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THERMODYNAMIC PRODUCT FORMULA FOR A TAUB–NUT BLACK HOLE

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We derive various important thermodynamic relations of the inner and outer horizons in the background of the Taub–NUT (Newman–Unti–Tamburino) black hole in four-dimensional Lorentzian geometry. We compare these properties with the properties of the Reissner–Nordstrøm black hole. We compute the area product, area sum, area subtraction, and area division of black hole horizons. We show that they all are not universal quantities. Based on these relations, we compute the area bound of all horizons. From the area bound, we derive an entropy bound and an irreducible mass bound for both horizons. We further study the stability of such black holes by computing the specific heat for both horizons. It is shown that due to the negative specific heat, the black hole is thermodynamically unstable. All these calculations might be helpful in understanding the nature of the black hole entropy (both interior and exterior) at the microscopic level.

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1. INTRODUCTION

There has recently been intense interest in the thermodynamic product formulas in both the general relativity community [1] and the string/M-theory community [2] to understand the black hole (BH) entropy [3] of multi-horizons at the microscopic level. Significant achievments have been made in the case of asymptotically flat supersymmetric BHs in four and five dimension, where the microscopic degrees of freedom could be explained in terms of a two-dimensional (2D) conformal field theory (CFT) [4]. For the microscopic entropy of extreme rotating BHs, the results can be found in [5].

In the case of a regular axisymmetric and stationary spacetime of Einstein–Maxwell gravity with surrounding matter, the area product formula of the event horizon ($\mathcal{H}^+$) and the Cauchy horizons ($\mathcal{H}^-$) is [1]

$$\frac{\mathcal{A}_+\mathcal{A}_-}{64\pi^2} = J^2 + \frac{Q^4}{4}, \tag{1}$$

and consequently the entropy product formula of $\mathcal{H}^\pm$ is

$$\frac{\mathcal{S}_+\mathcal{S}_-}{4\pi^2} = J^2 + \frac{Q^4}{4}. \tag{2}$$

In the absence of Maxwell gravity, these product formulas reduce to the form [6]

$$\frac{\mathcal{A}_+\mathcal{A}_-}{64\pi^2} = J^2 \tag{3}$$

and

$$\frac{\mathcal{S}_+\mathcal{S}_-}{4\pi^2} = J^2. \tag{4}$$

In the above formulas, the interesting point is that they are all independent of the mass, so-called the ADM (Arnowitt–Deser–Misner) mass of the background spacetime. Hence, they are universal quantities in this sense. If we incorporate the Bogomol'nyi–Prasad–Sommerfield (BPS) states, the area product formula becomes [2]

$$\frac{\mathcal{A}_+\mathcal{A}_-}{64\pi^2} = \left(\sqrt{N_1} \pm \sqrt{N_2}\right) = N, \quad N \in \mathbb{N}, \quad N_1 \in \mathbb{N}, \quad N_2 \in \mathbb{N}, \tag{5}$$

where the integers N_1 and N_2 can be considered as the excitation numbers of the left- and right-moving modes of a weakly coupled two-dimensional CFT, which depend explicitly on all the BH parameters.

Consequently, the entropy product formula of $\mathcal{H}^\pm$ should be

$$\frac{\mathcal{S}_+\mathcal{S}_-}{4\pi^2} = \left(\sqrt{N_1} \pm \sqrt{N_2}\right) = N, \quad N \in \mathbb{N}, \quad N_1 \in \mathbb{N}, \quad N_2 \in \mathbb{N}. \tag{6}$$

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It implies that the product of $\mathcal{H}^{\pm}$ can be expressed in terms of the underlying CFT, which can also be interpreted in terms of a level-matching condition. That means that the product of entropies of $\mathcal{H}^{\pm}$ is an integer quantity [7].

It is true that certain BHs have an inner horizon or a Cauchy horizon in addition to the outer horizon or the event horizon. Therefore, the inner horizon in BH thermodynamics must be relevant to the understanding of the microscopic nature of the inner BH entropy in comparison with the outer BH entropy. It is also true that the Cauchy horizon is a blue-shift region whereas the event horizon is a red-shift region by its own nature. Furthermore, the Cauchy horizon is highly unstable due to exterior perturbations [8]. Despite the above characteristics, the Cauchy horizon plays a crucial role in BH thermodynamics.

