rangle, \tag{130}$$

$$|v^{(1,n-1,l)}\rangle = c_0|w_0^{(1,n-1,l)}\rangle + c_1|w_1^{(1,n-1,l)}\rangle, \tag{131}$$

$$|d^{(1,n-1,l)}\rangle = c_1|w_0^{(1,n-1,l)}\rangle - c_0|w_1^{(1,n-1,l)}\rangle, \tag{132}$$

where

$$c_0 = \sqrt{\frac{(n-l+1/2)}{2(2n+l-1/2)}},\ c_1 = \sqrt{\frac{3(n+l-1/2)}{2(2n+l-1/2)}}, \tag{133}$$

$$|w_k^{(1,n-1,l)}\rangle = i|3/2,-1/2+2k\rangle|n-1,l+1/2-2k\rangle. \tag{134}$$

4.6. Transitions $J_0 = 3/2 \to J_1 = 1/2$

For the subspaces of the set $p=0$ (47) with

$$l = -(n+3/2)+2s,\ s=0,1,...,n+1, \tag{135}$$

at the values $s=0$ ($l=-(n+3/2)$) and $s=n+1$ ($l=(n+1/2)$:

$$N^{(0,n,l)} = N_d^{(0,n,l)} = 1,\ N^{(1,n-1,l)} = 0,$$

$$|d^{(0,n,-(n+3/2))}\rangle = |3/2,-3/2\rangle|n,-n\rangle, \tag{136}$$

$$|d^{(0,n,(n+1/2))}\rangle = |3/2,1/2\rangle|n,n\rangle, \tag{137}$$

while at the values $s=1,...,n$:

$$N^{(0,n,l)} = 2,\ N_d^{(0,n,l)} = 1,\ N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(1,n-1,l)}\rangle = i|1/2,-1/2\rangle|n-1,l+1/2\rangle, \tag{138}$$

$$|v^{(0,n,l)}\rangle = c_0|w_0^{(0,n,l)}\rangle + c_1|w_1^{(0,n,l)}\rangle, \tag{139}$$

$$|d^{(0,n,l)}\rangle = c_1|w_0^{(0,n,l)}\rangle - c_0|w_1^{(0,n,l)}\rangle, \tag{140}$$

where

$$c_0 = \sqrt{\frac{3(n+l+3/2)}{2(2n+l+5/2)}},\ c_1 = \sqrt{\frac{(n-l+1/2)}{2(2n+l+5/2)}}, \tag{141}$$

$$|w_k^{(0,n,l)}\rangle = |3/2,-3/2+2k\rangle|n,l+3/2-2k\rangle, \tag{142}$$

$$\xi^{(nl)} = \sqrt{(2n+l+5/2)/12}. \tag{143}$$

For the subspaces of the set $p=1$ (47) with

$$l = -(n+1/2)+2s,\ s=0,1,...,n+1, \tag{144}$$

at the values $s=0$ ($l=-(n+1/2)$) and $s=n+1$ ($l=(n+3/2)$:

$$N^{(0,n,l)} = N_d^{(0,n,l)} = 1,\ N^{(1,n-1,l)} = 0,$$

$$|d^{(0,n,-(n+1/2))}\rangle = |3/2,-1/2\rangle|n,-n\rangle, \tag{145}$$

$$|d^{(0,n,(n+3/2))}\rangle = |3/2,3/2\rangle|n,n\rangle, \tag{146}$$

while at the values $s=1,...,n$:

$$N^{(0,n,l)} = 2,\ N_d^{(0,n,l)} = 1,\ N^{(1,n-1,l)} = N_c^{(nl)} = 1,$$

$$|v^{(1,n-1,l)}\rangle = i|1/2,1/2\rangle|n-1,l-1/2\rangle, \tag{147}$$

$$|v^{(0,n,l)}\rangle = c_0|w_0^{(0,n,l)}\rangle + c_1|w_1^{(0,n,l)}\rangle, \tag{148}$$

$$|d^{(0,n,l)}\rangle = c_1|w_0^{(0,n,l)}\rangle - c_0|w_1^{(0,n,l)}\rangle, \tag{149}$$

where

$$c_0 = \sqrt{\frac{(n+l+1/2)}{2(2n-l+5/2)}},\ c_1 = \sqrt{\frac{3(n-l+3/2)}{2(2n-l+5/2)}}, \tag{150}$$

