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WALL-CROSSING INVARIANTS: FROM QUANTUM MECHANICS TO KNOTS

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We offer a pedestrian-level review of the wall-crossing invariants. The story begins from the scattering theory in quantum mechanics where the spectrum reshuffling can be related to permutations of S-matrices. In nontrivial situations, starting from spin chains and matrix models, the S-matrices are operator-valued and their algebra is described in terms of $\mathcal{R}$- and mixing (Racah) $\mathcal{U}$-matrices. Then the Kontsevich–Soibelman (KS) invariants are nothing but the standard knot invariants made out of these data within the Reshetikhin–Turaev–Witten approach. The $\mathcal{R}$ and Racah matrices acquire a relatively universal form in the semiclassical limit, where the basic reshufflings with the change of moduli are those of the Stokes line. Natural from this standpoint are matrices provided by the modular transformations of conformal blocks (with the usual identification $\mathcal{R} = T$ and $\mathcal{U} = S$), and in the simplest case of the first degenerate field $(2,1)$, when the conformal blocks satisfy a second-order Shrödinger-like equation, the invariants coincide with the Jones ($N = 2$) invariants of the associated knots. Another possibility to construct knot invariants is to realize the cluster coordinates associated with reshufflings of the Stokes lines immediately in terms of check-operators acting on solutions of the Knizhnik–Zamolodchikov equations. Then the $\mathcal{R}$-matrices are realized as products of successive mutations in the cluster algebra and are manifestly described in terms of quantum dilogarithms, ultimately leading to the Hikami construction of knot invariants.

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Contents

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1. INTRODUCTION

The string theory approach to any problem is to consider it together with all possible deformations and as a particular representation of some general structure appearing in many other, seemingly unrelated problems in other fields of science. One of the fresh applications of this approach is the study of the wall-crossing phenomena (phase transitions) and associated invariants, which remain the same after reshuffling. The outcome of this study is that the Kontsevich–Soibelman (KS) invariants [1] found so far on this way, are probably not that new: they belong to an old class of invariants of the Reshetikhin–Turaev–Witten type, of which the best known are knot invariants [2, 3]. At the same time, what naturally arises in wall-crossing problems is quantum $\mathcal{R}$-matrices in representations less trivial than the Verma modules of $SU_q(N)$, and this can further stimulate the study of knot invariants in nontrivial representations.

An archetypical example of the wall crossing is the spectrum dependence on the scattering potential in quantum mechanics. We consider a particle in the infinite well with some localized potential, for example:

$$\Big(-\partial_x^2 + u\delta(x) - k^2\Big)\psi(x) = 0, \quad \psi(-L_1) = \psi(L_2) = 0. \tag{1.1}$$

The spectrum $k(u)$ is defined by the spectral equation

$$\sin\Big(kL_1 + kL_2\Big) - \frac{u}{k}\sin(kL_1)\sin(kL_2) = 0 \tag{1.2}$$

and changes from the set

$$k_n = \frac{\pi n}{L_1 + L_2} \quad \text{at} \quad u = 0 \tag{1.3}$$

to the union of two sets[1)]

1) For $u > (L_1 + L_2)/L_1L_2$, there are also two bound states with $k = \pm i\kappa$, where κ solves the equation

$$\mathrm{sh}\Big(\kappa L_1 + \kappa L_2\Big) - \frac{u}{\kappa}\mathrm{sh}(\kappa L_1)\,\mathrm{sh}(\kappa L_2) = 0.$$

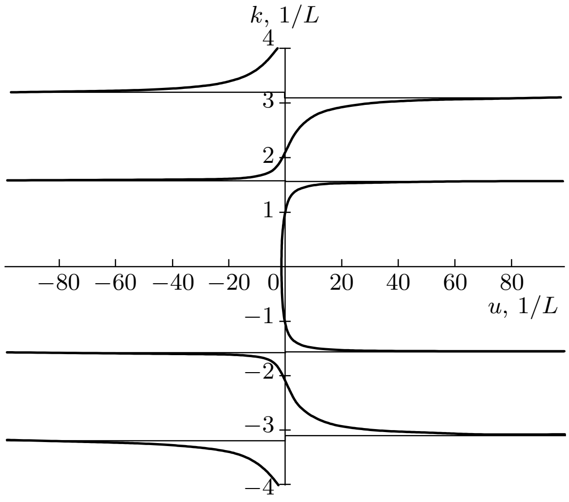


Fig. 1. The picture of energy levels as a function of u at $L_1 = 2L_2 := 2L$

$$k_n^I = \frac{\pi n}{L_1}, \quad k_n^{II} = \frac{\pi n}{L_2} \quad \text{at} \quad u = \infty. \tag{1.4}$$