In this work, we focus on thermodynamic properties of both inner and outer horizons of the Lorentzian Taub–NUT (Newman–Unti–Tamburino) BH. The special property of this BH is it is non-asymptotic type in comparison with the Reissner–Nordström (RN) BH, which is asymptotic type. The special features give a motivation to study them.

The plan of the paper is as follows. In Sec. 2, we describe various thermodynamic properties of the Taub–NUT BH. In this section, there are five subsections. In the first subsection, we derive the area bound of all horizons for the Taub–NUT BH. In the second subsection, we derive the entropy bound for the Taub–NUT BH. In the third subsection, we compute the irreducible mass bound of $\mathcal{H}^{\pm}$. In the fourth subsection, we derive a temperature bound of all horizons and finally, in last subsection, we compute a specific heat bound for both horizons. We conclude in Sec. 3.

2. THERMODYNAMIC PROPERTIES OF THE LORENTZIAN TAUB–NUT BH

The Lorentzian Taub–NUT BH is a stationary, spherically symmetric vacuum solution of the Einstein equations with the NUT parameter n. The NUT charge, or dual mass, has an intrinsic meaning in Einstein's general relativity, being a gravitational analogue of a magnetic monopole in Maxwell's electrodynamics [9]. The presence of the NUT parameter in a spacetime destroys its asymptotic structure, making it asymptotically nonflat, in contrast to the RN spacetime.

The metric is given by [10–12]

$$ds^2 = -\mathcal{B}(r)\left(dt + 2n\cos\theta d\phi\right)^2 + \frac{dr^2}{\mathcal{B}(r)} + \\ + \left(r^2+n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (7)$$

$$\mathcal{B}(r) = 1 - \frac{2(\mathcal{M}r + n^2)}{r^2+n^2}, \quad (8)$$

where $\mathcal{M}$ denotes the gravito-electric mass, or the ADM mass and n denotes the gravito-magnetic mass, or the dual mass or the magnetic mass of the spacetime. The idea of a dual mass can be found in Ref. [13]. It is evident that two types of singularities are present in metric (7). One type occurs at $B(r) = 0$, which gives us the Killing horizons or the BH horizons

$$r_{\pm} = \mathcal{M} \pm \sqrt{\mathcal{M}^2 + n^2} \text{ and } r_+ > r_-. \quad (9)$$

Here, r_+ is called the event horizon ($\mathcal{H}^+$) or the outer horizon and r_- is called the Cauchy horizon ($\mathcal{H}^-$) or the inner horizon. The other singularity types occur at $\theta = 0$ and $\theta = \pi$, where the determinant of the metric component vanishes. Misner [14] showed that in order to remove the apparent singularities at $\theta = 0$ and at $\theta = \pi$, t must be identified modulo $8\pi n$ if $r^2 + n^2 \neq 2(\mathcal{M}r + n^2)$. We note that the NUT parameter actually measures the deviation from the asymptotic flatness at infinity, which can be manifested in the off-diagonal components of the metric and hence occurs due to presence of the Dirac–Misner singularity type.

As long as

$$\mathcal{M}^2 + n^2 \geq 0, \quad (10)$$

the Taub–NUT metric describes a BH; otherwise, it has a naked singularity. When $\mathcal{M}^2 + n^2 = 0$, we find the extremal Taub–NUT BH.

The product and sum of horizon radii become

$$r_+ r_- = -n^2 \text{ and } r_+ + r_- = 2\mathcal{M}. \quad (11)$$

The area [15] of this BH is given by

$$\mathcal{A}_{\pm} = \int_0^{2\pi}\int_0^{\pi} \sqrt{g_{\theta\theta}g_{\phi\phi}}\Big|_{r=r_{\pm}} d\theta\, d\phi = \\ = 8\pi\left[(\mathcal{M}^2+n^2) \pm \mathcal{M}\sqrt{\mathcal{M}^2+n^2}\right]. \quad (12)$$

Their product [15] and sum are

$$\mathcal{A}_+\mathcal{A}_- = (8\pi n)^2\left(\mathcal{M}^2+n^2\right), \\ \mathcal{A}_+ + \mathcal{A}_- = 16\pi\left(\mathcal{M}^2+n^2\right). \quad (13)$$

It follows that the product is dependent on both the mass parameter and the NUT parameter. Hence, it is not a universal quantity. The conjecture "the area product is universal" does not therefore hold for a non-asymptotic Taub–NUT spacetime.