$$|w_k^{(0,n,l)}\rangle = |3/2,-1/2+2k\rangle|n,l+1/2-2k\rangle, \tag{151}$$

$$\xi^{(nl)} = \sqrt{(2n-l+5/2)/12}. \tag{152}$$

5. SINGLE-PHOTON TRANSITIONS

The dimensions of the subspaces $V^{(anl)}$ also do not exceed the value $n+1$ (53). So the eigenvalues and eigenvectors of operators $\hat{D}_a$ in these subspaces may be easily obtained analytically in the case of single-photon transitions $n=1$.

For the subspaces with

$$l=-(J_0-1),-J_0,...,J_0-1, \tag{153}$$

$N^{(0,1,l)}=2$, $N_d^{(0,1,l)}=1$, $N^{(1,0,l)}=N_c^{(1,l)}=1$,

$$|v^{(0,1,l)}\rangle=c_0|w_0^{(0,1,l)}\rangle+c_1|w_1^{(0,1,l)}\rangle, \tag{154}$$

$$|d^{(0,1,l)}\rangle=c_1|w_0^{(0,1,l)}\rangle-c_0|w_1^{(0,1,l)}\rangle, \tag{155}$$

$$|v^{(1,0,l)}\rangle=i|J_1,l\rangle|0,0\rangle, \tag{156}$$

where

$$c_0=\frac{\xi_0^{(l)}}{\xi^{(l)}},\ c_1=\frac{\xi_1^{(l)}}{\xi^{(l)}},\ \xi^{(l)}=\xi^{(1l)}=\sqrt{\xi_0^{(l)2}+\xi_1^{(l)2}}, \tag{157}$$

$$\xi_0^{(l)}=f_1(l-1),\ \xi_1^{(l)}=f_2(l+1), \tag{158}$$

$$|w_k^{(0,1,l)}\rangle=|J_0,l-1+2k\rangle|1,1-2k\rangle,\ k=0,1, \tag{159}$$

while $f_1(m)$ and $f_2(m)$ are defined by the equations (21), (22).

In the case of transitions $J_0=J\to J_1=J-1$:

$$\xi_0^{(l)}=\left[\frac{(J-l)(J-l+1)}{(2J+1)2J(2J-1)}\right]^{1/2}, \tag{160}$$

$$\xi_1^{(l)}=\left[\frac{(J+l)(J+l+1)}{(2J+1)2J(2J-1)}\right]^{1/2}, \tag{161}$$

in the case of transitions $J_0=J\to J_1=J$:

$$\xi_0^{(l)}=\left[\frac{(J+l)(J-l+1)}{(J+1)2J(2J+1)}\right]^{1/2}, \tag{162}$$

$$\xi_1^{(l)}=-\left[\frac{(J-l)(J+l+1)}{(J+1)2J(2J+1)}\right]^{1/2}, \tag{163}$$

in the case of transitions $J_0=J\to J_1=J+1$:

$$\xi_0^{(l)}=\left[\frac{(J+l)(J+l+1)}{(2J+3)(2J+2)(2J+1)}\right]^{1/2}, \tag{164}$$

$$\xi_1^{(l)}=\left[\frac{(J-l)(J-l+1)}{(2J+3)(2J+2)(2J+1)}\right]^{1/2}. \tag{165}$$

For the subspaces with $l=\pm J_0$ in the case of transitions $J_0=J\to J_1=J-1$: $N^{(1,0,l)}=0$, $N^{(0,1,l)}=N_d^{(0,1,l)}=1$,

$$|d^{(0,1,\pm J)}\rangle=|J_0,\pm(J-1)\rangle|1,\pm1\rangle, \tag{166}$$

in the case of transitions $J_0=J\to J_1=J$:

$$N^{(1,0,l)}=N^{(0,1,l)}=N_c^{(1,l)}=1,$$

$$|v^{(0,1,\pm J)}\rangle=\pm|J_0,\pm(J-1)\rangle|1,\pm1\rangle, \tag{167}$$

$$|v^{(1,0,\pm J)}\rangle=i|J_1,\pm J\rangle|0,0\rangle, \tag{168}$$

$$\xi^{(l)}=\frac{1}{\sqrt{(J+1)(2J+1)}}, \tag{169}$$

in the case of transitions $J_0=J\to J_1=J+1$:

