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TIME FRACTIONAL EFFECT ON ION ACOUSTIC SHOCK WAVES IN ION-PAIR PLASMA

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The nonlinear properties of ion acoustic shock waves are studied. The Burgers equation is derived and converted into the time fractional Burgers equation by Agrawal's method. Using the Adomian decomposition method, shock wave solutions of the time fractional Burgers equation are constructed. The effect of the time fractional parameter on the shock wave properties in ion-pair plasma is investigated. The results obtained may be important in investigating the broadband electrostatic shock noise in D- and F-regions of Earth's ionosphere.

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1. INTRODUCTION

In last few years, ion acoustic waves in ion-pair plasmas were investigated both numerically and experimentally [1–3]. Many observations clearly indicate the presence of positive–negative ion structures in a variety of astrophysical plasma environments [4]. More specifically, negative ions are present at D-region altitudes of the ionosphere of Earth in coexistence with electrons as they are formed primarily by electrons added to electronegative species [5]. In the past decade, acoustic shocks in astrophysical plasma have been investigated [6–10]. The propagation of dust ion acoustic shocks in multi-ion plasmas was studied in [11]. It is noted that the acoustic shocks are modified by the heavy-to-light ion number density parameter. Also, potentials of both polarities exist in the plasma system [11]. On the other hand, applications of nonlinear fractional partial differential equations have received much attention in fluid mechanics and the physics of plasma [12–15]. In [16], electrostatic Viking satellite electron acoustic solitons observed in the dayside auroral zone were investigated by using the nonlinear time fractional Korteweg–de Vries (KdV) equation. Accordingly, the effect of trapped hot electrons on the dusty ion acoustic waves was discussed using the modified KdV equation with a time fractional term [17]. The progress of ion waves in an ion-pair plasma model was studied in [18] by means of the Gardner equation with a time fraction term. The method of variational iterations was used in [18] to investigate the effect of nonthermal electrons on the produced wave.

The studies on plasma physics using fractional nonlinear evolution equations have been discussed by many authors [19–22]. Later, the properties of dust acoustic shock waves have been studied in two-temperature dust plasmas using the Burgers equation with a time fractional order. The time fractional parameter effect on shock wave features was discussed using the variational iteration technique [23]. Furthermore, shock waves in dusty plasmas were studied in [24] using the space–time fractional KdV–Burgers equation. It was noted that the space–time fractional parameter affects the coexistence of shocks [24]. In this paper, an ion acoustic model with nonthermal electrons and ion pairs is considered. The KdV equation is derived and Agrawal's method [12, 25–27] is applied to formulate the time fractional KdV equation, and the Adomian decomposition method [28, 29] is used to solve it. In Sec. 2, we present the basic set of fluid equations for the system, and the Burgers equation with a time-fractional term is derived in Sec. 3. In Sec. 4, the Adomian decomposition method is used to solve the time-fractional Burgers equation. Section 5 contains the results and a discussion.

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2. BASIC EQUATIONS

In the model considered here, a three-component collisionless nonmagnetized plasma consists of viscous fluids of positive and negative ions and a nonthermal electron density distribution n_e. The normalized system of equations for a small but finite amplitude is given by [30, 31]

$$\frac{\partial n_\pm(x,t)}{\partial t} + \frac{\partial}{\partial x}(n_\pm(x,t)u_\pm(x,t)) = 0, \quad (1)$$

$$\left(\frac{\partial}{\partial t} + u_+(x,t)\frac{\partial}{\partial x}\right)u_+(x,t) = = -\frac{\partial \phi(x,t)}{\partial x} - \eta_1 \frac{\partial^2 u_+(x,t)}{\partial x^2}, \quad (2)$$

$$\left(\frac{\partial}{\partial t} + u_-(x,t)\frac{\partial}{\partial x}\right)u_-(x,t) = = \frac{Z_-}{Z_+}Q\frac{\partial \phi(x,t)}{\partial x} - \eta_2 \frac{\partial^2 u_-(x,t)}{\partial x^2}, \quad (3)$$