For completeness, we further compute the area difference and ratio:

$$\mathcal{A}_\pm - \mathcal{A}_\mp = 8\pi\mathcal{M}T_\pm\mathcal{A}_\pm \tag{14}$$

and

$$\frac{\mathcal{A}_+}{\mathcal{A}_-} = \frac{r_+^2+n^2}{r_-^2+n^2}. \tag{15}$$

Also, the sum of inverse areas is found to be

$$\frac{1}{\mathcal{A}_+} + \frac{1}{\mathcal{A}_-} = \frac{1}{4\pi n^2}, \tag{16}$$

and the difference of inverse areas is

$$\frac{1}{\mathcal{A}_\pm} - \frac{1}{\mathcal{A}_\mp} = \mp\frac{\mathcal{M}}{8\pi n^2\sqrt{\mathcal{M}^2+n^2}}. \tag{17}$$

It follows that they are all mass-dependent relations.

Likewise, the entropy product [15] and the entropy sum of $\mathcal{H}^\pm$ become

$$\begin{gathered} \mathcal{S}_-\mathcal{S}_+ = (2\pi n)^2\left(\mathcal{M}^2+n^2\right), \\ \mathcal{S}_- + \mathcal{S}_+ = 4\pi\left(\mathcal{M}^2+n^2\right). \end{gathered} \tag{18}$$

For our record, we also compute the entropy difference of $\mathcal{H}^\pm$ as

$$\mathcal{S}_\pm - \mathcal{S}_\mp = 8\pi\mathcal{M}T_\pm\mathcal{S}_\pm \tag{19}$$

and the entropy ratio of $\mathcal{H}^\pm$ as

$$\frac{\mathcal{S}_+}{\mathcal{S}_-} = \frac{r_+^2+n^2}{r_-^2+n^2}. \tag{20}$$

The sum of inverse entropies is found to be

$$\frac{1}{\mathcal{S}_+} + \frac{1}{\mathcal{S}_-} = \frac{1}{\pi n^2}, \tag{21}$$

and the difference of inverse entropies is

$$\frac{1}{\mathcal{S}_\pm} - \frac{1}{\mathcal{S}_\mp} = \mp\frac{\mathcal{M}}{2\pi n^2\sqrt{\mathcal{M}^2+n^2}}. \tag{22}$$

The Hawking temperatures [16] of $\mathcal{H}^\pm$ are

$$T_\pm = \frac{r_\pm - r_\mp}{8\pi\left(\mathcal{M}r_\pm + n^2\right)}, \quad T_+ > T_-. \tag{23}$$

Their product [15] and sum are

$$\begin{gathered} T_+T_- = -\frac{1}{4\pi n^2}, \\ T_+ + T_- = -\frac{\mathcal{M}}{2\pi n^2}. \end{gathered} \tag{24}$$

It may be noted that the surface temperature product is independent of the mass, while the sum depends on the mass, and hence the product is universal whereas the sum is not. It is also shown for the Taub–NUT BH that

$$T_+\mathcal{S}_+ + T_-\mathcal{S}_- = 0. \tag{25}$$

It is evident that this is a mass-independent (universal) relation, which implies that $T_+\mathcal{S}_+ = -T_-\mathcal{S}_-$ may be taken as a criterion of whether there is a 2D CFT dual for the BHs in Einstein gravity and other diffeomorphism gravity theories [17,18]. This universal relation also indicates that the left and right central charges are equal, $c_L = c_R$, which is holographically dual to 2D CFT.

2.1. Area bound of the Taub–NUT BH for $\mathcal{H}^\pm$

Now we are ready to derive the area bound relations for both the horizons. Using the ineqality in Eq. (10), we can obtain $\mathcal{M}^2 \geq -n^2$. Because $r_+ \geq r_-$, we obtain $\mathcal{A}_+ \geq \mathcal{A}_- \geq 0$. Then the area product gives

$$\mathcal{A}_+ \geq \sqrt{\mathcal{A}_+\mathcal{A}_-} = 8\pi n\sqrt{\mathcal{M}^2+n^2} \geq \mathcal{A}_-, \tag{26}$$

and the area sum is

$$\begin{aligned} 16\pi\left(\mathcal{M}^2+n^2\right) = \mathcal{A}_+ + \mathcal{A}_- \geq \mathcal{A}_+ \geq \\ \geq \frac{\mathcal{A}_+ + \mathcal{A}_-}{2} = 8\pi\left(\mathcal{M}^2+n^2\right). \end{aligned} \tag{27}$$