$$N^{(1,0,l)}=N^{(0,1,l)}=N_c^{(1,l)}=1,$$

$$|v^{(0,1,\pm J)}\rangle=|J_0,\pm(J-1)\rangle|1,\pm1\rangle, \tag{170}$$

$$|v^{(1,0,\pm J)}\rangle=i|J_1,\pm J\rangle|0,0\rangle, \tag{171}$$

$$\xi^{(l)}=\sqrt{\frac{J}{(J+1)(2J+3)}}. \tag{172}$$

For the subspaces with $l=\pm(J_0+1)$ in the case of transitions $J_0=J\to J_1=J-1$ and $J_0=J\to J_1=J$:

$$N^{(1,0,l)}=0,\ N^{(0,1,l)}=N_d^{(0,1,l)}=1,$$

$$|d^{(0,1,\pm(J+1))}\rangle=|J_0,\pm J\rangle|1,\pm1\rangle, \tag{173}$$

in the case of transitions $J_0=J\to J_1=J+1$:

$$N^{(1,0,l)}=N^{(0,1,l)}=N_c^{(1,l)}=1,$$

$$|v^{(0,1,\pm(J+1))}\rangle=|J_0,\pm J\rangle|1,\pm1\rangle, \tag{174}$$

$$|v^{(1,0,\pm(J+1))}\rangle=i|J_1,\pm(J+1)\rangle|0,0\rangle, \tag{175}$$

$$\xi^{(l)}=\frac{1}{\sqrt{2J+3}}. \tag{176}$$

Let us consider an example, when initially there is a single photon with the σ^+ polarization $\mathbf{s}_1$ in the cavity, while the atom is at its ground level, so that the initial density matrix of the system is as follows:

$$\hat{\rho}(0)=\sum_{m=-J_0}^{J_0}n_m^{(0)}|J_0,m\rangle|1,1\rangle\langle1,1|\langle J_0,m|. \tag{177}$$

The probability to find in the cavity at the instant of time t the photon with the orthogonal σ^- polarization $\mathbf{s}_2$ is given by the formula

$$w(t)=tr\{\hat{\rho}(t)|1,-1\rangle\langle1,-1|\}. \tag{178}$$

Then from the equations (55), (65) with an account of (153)–(176) we obtain

$$w(t)=\sum_{m=-J_0}^{J_0}n_m^{(0)}|F_{m+1}(t)|^2, \tag{179}$$

where

$$F_m(t) = \frac{\xi_0^{(m)}\xi_1^{(m)}}{\xi^{(m)2}}\left[C^{(m)}(t) - e^{i\Delta t/2}\right], \quad (180)$$

$$C^{(m)}(t) = \cos\left(\frac{\Omega^{(m)}t}{2}\right) + i\frac{\Delta}{\Omega^{(m)}}\sin\left(\frac{\Omega^{(m)}t}{2}\right), \quad (181)$$

$$\Omega^{(m)} = \sqrt{\Delta^2 + \theta^2\xi^{(m)2}}, \quad (182)$$

while $\xi_0^{(m)}$, $\xi_1^{(m)}$ and $\xi^{(m)}$ are defined by (157), (158). In the case of the exact resonance $\Delta = 0$ and the atom initially prepared at the pure state $n_m^{(0)} = \delta_{m,-1}$:

$$w(t) = \frac{1}{4}\left[1 - \cos\left(\frac{\Omega^{(0)}t}{2}\right)\right]^2, \Omega^{(0)} = \theta f_2(1)\sqrt{2}, \quad (183)$$

so that this probability attains the unity value at the instant of time $t = 2\pi/\Omega^{(0)}$.