The smooth evolution with u is shown in Fig. 1, but the net result is the rather radical reshuffling of (1.3) into (1.4). The task can be to study this reshuffling and to ask if there are quantities that remain the same after the reshuffling.

The question is actually uninvestigated, but its more sophisticated versions were studied and some invariants were revealed (although it is unclear if they reduce to triviality in this original problem).

The point is that there is nothing special about the δ-function potential: the pattern remains the same for an arbitrary barrier vanishing in the vicinity of the box walls. Any such problem is described in terms of the 2×2 scattering matrix

$$\mathcal{S}: \quad e^{\pm ikx}\big|_{x \text{ near } -L_1} \to \alpha_\pm(k)e^{\pm ikx} + \beta_\pm(k)e^{\mp ikx}\big|_{x \text{ near } L_2}. \tag{1.5}$$

The spectral equation states that $\mathcal{S}$ converts $\sin\Big(k(x+L_1)\Big) \sim e^{ik(x+L_1)} - e^{-ik(x+L_1)}$ into $\sin\Big(k(x-L_2)\Big) \sim$ $\sim e^{ik(x-L_2)} - e^{-ik(x-L_2)}$. For the δ-function potential, the scattering matrix is just

$$\mathcal{S} = \begin{pmatrix} 1 - \dfrac{u}{2ik} & -\dfrac{u}{2ik} \\ \dfrac{u}{2ik} & 1 + \dfrac{u}{2ik} \end{pmatrix}. \tag{1.6}$$

For the two isolated barriers, the scattering matrix is a product

$$\mathcal{S}_1 \circ \mathcal{S}_2 = \mathcal{S}_1 \cdot \begin{pmatrix} e^{ikl_{12}} & 0 \\ 0 & e^{-ikl_{12}} \end{pmatrix} \cdot \mathcal{S}_2, \tag{1.7}$$

where l_{12} is the distance between the two, and so on: in general, we have the product $\circ_i \mathcal{S}_i$.

Changing the shape of the potential reduces to a composition of reshufflings, when constituents of the product change order, i. e., to a composition of operations:

$$\mathcal{K}_j: \quad \circ_{i<j}\mathcal{S}_i \circ \mathcal{S}_j \circ \mathcal{S}_{j+1} \circ_{i>j+1} \mathcal{S}_i \to \circ_{i<j}\mathcal{S}_i \circ \mathcal{S}_{j+1} \circ \mathcal{S}_j \circ_{i>j+1} \mathcal{S}_i. \tag{1.8}$$

This operation is of course very familiar: in the theory of quantum groups, if the $\mathcal{S}_i$ are group elements [4, 5], this permutation is described by the quantum $\mathcal{R}$-matrix, and we can therefore suspect that actually $\mathcal{K}_j = \mathcal{R}_j$ and, further, $\mathcal{R}_{j+1} = \mathcal{U}_j \mathcal{R}_j \mathcal{U}_j^\dagger$, where $\mathcal{U}$ is the quantum mixing (Racah) matrix (see [6]). From this data, it is then straightforward to build invariants: these are the ordinary knot invariants in the Reshetikhin–Turaev–Witten (RTW) formalism (graded traces and certain matrix elements of the ordered products of the $\mathcal{R}$-matrices), which in the context of wall-crossing theory are known as KS invariants.

To make this story really nontrivial, we need to promote scattering matrices $\mathcal{S}$ to operator-valued quantities $\hat{\mathcal{S}}$. In quantum group theory, this is achieved by making the elements of the algebra of functions noncommutative; in quantum mechanics it suffices to introduce internal degrees of freedom like spins or, more generally, to consider matrix models (e. g., matrix quantum mechanics). This makes $\mathcal{R}$- and $\mathcal{U}$-matrices different from just ordinary permutation matrices: they start to realize the far less simple braid group structures.