$$Z_+\frac{\partial^2 \phi(x,t)}{\partial x^2} = Z_- n_- \phi(x,t) + n_e\ \phi(x,t) - - Z_+ n_+ \phi(x,t) \quad (4)$$

for positive (+) and negative (−) ions. Here, n_e is the electron density that obeys the nonthermal distribution [32]

$$n_e(x,t) = \mu\left(1 - \beta\frac{Z_-}{Z_+}\phi(x,t) + \beta\left(\frac{Z_-}{Z_+}\right)^2\phi^2(x,t)\right)\times \times \exp\left(\frac{Z_-}{Z_+}\phi(x,t)\right), \quad (5)$$

$$\beta = 4\delta/(1+3\delta), \quad (6)$$

where δ is the electron nonthermal parameter and $\mu = n_{e0}/n_{+0}$ is the unperturbed electron-to-positive-ion ratio. In the above equations, n_+ (n_-) is the positive (negative) ion number density normalized by $n_{+0}(n_{-0})$, u_+ (u_-) is the positive (negative) ion fluid velocity normalized by the ion sound speed $C_s = (k_B T_e/m_+)^{1/2}$, and ϕ is the electrostatic wave potential (normalized by $k_B T_e/eZ_+$). Also, $x(t)$ is the space (time) coordinate with x normalized to the Debye length of hot electrons $\lambda_{Di} = (k_B T_e/4\pi e^2 Z_+^2 n_{+0})^{1/2}$ and t normalized by the inverse plasma frequency of cold electrons $\omega_{pi}^{-1} = (4\pi e^2 Z_+^2 n_{+0}/m_+)^{-1/2}$, where k_B is Boltzmann's constant. The normalized kinematic viscosity coefficients are $\eta_1 = \eta_+\omega_{pi}\lambda_{Di}^2$ and $\eta_2 = \eta_-\omega_{pi}\lambda_{Di}^2$; T_e is the electron temperature; $\nu = n_{-0}/n_{+0}$; the mass ratio $Q = m_+/m_-$, where m_+ and m_- are the positive and negative ion fluid masses, and $Z_\pm$ is the ionic charge number. The neutrality condition implies that

$$Z_+ = Z_-\nu + \mu. \quad (7)$$

For simplicity, we assume that $Z_+ = Z_- = 1$.

3. NONLINEAR ANALYSIS

The slow stretched coordinates introduced using the RPT method [33] are

$$\tau = \epsilon^{3/2}t, \quad \xi = \epsilon^{1/2}(x - \lambda t), \quad (8)$$

where λ is the normalized speed of the wave (to be determined later) and ϵ is a very small perturbation parameter. We expand the quantities in Eqs. (1)–(6) about their equilibrium values:

$$\begin{aligned} n_+(\xi,\tau) &= 1 + \epsilon N_{+1}(\xi,\tau) + \epsilon^2 N_{+2}(\xi,\tau) + \epsilon^3 N_{+3}(\xi,\tau) + \ldots, \\ u_+(\xi,\tau) &= \epsilon U_{+1}(\xi,\tau) + \epsilon^2 U_{+2}(\xi,\tau) + \epsilon^3 U_{+3}(\xi,\tau) + \ldots, \\ n_-(\xi,\tau) &= 1 + \epsilon N_{-1}(\xi,\tau) + \epsilon^2 N_{-2}(\xi,\tau) + \epsilon^3 N_{-3}(\xi,\tau) + \ldots, \\ u_-(\xi,\tau) &= \epsilon U_{-1}(\xi,\tau) + \epsilon^2 U_{-2}(\xi,\tau) + \epsilon^3 U_{-3}(\xi,\tau) + \ldots, \\ \phi(\xi,\tau) &= \epsilon\phi_1(\xi,\tau) + \epsilon^2\phi_2(\xi,\tau) + \epsilon^3\phi_3(\xi,\tau) + \ldots \end{aligned} \quad (9)$$