6. SINGLE-PHOTON EMISSION

Now let us consider the emission of the photon by an atom at excited state into an empty cavity, so that the initial density matrix of the system is as follows:

$$\hat{\rho}(0) = \sum_{m,m'=-J_1}^{J_1} n_{mm'}^{(1)}|J_1, m\rangle|0,0\rangle\langle 0,0|\langle J_1, m'|. \quad (184)$$

While considering the photon emission let us take into account the relaxation processes. Then in the equation (54) for the system density matrix the relaxation terms must be added:

$$\frac{d\hat{\rho}}{dt} = \frac{i}{2}\left[\hat{\Omega}, \hat{\rho}\right] + \hat{L}_c + \hat{L}_a, \quad (185)$$

where the term

$$\hat{L}_c = -\frac{\gamma_c}{2}\sum_{i=1}^{2}\left(\hat{a}_i^\dagger\hat{a}_i\hat{\rho} + \hat{\rho}\hat{a}_i^\dagger\hat{a}_i - 2\hat{a}_i\hat{\rho}\hat{a}_i^\dagger\right), \quad (186)$$

describes the relaxation of the cavity field with the rate γ_c, while the term

$$\hat{L}_a = -\frac{\gamma_a}{2}(2J_1+1)\sum_{i=1}^{3}\left(\hat{g}_i^\dagger\hat{g}_i\hat{\rho} + \hat{\rho}\hat{g}_i^\dagger\hat{g}_i - 2\hat{g}_i\hat{\rho}\hat{g}_i^\dagger\right), \quad (187)$$

describes the spontaneous emission of the atom at the transitions $J_1 \to J_0$ into a free-space modes with the rate γ_a. In the equation (186) the summation is carried out over two cavity modes with unit polarization vectors $\mathbf{s}_1$ (σ^+) and $\mathbf{s}_2$ (σ^-), while in the equation (187) the summation is carried out over all three unit polarization vectors $\mathbf{s}_1$ (σ^+), $\mathbf{s}_2$ (σ^-) and $\mathbf{s}_3$ (π),

$$\sum_{i=1}^{2}\hat{a}_i^\dagger\hat{a}_i = \hat{n}, \quad \sum_{i=1}^{3}\hat{g}_i^\dagger\hat{g}_i = \hat{P}_A^{(1)}/(2J_1+1), \quad (188)$$

$\hat{n}$ and $\hat{P}_A^{(1)}$ being the photon number operator and the projector on the subspace of states referring to the excited atomic level. The presence of thermal photons in the relaxation terms is neglected since the photon emission in the optical domain is considered.

The probability w to find the photon in the cavity and photon polarization matrix $\hat{\sigma}$ are described by the 2×2 Hermitian density matrix of the photon $\hat{w}$:

$$w = \sum_{q=\pm 1} w_{qq}, \quad \sigma_{qq'} = w_{qq'}/w, \quad (189)$$

$$w_{qq'} = tr\left\{\hat{\rho}|1,q'\rangle\langle 1,q|\right\}, \quad (190)$$

where $\hat{\rho}$ is the system density matrix.

The orthonormal basis of the space of system states, which contribute to the photon dynamics, consists of the eigenvectors of the interaction operator:

$$|u_1^{(m)}\rangle = c_m^{(+)}|v^{(0,1,m)}\rangle + c_m^{(-)}|v^{(1,0,m)}\rangle, \quad (191)$$

$$|u_2^{(m)}\rangle = c_m^{(-)}|v^{(0,1,m)}\rangle - c_m^{(+)}|v^{(1,0,m)}\rangle, \quad (192)$$

$$c_m^{(\pm)} = \sqrt{\frac{1}{2}\left(1 \pm \frac{\Delta}{\Omega^{(m)}}\right)}, \quad (193)$$

where $|v^{(0,1,m)}\rangle$, $|v^{(1,0,m)}\rangle$, and $\Omega^{(m)}$ are defined by (154), (156) and (182) correspondingly. The master equation (185)–(187) for the system density matrix elements $\rho_{kk'}^{mm'} = \langle u_k^{(m)}|\hat{\rho}|u_{k'}^{(m')}\rangle$, $(k,k'=1,2)$ in the basis of these states is as follows:

$$\frac{d}{dt}\rho_{kk'}^{mm'} = \frac{i}{2}\left(\lambda_k^m - \lambda_{k'}^{m'}\right)\rho_{kk'}^{mm'} + L_{kk'}^{mm'}, \quad (194)$$

$$L_{kk'}^{mm'} = -\frac{1}{2}\sum_{s=1}^{2}\left(\gamma_{ks}^m\rho_{sk'}^{mm'} + \rho_{ks}^{mm'}\gamma_{sk'}^{m'}\right), \quad (195)$$

where

$$\lambda_1^m = \Omega^{(m)}, \quad \lambda_2^m = -\Omega^{(m)}, \quad (196)$$

$$\gamma_{11}^m = \gamma_c c_m^{(+)2} + \gamma_a c_m^{(-)2}, \gamma_{22}^m = \gamma_a c_m^{(+)2} + \gamma_c c_m^{(-)2}, \quad (197)$$

$$\gamma_{12}^m = \gamma_{21}^m = c_m^{(+)}c_m^{(-)}(\gamma_c - \gamma_a). \quad (198)$$