Therefore, the area bound for $\mathcal{H}^+$ is

$$8\pi\left(\mathcal{M}^2+n^2\right) \leq \mathcal{A}_+ \leq 16\pi\left(\mathcal{M}^2+n^2\right), \tag{28}$$

and the area bound for $\mathcal{H}^-$ is

$$0 \leq \mathcal{A}_- \leq 8\pi n\sqrt{\mathcal{M}^2+n^2}. \tag{29}$$

2.2. Entropy bound for $\mathcal{H}^\pm$

Similarly, because $r_+ \geq r_-$, we get $\mathcal{S}_+ \geq \mathcal{S}_- \geq 0$. Therefore, the entropy product gives

$$\mathcal{S}_+ \geq \sqrt{\mathcal{S}_+\mathcal{S}_-} = 2\pi n\sqrt{\mathcal{M}^2+n^2} \geq \mathcal{S}_-, \tag{30}$$

and the entropy sum gives

$$\begin{aligned} 4\pi\left(\mathcal{M}^2+n^2\right) = \mathcal{S}_+ + \mathcal{S}_- \geq \mathcal{S}_+ \geq \\ \geq \frac{\mathcal{S}_+ + \mathcal{S}_-}{2} = 2\pi\left(\mathcal{M}^2+n^2\right). \end{aligned} \tag{31}$$

Thus, the entropy bound for $\mathcal{H}^+$ is given by

$$2\pi\left(\mathcal{M}^2+n^2\right) \leq \mathcal{S}_+ \leq 4\pi\left(\mathcal{M}^2+n^2\right), \tag{32}$$

and the entropy bound for $\mathcal{H}^-$ is

$$0 \leq \mathcal{S}_- \leq 2\pi n\sqrt{\mathcal{M}^2+n^2}. \tag{33}$$

2.3. Irreducible mass bound for $\mathcal{H}^{\pm}$

Christodoulou [19] derived a relation between the surface area of $\mathcal{H}^{+}$ and the irreducible mass, which can be written as

$$\mathcal{M}_{irr,\pm} = \sqrt{\frac{\mathcal{A}_{\pm}}{16\pi}} = \sqrt{\frac{\mathcal{S}_{\pm}}{4\pi}}. \tag{34}$$

The product and the sum of the irreducible masses for both horizons are

$$\begin{gathered}\mathcal{M}_{irr,+}\mathcal{M}_{irr,-} = \frac{n\sqrt{\mathcal{M}^2+n^2}}{2}, \\ \mathcal{M}^2_{irr,+} + \mathcal{M}^2_{irr,-} = \mathcal{M}^2 + n^2.\end{gathered} \tag{35}$$

From the area bound, we obtain the irreducible mass bound for the Taub–NUT BH:

$$\frac{\sqrt{\mathcal{M}^2+n^2}}{\sqrt{2}} \le \mathcal{M}_{irr,+} \le \sqrt{\mathcal{M}^2+n^2} \tag{36}$$

for $\mathcal{H}^{+}$, and

$$0 \le \mathcal{M}_{irr,-} \le \frac{\left[n^2(\mathcal{M}^2+n^2)\right]^{1/4}}{\sqrt{2}} \tag{37}$$

for $\mathcal{H}^{-}$. Equation (36) is nothing but the Penrose inequality, which is the first geometric inequality for BHs [20].

2.4. Temperature bound for $\mathcal{H}^{\pm}$

In BH thermodynamics, temperature is an important thermodynamic parameter. Hence, there must exist temperature bound relation on the horizons. When $r_+ \ge r_-$, we must obtain $|T_+| \ge |T_-| \ge 0$. Therefore, the temperature product gives

$$|T_+| \ge \sqrt{|T_+T_-|} = \frac{1}{4\pi n} \ge |T_-|, \tag{38}$$

and the temperature sum gives

$$\frac{\mathcal{M}}{2\pi n^2} = |T_+ + T_-| \ge |T_+| \ge \frac{|T_+ + T_-|}{2} = \frac{\mathcal{M}}{4\pi n^2}. \tag{39}$$