The last equations are valid under the conditions $|\xi| \to \infty$, $n_- = n_+ = 1$, $u_+ = 0$, and $\phi = 0$. Using Eqs. (8) and (9) in Eqs. (1)–(6) and applying the perturbation conditions, the lowest-order in ϵ gives

$$\begin{aligned} N_{+1}(\xi,\tau) &= \frac{\phi_1(\xi,\tau)}{\lambda^2}, \quad U_{+1}(\xi,\tau) = \frac{\phi_1(\xi,\tau)}{\lambda}, \\ N_{-1}(\xi,\tau) &= -\frac{Q\nu\phi_1(\xi,\tau)}{\lambda^2}, \\ U_{-1}(\xi,\tau) &= -\frac{Q\ \phi_1(\xi,\tau)}{\lambda}. \end{aligned} \quad (10)$$

The dispersion form is given by

$$\frac{(\beta-1)\lambda^2\mu + \nu Q + 1}{\lambda^2} = 0. \quad (11)$$

The next equations, of the order $O(\epsilon^2)$, yields

$$
\begin{aligned}
&-\lambda\frac{\partial N_{+2}(\xi,\tau)}{\partial\xi}+\frac{\partial U_{+2}(\xi,\tau)}{\partial\xi}+\\
&+\frac{\partial N_{+1}(\xi,\tau)}{\partial\tau}+N_{+1}(\xi,\tau)\frac{\partial U_{+1}(\xi,\tau)}{\partial\xi}+\\
&+U_{+1}(\xi,\tau)\frac{\partial N_{+1}(\xi,\tau)}{\partial\xi}=0,\\
&-\lambda\frac{\partial U_{+2}(\xi,\tau)}{\partial\xi}+\frac{\partial U_{+1}(\xi,\tau)}{\partial\tau}+\frac{\partial\phi_2(\xi,\tau)}{\partial\xi}+\\
&+U_{+1}(\xi,\tau)\frac{\partial U_{+1}(\xi,\tau)}{\partial\xi}+\eta_1\frac{\partial^2U_{+1}(\xi,\tau)}{\partial\xi^2}=0,\\
&-\lambda\frac{\partial N_{-2}(\xi,\tau)}{\partial\xi}+\nu\frac{\partial U_{-2}(\xi,\tau)}{\partial\xi}+\frac{\partial N_{-1}(\xi,\tau)}{\partial\xi}+\\
&+N_{-1}(\xi,\tau)\frac{\partial U_{-1}(\xi,\tau)}{\partial\xi}+U_{-1}(\xi,\tau)\frac{\partial N_{-1}(\xi,\tau)}{\partial\xi}=0,\\
&-\lambda\frac{\partial U_{-2}(\xi,\tau)}{\partial\xi}+\frac{\partial U_{-1}(\xi,\tau)}{\partial\tau}-Q\frac{\partial\phi_2(\xi,\tau)}{\partial\xi}+\\
&+U_{-1}(\xi,\tau)\frac{\partial U_{-1}(\xi,\tau)}{\partial\xi}+\eta_2\frac{\partial^2U_{-1}(\xi,\tau)}{\partial\xi^2}=0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\mu\phi_1^2(\xi,\tau)+(\mu-\beta\mu)\phi_2(\xi,\tau)+\\
&\qquad+N_{-2}(\xi,\tau)-N_{+2}(\xi,\tau)=0. \quad (12)
\end{aligned}
$$

Eliminating $N_{+2}(\xi,\tau)$, $N_{-2}(\xi,\tau)$, $\phi_2(\xi,\tau), U_{+2}(\xi,\tau)$, and $U_{-2}\ (\xi,\tau)$ from Eqs. (12), we obtain the Burgers equation for ϕ_1:

$$
\frac{\partial\phi_1(\xi,\tau)}{\partial\tau}+A\,\phi_1\frac{\partial\phi_1(\xi,\tau)}{\partial\xi}+B\frac{\partial_1^2\phi(\xi,\tau)}{\partial\xi^2}=0, \quad (13)
$$

where

$$
A=-\frac{\lambda^4\mu+3\nu Q^2-3}{2(\lambda+\lambda\nu Q)},\qquad B=\frac{\eta_1+\eta_2\nu Q}{2\nu Q+2}.
$$