In the case of strong coupling ($\Omega^{(m)} \gg \gamma_c, \gamma_a$) the solution of the equations (194), (195) is easily obtained:

$$\rho_{kk'}^{mm'}(t) = \rho_{kk'}^{mm'}(0)f_{mk}^*(t)f_{m'k'}(t), \quad (199)$$

where

$$f_{mk}(t) = \exp\left\{-\frac{1}{2}\left(\gamma_{kk}^{m} + i\lambda_{k}^{m}\right)t\right\}. \qquad (200)$$

Then we obtain the following expression for the photon density matrix (190) in this approximation:

$$w_{q'q} = e^{-\gamma t}\sum_{m,m'} n_{mm'}^{(1)} h_{mq}^{*} h_{m'q'}\delta_{m'-m,q'-q}, \qquad (201)$$

$$h_{qm} = b_{q}^{(m)}\frac{\theta}{\Omega^{(m)}}\left(p_m + iq_m\right), \qquad (202)$$

$$p_m = \mathrm{sh}\left(\frac{\Delta^{(m)}t}{2}\right)\cos\left(\frac{\Omega^{(m)}t}{2}\right), \qquad (203)$$

$$q_m = \mathrm{ch}\left(\frac{\Delta^{(m)}t}{2}\right)\sin\left(\frac{\Omega^{(m)}t}{2}\right), \qquad (204)$$

$$\gamma = \frac{(\gamma_c + \gamma_a)}{2},\ \Delta^{(m)} = \frac{\Delta}{\Omega^{(m)}}\frac{(\gamma_c - \gamma_a)}{2}, \qquad (205)$$

where $b_1^{(m)} = \xi_0^{(m)}$, $b_{-1}^{(m)} = \xi_1^{(m)}$, while $\xi_0^{(m)}$, $\xi_1^{(m)}$, and $\Omega^{(m)}$ are defined by (158) and (182) correspondingly.

For example, in the case of transitions $J_0 = 0 \to J_1 = 1$ we obtain from (201)–(205) the following expressions for the photon polarization matrix $\hat{\sigma}$ and the probability to find the photon into the cavity $w(t)$ (189):

$$\sigma_{qq'} = n_{qq'}^{(1)}/n_0,\ n_0 = n_{11}^{(1)} + n_{-1,-1}^{(1)}, \qquad (206)$$

$$w(t) = n_0\frac{\theta^2\xi^{(1)2}}{\Omega^{(1)2}}F(t),\ \xi^{(1)} = \frac{1}{\sqrt{3}}, \qquad (207)$$

$$F(t) = \sin^2\left(\frac{\Omega^{(1)}t}{2}\right) + \mathrm{sh}^2\left(\frac{\Delta^{(1)}t}{2}\right), \qquad (208)$$

where $\Omega^{(1)}$ and $\Delta^{(1)}$ are defined by (182) and (205).

7. CONCLUSIONS

The problem of diagonalization of the Hamiltonian of the Jaynes – Cummings model with degenerate atomic levels and polarization-degenerate field mode may be reduced to the diagonalization of Hermitian matrix which dimension does not exceed the degree of degeneracy of atomic levels. Such diagonalization may be easily performed analytically for low values $(J_0, J_1 \leq 3/2)$ of atomic angular momentum or numerically otherwise. The evolution operator of the system is decomposed into blocks determined by the photon number n and the projection l of the system angular momentum on the cavity axis, such blocks being expressed through the corresponding eigenvalues and eigenvectors of the system Hamiltonian. The obtained expressions may be used for analytical or numerical studies of the atom-field dynamics in micro-cavities with arbitrary angular momenta of atomic levels and arbitrary polarization of the field mode.

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