The emergence of braids is, of course, not universal for quantum mechanical problems, they arise only when the space is 2-dimensional and there are topologically different ways to adiabatically carry one point around another, producing a Berry phase. However, this is a generic situation for algebraically integrable dynamics, where the separation of variables reduces the study to a complex curve (sometimes called the spectral or Seiberg–Witten curve, the Liouville torus being its Jacobian). Although formally integrable systems are pretty rare, there is a growing evidence that typical effective field theory obtained after integration over fast variables is integrable [5, 7], and this explains the growing interest in this type of theories.

Even in this integrability context, naturally appearing representations of the braid group can be quite sophisticated and difficult to study. Still, there are two immediate classes of examples (in addition to the ordinary Verma modules of ordinary quantum groups like $SU_q(N)$, widely used in conventional knot theory). One of them is provided by the WKB limit of quantum mechanics, where the $\mathcal{R}$-matrices actually describe reshufflings of the Stokes lines. The other is provided by modular transformations of conformal blocks: the modular kernels T_i and S_i provide an interesting set of $\mathcal{R}_i$ and $\mathcal{U}_i$ matrices, which can be used to construct a priori new families of knot invariants. In the simplest case, however, the family is not new: it yields just the ordinary Jones polynomials (and probably HOMFLY at the next step), but more sophisticated examples seem to be capable of providing a long-awaited group theory (RTW) interpretation of the Hikami invariants.

The plan of this paper is as follows.

We begin in Sec. 2 with a general review of the WKB approach involving the theory of Stokes lines, their reshufflings, and KS invariants. Then, in Sec. 3, we consider the standard example of the double-well potential from this perspective. After that, in Sec. 4, we switch to matrix models and reformulate the problem in terms of operator-valued (check) resolvents. In Sec. 5, we consider KS/RTW invariants associated with the simplest knots and links and show that the $\mathcal{R}$-matrices provided by the modular transformations of conformal blocks give rise to various types of knot invariants: the Jones polynomials and the Hikami integrals.

A natural part of this presentation are the distinguished (Fock–Goncharov [8]) coordinates on the moduli space provided by the WKB theory, where the $\mathcal{R}$-matrices act via peculiar rational transformations (also known as mutations in cluster algebra [9] and related to discrete changes of coordinates in the algebra of functions [10]).

The conclusions in Sec. 6 describe an (incomplete) list of relations between different subjects in theoretical physics, which are brought together by consideration of the wall-crossing phenomena.

2. WALL CROSSING FORMULAS AS A PIECE OF THE WKB THEORY

2.1. Asymptotic behavior

In the Seiberg–Witten (SW) theory describing the low-energy limit of $N = 2$ supersymmetric gauge theory [11], the central charge and the mass of an excitation are given by contour integrals of the SW differential λ:

$$Z_\gamma = \oint\limits_\gamma \lambda, \quad M_\gamma = \oint\limits_\gamma |\lambda|. \tag{2.1}$$

The BPS states have $M = |Z|$, and therefore they are associated with the Stokes contours $\sigma(x) \in \Sigma$ on the spectral surface Σ, such that the SW differential λ (which is a meromorphic differential on Σ whose variations with respect to the moduli are holomorphic) along these contours has a definite phase ϕ_σ:

$$\mathrm{Arg}\left(e^{-i\phi_\sigma}\lambda\Big[\sigma(x)\Big]\right) = 0, \tag{2.2}$$

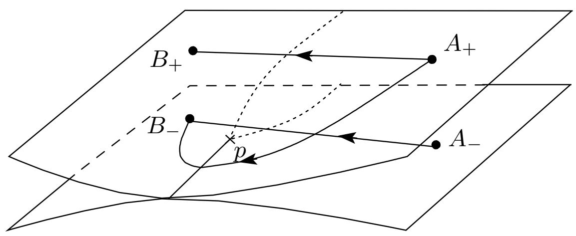


Fig. 2. Intersecting one Stokes line

whence

$$\operatorname{Arg}\left(e^{-i\phi_\sigma}\int\limits_{\sigma(x)}\lambda\right)=0. \tag{2.3}$$