Equation (13) admits the ion acoustic shock wave solution in the form

$$
\Phi_1=\frac{2B}{A}\left(1+\tanh\chi\right), \quad (14)
$$

whose amplitude equals $2B/A$. In Eq. (13), $\phi_1(\xi,\tau)$ is a field variable, $\tau\in T=[0,T_0]$ is the time variable, and ξ is a space coordinate in the propagation direction of the field. In [23], the time fractional Burgers equation in time and one space dimension was derived in the form

$$
\begin{aligned}
&{}_0^RD_\tau^{-\alpha}\phi_1(\xi,\tau)+A\,\phi_1(\xi,\tau)\frac{\partial\phi_1(\xi,\tau)}{\partial\xi}+\\
&\qquad+B\frac{\partial^2\phi_1(\xi,\tau)}{\partial\xi^2}=0, \quad (15)
\end{aligned}
$$

where the Riesz fractional operator is represented by

$$
\begin{aligned}
{}_0^RD_\tau^{-\alpha}\phi_1(\xi,\tau)&=\frac{1}{2}\frac{1}{\Gamma(1-\alpha)}\times\\
&\times\left[\frac{d}{d\tau}\int\limits_0^\tau dt(\tau-t)^{-\alpha}\phi_1(\xi,\tau)-\right.\\
&\left.-\frac{d}{d\tau}\int\limits_\tau^{T_0}dt(t-\tau)^{-\alpha}\phi_1(\xi,\tau)\right]=\\
&=\frac{1}{2}\frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int\limits_0^{T_0}dt\,|\tau-t|^{-\alpha}\,\phi_1(\xi,\tau). \quad (16)
\end{aligned}
$$

Equation (15) represents the Burgers equation with a time-fractional term that is formulated using the Euler–Lagrange variational technique, which we solve using the Adomian decomposition method (ADM) in the next section.

4. SOLUTION OF THE TIME-FRACTIONAL BURGERS EQUATION

The ADM [28, 29] consists in decomposing the solution or the unknown function $\psi(\xi,\tau)$ in a nonlinear equation into an infinite series in the components $\psi_n(\xi,\tau)$ defined by

$$
\psi(\xi,\tau)=\sum_{n=0}^{\infty}\psi_n(\xi,\tau), \quad (17a)
$$

and representing the nonlinear term $\Theta(\psi(\xi,\tau))$ by the Adomian series as

$$
\Theta(\psi(\xi,\tau))=\sum_{n=0}^{\infty}A_n, \quad (17b)
$$

where A_n are the Adomian polynomials, which are given by

$$
A_0=\Theta(\psi_0(\xi,\tau)), \quad (18a)
$$

$$
A_1=\psi_1\frac{\partial\Theta(\psi_0)}{\partial\psi_0}, \quad (18b)
$$

$$
A_2=\psi_2\frac{\partial\Theta(\psi_0)}{\partial\psi_0}+\frac{1}{2}\psi_1^2\frac{\partial^2\Theta(\psi_0)}{\partial\psi_0^2}, \quad (18c)
$$

and the other polynomials can be generated in the same manner. Applying the operator ${}_0^RD_\tau^{-\alpha}$ to both sides of Eq. (15) yields the equation

$$
\begin{aligned}
&\phi_1(\xi,\tau)=\phi_1(\xi,0)-{}_0^RD_\tau^{-\alpha}\times\\
&\times\left[A\,\phi_1(\xi,\tau)\frac{\partial\phi_1(\xi,\tau)}{\partial\xi}+B\frac{\partial^2\phi_1(\xi,\tau)}{\partial\xi^2}\right], \quad (19)
\end{aligned}
$$