Hence we have the temperature bound for $\mathcal{H}^{+}$

$$\frac{\mathcal{M}}{4\pi n^2} \le |T_+| \le \frac{\mathcal{M}}{2\pi n^2} \tag{40}$$

and the temperature bound for $\mathcal{H}^{-}$

$$0 \le |T_-| \le \frac{1}{4\pi n}. \tag{41}$$

2.5. Bound on the heat capacity $C_{\pm}$ for $\mathcal{H}^{\pm}$

The specific heat can be defined as

$$C_{\pm} = \frac{\partial \mathcal{M}}{\partial T_{\pm}}. \tag{42}$$

After some calculations, we obtain the specific heat for both horizons:

$$C_{\pm} = -2\pi(r_{\pm}^2 + n^2). \tag{43}$$

Their product and sum on $\mathcal{H}^{\pm}$ yield

$$C_+C_- = (4\pi n)^2\left(\mathcal{M}^2+n^2\right) \tag{44}$$

and

$$C_+ + C_- = -8\pi\left(\mathcal{M}^2+n^2\right). \tag{45}$$

Using the inequality $\mathcal{M}^2 \ge -n^2$ with the product and sum of the heat capacities, we obtain a bound on the heat capacity for both horizons:

$$4\pi\left(\mathcal{M}^2+n^2\right) \le |C_+| \le 8\pi\left(\mathcal{M}^2+n^2\right) \tag{46}$$

for $\mathcal{H}^{+}$ and

$$0 \le |C_-| \le 4\pi n\sqrt{\mathcal{M}^2+n^2} \tag{47}$$

for $\mathcal{H}^{-}$. Using Eqs. (13), (18), and (44), we find a new relation among the specific heat product, the area product, and the entropy product:

$$C_+C_- = \frac{\mathcal{A}_+\mathcal{A}_-}{4} = 4\mathcal{S}_+\mathcal{S}_-. \tag{48}$$

So far we have calculated different thermodynamic quantities, and these formulae might be useful in further understanding the microscopic nature of BH entropy, both exterior and interior. Again, the entropy product of the inner and outer horizons could be used to determine whether the classical BH entropy can be written as a Cardy formula, giving some evidence for a holographic description of the BH/CFT correspondence [18, 21]. The above thermodynamic properties including the Hawking temperature and the area of both horizons may therefore be expected to play a crucial role in understanding the BH entropy of $\mathcal{H}^{\pm}$ at the microscopic level.

3. CONCLUSION

To understand the black hole entropy at the microscopic level, we have studied both inner-horizon and outer-horizon thermodynamics for the Taub–NUT BH in four-dimensional Lorentzian geometry. We also computed various thermodynamic relations, such as product, sum, difference, and ratio of the inner and outer horizons. Due to the presence of the NUT parameter, we have found that all the thermodynamic relations

Table

Parameter ($i=+,-$)	RN BH	Taub–NUT BH
$r_\pm$	$\mathcal{M}\pm\sqrt{\mathcal{M}^2-Q^2}$	$\mathcal{M}\pm\sqrt{\mathcal{M}^2+n^2}$
$\sum r_i$	$2\mathcal{M}$	$2\mathcal{M}$
$\prod r_i$	Q^2	$-n^2$
$\mathcal{A}_\pm$	$4\pi\left(2\mathcal{M}r_\pm-Q^2\right)$	$8\pi\left(\mathcal{M}r_\pm+n^2\right)$
$\sum \mathcal{A}_i$	$8\pi\left(2\mathcal{M}^2-Q^2\right)$	$16\pi\left(\mathcal{M}^2+n^2\right)$
$\prod \mathcal{A}_i$	$(4\pi Q^2)^2$	$(8\pi n)^2\left(\mathcal{M}^2+n^2\right)$
$\mathcal{S}_\pm$	$\pi\left(2\mathcal{M}r_\pm-Q^2\right)$	$2\pi\left(\mathcal{M}r_\pm+n^2\right)$
$\sum \mathcal{S}_i$	$2\pi(2\mathcal{M}^2-Q^2)$	$4\pi\left(\mathcal{M}^2+n^2\right)$
$\prod \mathcal{S}_i$	π^2Q^4	$(2\pi n)^2\left(\mathcal{M}^2+n^2\right)$
$\kappa_\pm$	$\dfrac{r_\pm-r_\mp}{2\left(2\mathcal{M}r_\pm-Q^2\right)}$	$\dfrac{r_\pm-r_\mp}{4\left(\mathcal{M}r_\pm+n^2\right)}$
$\sum \kappa_i$	$\dfrac{4\mathcal{M}(Q^2-\mathcal{M}^2)}{Q^4}$	$-\dfrac{\mathcal{M}}{n^2}$
$\prod \kappa_i$	$\dfrac{Q^2-\mathcal{M}^2}{Q^4}$	$-\dfrac{1}{4n^2}$
$T_\pm$	$\dfrac{r_\pm-r_\mp}{4\pi r_\pm^2}$	$\dfrac{r_\pm-r_\mp}{4\pi(r_\pm^2+n^2)}$
$\sum T_i$	$\dfrac{2\mathcal{M}(Q^2-\mathcal{M}^2)}{\pi Q^4}$	$-\dfrac{\mathcal{M}}{2\pi n^2}$
$\prod T_i$	$\dfrac{Q^2-\mathcal{M}^2}{4\pi^2(Q^4)}$	$-\dfrac{1}{(4\pi n)^2}$
$\mathcal{M}_{irr,\pm}$	$\sqrt{\dfrac{\mathcal{A}_\pm}{16\pi}}$	$\sqrt{\dfrac{\mathcal{A}_\pm}{16\pi}}$
$\sum \mathcal{M}_{irr}^2$	$\mathcal{M}^2-\dfrac{Q^2}{2}$	$\mathcal{M}^2+n^2$
$\prod \mathcal{M}_{irr}$	$\dfrac{Q^2}{4}$	$\sqrt{\dfrac{n^2(\mathcal{M}^2+n^2)}{4}}$