If the gauge theory is described via M-theory [12], then these σ are interpreted as intersections of the main $M5$-brane with the $M2$-branes (see [13]). The spectral surface Σ is a ramified covering of the original bare curve Σ_0, and λ is an eigenvalue of the Lax 1-form [14]. The mass of the BPS state is given by the absolute value of the same integral, and the mass is finite when the contour is closed. To make the Stokes contour closed, we should adjust the phase ϕ_σ; this possibility to adjust the phase of the Planck constant is the main new peculiarity of the BPS state counting as compared to the usual WKB theory. When the moduli of the spectral surface change, so do the phase and the shape of the contour, and at some values of moduli the change can be abrupt: discontinuous. Such a jump in the multiplicities of the BPS states occurs along real-codimension-one surfaces in the moduli space and is called the wall-crossing phenomenon. What remains invariant is peculiar combinations of multiplicities, encoded in the form of the Kontsevich–Soibelman formula.

Our purpose in this paper is to discuss a pedestrian approach to this kind of problems, relating them to the elementary textbook consideration of Stokes phenomena for the WKB approximation.

For this purpose, we consider the Wilson line

$$W_\Gamma(\phi)=P\exp\left(\frac{ie^{i\phi}}{\hbar}\int\limits_\Gamma\mathcal{L}\right) \tag{2.4}$$

of the $N\times N$-matrix valued Lax form along an open contour $\Gamma\in\Sigma_0$ on the bare Riemann surface Σ_0. The N eigenvalues of $\mathcal{L}$ define the SW differential λ_i on the N sheets of the spectral surface Σ that N times covers Σ_0, and we hence can roughly write $W_\Gamma(\phi)$ as

$$w_\Gamma(\phi)=\operatorname{diag}\left\{\exp\left(\frac{ie^{i\phi}}{\hbar}\int\limits_{\gamma_i}\lambda_i\right)\right\} \tag{2.5}$$

where γ_i are pre-images of Γ on Σ. However, if we wish to treat w_Γ as a semiclassical approximation to an evolution operator for some quantum mechanical system, then we should switch between different branches i as Γ crosses the Stokes lines. For example, if we have a two-sheet covering with $\lambda_\pm=\pm\lambda$, and consider the evolution along the contour that goes from point A to B and crosses a Stokes line originating at ramification point P (see Fig. 2), then

$$\Psi(B)=\begin{pmatrix}\psi(B_+)\\ \psi(B_-)\end{pmatrix}=\begin{pmatrix}\exp\left(\dfrac{ie^{i\phi}}{\hbar}\int\limits_{A_+}^{B_+}\lambda\right) & \exp\left(\dfrac{ie^{i\phi}}{\hbar}\left(\int\limits_{P}^{B_+}\lambda-\int\limits_{A_-}^{P}\lambda\right)\right)\\ 0 & \exp\left(-\dfrac{ie^{i\phi}}{\hbar}\int\limits_{A_-}^{B_-}\lambda\right)\end{pmatrix}\begin{pmatrix}\psi(A_+)\\ \psi(A_-)\end{pmatrix} \tag{2.6}$$

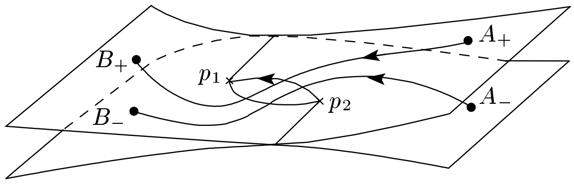


Fig. 3. Intersecting two Stokes lines

while for the inverse path from B to A, we would rather encounter a low-triangular matrix. Thus, instead of the naive w_Γ, we obtain a semiclassical Abelianization of the Wilson operator $W_{AB}(\phi)$ given by the sum of three elementary matrices

$$\Psi(B) = \Big(w_{B_+A_+}(\phi) + w_{B_+PA_-}(\phi) + w_{B_-A_-}(\phi)\Big)\Psi(A). \tag{2.7}$$

2.2. Jumps of WKB network topology on the curve

We call the contours σ satisfying (2.2) the set of WKB lines (WKB network) $\Gamma \in \Sigma_0$ such that its preimages compose. The WKB network gives a triangulation of the spectral surface Σ_0. This triangulation depends on the phase ϕ and it can jump at some specific critical values ϕ_c, making a kind of flips of the triangulation.