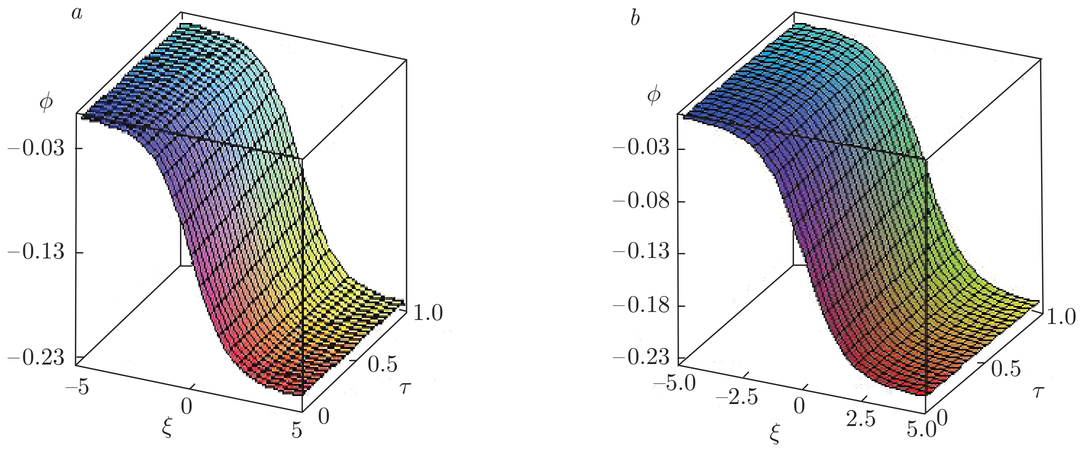

Fig. 1. (Color online) The electrostatic potential versus the position ξ and time τ at $\nu = 0.3$, $\delta = 0.3$, $Q = 0.03$, $\eta_1 = 0.5$, $\eta_2 = 0.2$, and $\mu = 0.5$; the comparison between (a) the time-fractional Burgers equation with an ADM solution at $\alpha = 1.0$ and (b) the classical Burgers solution

where ${}_0^R D_\tau^{-\alpha}$ is the Riemann–Liouville fractional integral, which is defined using the formula [32, 33]

$$ {}_0^R D_\tau^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{g(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau, \quad 0 \le \alpha < 1. \tag{20} $$

Substituting (17) in (19) leads to

$$ \sum_{n=0}^{\infty} \phi_{1n}(\xi,\tau) = \phi_1(\xi,0) - {}_0^R D_\tau^{-\alpha} \times \times \left[A\,\phi_1(\xi,\tau) \frac{\partial \phi_1(\xi,\tau)}{\partial \xi} + B \frac{\partial^2 \phi_1(\xi,\tau)}{\partial \xi^2} \right]. \tag{21} $$

The components $\phi_{1n}(\xi,\tau)$ of the solution $\phi_1(\xi,0)$ can be computed using the recursive relation

$$ \phi_{10}(\xi,0) = \phi_1(\xi,0), \tag{22a} $$

$$ \phi_{1k+1}(\xi,\tau) = -{}_0^R D_\tau^{-\alpha} \left[A\, A_k + B \frac{\partial^2 \phi_1(\xi,\tau)}{\partial \xi^2} \right], \quad k \ge 1. \tag{22b} $$

The initial condition is taken as

$$ \phi_1(\xi,0) = \phi_m \left[1 + \tanh(c\xi) \right], \tag{23a} $$

where ϕ_m and c are the amplitude and the width of the wave, which are dependent on the physical parameters of the system:

$$ \phi_m = \nu/A, \quad c = \nu/2B. \tag{23b} $$

The zeroth-order Adomian decomposition solution can be taken as the initial value of the state variable, which is taken as the solution of the regular equation at time equal to zero, in this case as

$$ \phi_{10}(\xi,0) = \phi_1(\xi,0) = \phi_m \left[1 + \tanh(c\,\xi) \right]. \tag{24a} $$

Substituting this zeroth order in recursive relation (22b) leads to the first recursive equation

$$ \phi_{11}(\xi,\tau) = -cA\phi_m^2 \operatorname{sech}^2(\xi) \frac{\tau^\alpha}{\Gamma(\alpha+1)}. \tag{24b} $$