are mass dependent, which means that they are not universal.

Using the proposed relations, we also compute different thermodynamic bounds, such as the area bound, entropy bound, temperature bound, etc. We also derived a relation between the specific heat product, the area product, and the entropy product. The thermodynamic relations that we have derived in this note must have implications. We suggest that these thermodynamic formulas give a further clue to the understanding of the microscopic nature of the BH entropy (both inner and outer) for both the horizons. In [22], we derived a thermodynamic product formula for another non-asymptotic type of BHs like the Kehagias–Sfetsos BH in Horava–Lifshitz gravity, where the area product is univesal in nature.

APPENDIX

In Table we have given various thermodynamic relations for the Taub–NUT BH in comparison with the RN BH.

REFERENCES

1. M. Ansorg and J. Hennig, Phys. Rev. Lett. **102**, 221102 (2009).
2. M. Cvetic, G. W. Gibbons, and C. N. Pope, Phys. Rev. Lett. **106**, 121301 (2011).
3. J. D. Bekenstein, Lett. Nuovo Cim. **4**, 737 (1972).
4. A. Strominger and C. Vapa, Phys. Lett. B **379**, 99 (1996).
5. M. Guica, T. Hartman, W. Song, and A. Strominger, Phys. Rev. D **80**, 124008 (2009).
6. A. Curir, Nuovo Cim. B **51**, 262 (1979).
7. F. Larsen, Phys. Rev. D **56**, 1005 (1997).
8. S. Chandrashekar, *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford (1983).
9. D. Lynden-Bell and M. Nouri-Zonoz, Rev. Mod. Phys. **70**, 447 (1998).
10. C. W. Misner and A. H. Taub, Zh. Eksp. Teor. Fiz. **55**, 233 (1968).
11. J. G. Miller, M. D. Kruskal, and B. B. Godfrey, Phys. Rev. D **4**, 2945 (1971).
12. E. T. Newman, L. Tamburino, and T. Unti, J. Math. Phys. **4**, 7 (1963).
13. S. Ramaswamy and A. Sen, J. Math. Phys. **22**, 2612 (1981).
14. C. W. Misner, J. Math. Phys. **4**, 924 (1963).
15. P. Pradhan, arXiv:1408.2973 [gr-qc].
16. J. M. Bardeen, B. Carter, and S. W. Hawking, Comm. Math. Phys. **31**, 161 (1973).
17. A. Castro and M. J. Rodriguez, Phys. Rev. D **86**, 024008 (2012).
18. B. Chen, S. X. Liu, and J. J. Zhang, J. High Energy Phys. **1211**, 017 (2012).
19. D. Christodoulou, Phys. Rev. Lett. **25**, 1596 (1970).
20. R. Penrose, Ann. N. Y. Acad. Sci. **224**, 125 (1973).
21. W. Xu, J. Wang, and X. Meng, Phys. Lett. B **746**, 53 (2015).
22. P. Pradhan, Phys. Lett. B **747**, 64 (2015).