When a flip occurs, two different Stokes lines merge into a single line of finite length (we let γ_c denote its pre-image on Σ), and hence integral (2.1) giving the central charge $Z_{\gamma_c} = \oint_{\gamma_c} \lambda$ becomes convergent. We can immediately define the value of the critical phase as

$$\phi_c = \operatorname{Arg} Z_{\gamma_c}. \tag{2.8}$$

Now we observe how the value of the asymptotic expansion of a Wilson operator changes. If two Stokes lines that originate at ramification points P_1 and P_2 were crossed on the way from A to B (see Fig. 3), we would obtain a Wilson operator

$$\begin{aligned} W_{AB}(\phi_-) = w_{B_+A_+}(\phi_-) + w_{B_-A_-}(\phi_-) + w_{B_+P_1A_-}(\phi_-) + w_{B_-P_2A_+}(\phi_-) + \underline{w_{B_-P_2P_1A_-}(\phi_-)} = \\ = w_{B_+A_+}(\phi_c) + (1+w_\sigma)w_{B_-A_-}(\phi_c) + w_{B_+P_1A_-}(\phi_c) + w_{B_+P_2A_-}(\phi_c), \end{aligned} \tag{2.9}$$

where $\phi_- = \phi_c - 0$. For the other value of the phase $\phi_+ = \phi_c + 0$, the configuration of the Stokes lines can be different, and we obtain another decomposition

$$\begin{aligned} W_{AB}(\phi_+) = w_{B_+A_+}(\phi_+) + w_{B_+P_1A_-}(\phi_+) + w_{B_-P_2A_+}(\phi_+) + \underline{w_{B_-P_1P_2A_-}(\phi_+)} = \\ = (1+w_\sigma)w_{B_+A_+}(\phi_c) + w_{B_-A_-}(\phi_c) + w_{B_+P_1A_-}(\phi_c) + w_{B_+P_2A_-}(\phi_c) \end{aligned} \tag{2.10}$$

differing not only in the value of ϕ but also in reordering of the points P_1 and P_2. At the critical value ϕ_c of the phase, where reshuffling of the Stokes lines occurs and a closed Stokes line σ appears, the two expressions differ exactly by w_σ, i.e., an abrupt reshuffling occurs

$$W_{AB} \to \hat{K}_\sigma W_{AB}, \tag{2.11}$$

where a morphism $\hat{K}$ acts as

$$\hat{K}_\sigma : \quad w_\alpha = (1+w_\sigma)^{\langle\sigma,\alpha\rangle} w_\alpha. \tag{2.12}$$

We note that this is not an ordinary operator. This morphism acts as a change of coordinates on the moduli space of flat connections. We can therefore apply it to every term in the sum independently. In detail,

$$\hat{K}_\sigma(W_{AB}(\phi_-)) = \hat{K}_\sigma(w_{B_+A_+}(\phi_c)) + (1+\hat{K}_\sigma(w_\sigma))\hat{K}_\sigma(w_{B_-A_-}(\phi_c)) + \\ + \hat{K}_\sigma(w_{B_+P_1A_-}(\phi_c)) + \hat{K}_\sigma(w_{B_+P_2A_-}(\phi_c)) = (1+w_\sigma)^{\overbrace{\langle\sigma,B_+A_+\rangle}^{1}} w_{B_+A_+}(\phi_c) + \\ + \left(1+(1+w_\sigma)^{\overbrace{\langle\sigma,\sigma\rangle}^{0}} w_\sigma\right)(1+w_\sigma)^{\overbrace{\langle\sigma,B_-A_-\rangle}^{-1}} w_{B_-A_-}(\phi_c) + (1+w_\sigma)^{\overbrace{\langle\sigma,B_+P_1A_-\rangle}^{0}} w_{B_+P_1A_-}(\phi_c) + \\ + (1+w_\sigma)^{\overbrace{\langle\sigma,B_+P_2A_-\rangle}^{0}} w_{B_+P_2A_-}(\phi_c) = W_{AB}(\phi_+). \quad (2.13)$$

2.3. Nontrivial moduli space invariants: wall-crossing formulas in the moduli space

As we have seen, the asymptotics of Wilson lines is **not** smooth. The discontinuity is of the order of $w_\sigma(\phi_c)$, which is asymptotically small.