Substituting this first order in recursive relation (22b) gives the second order correction as

$$ \phi_{12}(\xi,\tau) = -8B^2c^4\ \phi_m \operatorname{sech}^3(\xi) \sinh(\xi) \times \times \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{24c} $$

In the same manner, the higher recursive orders can be calculated using the Maple or the Mathematica package. Substituting these different recursive orders in (17a) yields the solution of the time-fractional Burgers equation as

$$ \phi_1(\xi,\tau) = \phi_{12}(\xi,\tau) = \phi_m \left[1 + \tanh(c\xi) \right] - - cA\phi_m^2 \operatorname{sech}^2(\xi) \frac{\tau^\alpha}{\Gamma(\alpha+1)} - - 8B^2c^4\phi_m \operatorname{sech}^3(\xi) \sinh(\xi) \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots \tag{25} $$

5. MODEL RESULTS AND DISCUSSIONS

Numerical investigation have been done using the plasma parameters for the D and F regions of the

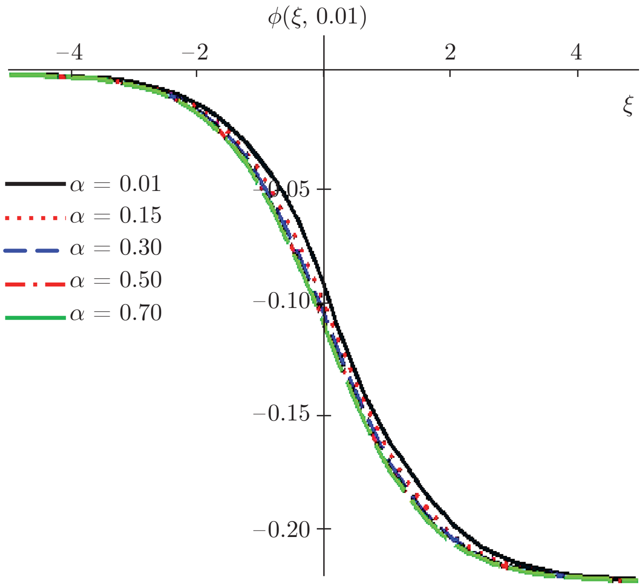

Fig. 2. (Color online) The electrostatic potential $\phi(\xi, 0.01)$ versus the position ξ at $\nu = 0.3$, $\delta = 0.3$, $Q = 0.1$, $\eta_1 = 0.5$, $\eta_2 = 0.3$, and $\mu = 0.5$ for different values of fractional parameter α

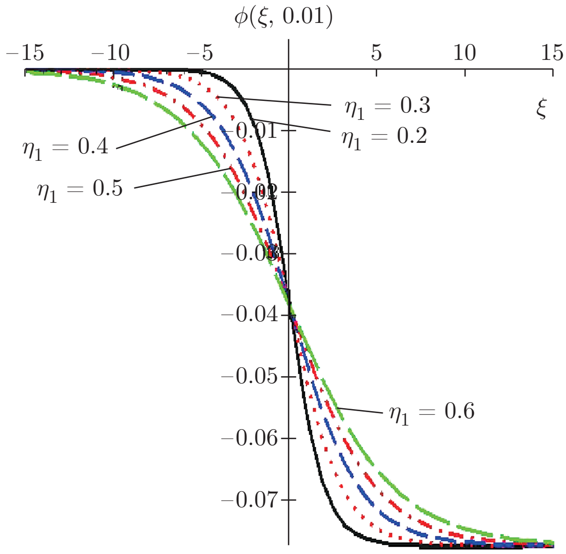

Fig. 3. (Color online) The electrostatic potential $\phi(\xi, 0.01)$ versus the position ξ at $\nu = 0.1$, $\delta = 0.3$, $Q = 0.03$, $\eta_2 = 0.5$, $\alpha = 0.5$, and $\mu = 0.5$ for different values of the kinematic viscosity coefficient of positive ions η_1

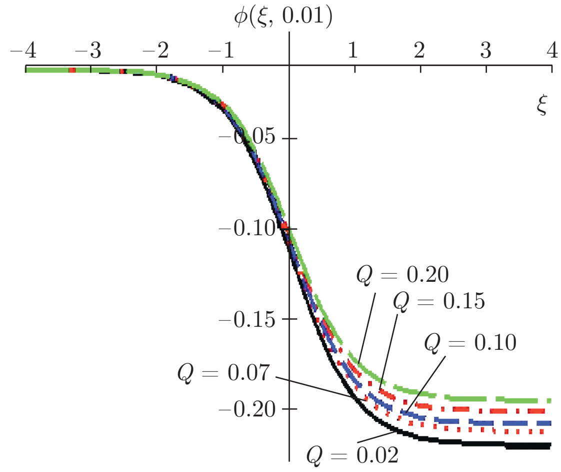

Fig. 4. (Color online) The electrostatic potential $\phi(\xi, 0.01)$ versus the position ξ at $\nu = 0.3$, $\delta = 0.3$, $\eta_1 = 0.3$, $\eta_2 = 0.2$, $\alpha = 0.5$, and $\mu = 0.5$ for different values of the positive–negative mass ration Q

Earth's ionosphere [30, 31, 34]. Our aim was to study the effect of the fractional parameter of time α, the positive–negative mass ratio Q, the electron nonthermal parameter δ, and the kinematic viscosity coefficients η_1 and η_2 of positive and negative ions on the propagating ion acoustic shocks. In our model, the ion acoustic shock waves are described by time-fractional Burgers equation (13). According to the wave dissipation caused by kinematic viscosity, this system supports ion shock waves.

The comparison between shock solutions of time-fractional Burgers equation with the ADM method and the classical Burgers equation is shown in Fig. 1. The effect of the time-fractional parameter α on the shock wave properties is examined in Fig. 2. It is noted that introducing the time fractionality decreases the shock wave steepness. The effect of the kinematic viscosity coefficient η_1 of positive ions and the positive–negative mass ratio Q on the amplitude and steepness of shock waves are depicted in Figs. 3 and 4. It is found that the increase in the parameters η_1 and Q decreases the steepness and amplitude of the shock waves. Finally, an increase in the electron nonthermal parameter δ decreases the steepness and amplitude of the ion acoustic shock waves (Fig. 5).

In summary, the time fractional factor α would modulate the shape and existence of the shock wave profile. Also, it has been shown that the obtained shock wave is sensitive to the positive–negative mass ratio Q, the electron nonthermal parameter δ, and the kinematic viscosity coefficients of positive and negative ions η_1. The results obtained here may be useful in understanding the features of broadband electrostatic shock noise in D- and F-regions of the Earth's ionosphere.

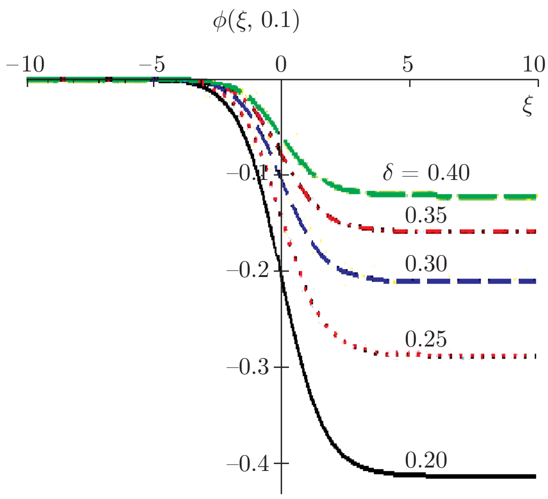

Fig. 5. (Color online) The electrostatic potential $\phi(\xi, 0.1)$ versus the position ξ at $\nu = 0.3$, $\eta_1 = 0.5$, $Q = 0.1$, $\eta_2 = 0.3$, $\alpha = 0.5$, and $\mu = 0.5$ for different values of the electron nonthermal parameter δ

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