Nevertheless, the value of the observable itself is expected to be smooth. This fact allows constructing nontrivial invariants of the morphisms $\hat{K}$ on the Coulomb branch, called spectrum generators [53], tightly related to the spectra of BPS states arising in the effective theory.

In general, as we start increasing the phase ϕ from 0 to π, a number of reshufflings occur, when particular closed Stokes lines σ_a appear at critical values ϕ_a, and disappear with the further increase in ϕ. This provides a sequence of actions

$$\overleftarrow{\prod_a} \hat{K}_{\sigma_a}. \quad (2.14)$$

The number of factors here is actually the number of BPS states on a given spectral curve, i. e., at a given point of the moduli space. If we now start changing moduli of the spectral curve, this product can change, reflecting the change of the ordered set $\mathcal{A}$ of the BPS states, including their number (the number of factors in the product), and the order in which they occur with as the phase ϕ increases. However, on the domain wall in moduli space (on a hypersurface of marginal stability) given by the condition $\phi = \phi_c$, the two different products should coincide:

$$\boxed{\overleftarrow{\prod_{a\in\mathcal{A}}} \hat{K}_{\sigma_a} = \overleftarrow{\prod_{b\in\mathcal{B}}} \hat{K}_{\sigma_b}.} \quad (2.15)$$

Thus, we obtain the Kontsevich–Soibelman (KS) invariant, taking values in functors acting on the space of w-variables often called the Fock–Goncharov coordinates of the flat connection moduli space.

Basic example: For two conjugate A and B cycles on a torus with $\langle A,B\rangle = 1$, the KS relation states that

$$\hat{K}_A\hat{K}_B = \hat{K}_B\hat{K}_{A+B}\hat{K}_A, \quad (2.16)$$

where the operator action is defined as

$$\hat{K}_{mA+nB}w_\gamma = (1+w_A^m w_B^n)^{m\langle A,\gamma\rangle + n\langle B,\gamma\rangle} w_\gamma. \quad (2.17)$$

Note that the coordinates $w_Aw_B = w_Bw_A$ commute, and also $w_{A+B} = w_Aw_B$, while neither of these is true for the operators $\hat{K}$. With these definitions, Eq. (2.16) is just an identity: indeed, applying both sides, e. g., to w_A, we obtain

$$\hat{K}_A\hat{K}_B w_A = \hat{K}_A \frac{1}{1+w_B} w_A = \frac{1}{1+(1+w_A)w_B} w_A = \frac{w_A}{1+w_B+w_Aw_B}, \quad (2.18)$$

and

$$\hat{K}_B\hat{K}_{A+B}\hat{K}_A w_A = \hat{K}_B\hat{K}_{A+B} w_A = \hat{K}_B \frac{1}{1+w_Aw_B} w_A = \\ = \frac{1}{1+\frac{1}{1+w_B}w_Aw_B}\frac{1}{1+w_B} w_A = \frac{w_A}{1+w_B+w_Aw_B}. \quad (2.19)$$

Similarly, in application to w_B:

$$\hat{K}_A\hat{K}_B w_B = (1+w_A)w_B, \tag{2.20}$$

and

$$\hat{K}_B\hat{K}_{A+B}\hat{K}_A w_B = \hat{K}_B\hat{K}_{A+B}(1+w_A)w_B = \hat{K}_B\left(1+\frac{1}{1+w_Aw_B}w_A\right)(1+w_Aw_B)w_B =$$
$$= \left(1+\frac{1}{1+\frac{1}{1+w_B}w_Aw_B}\frac{1}{1+w_B}w_A\right)\left(1+\frac{1}{1+w_B}w_Aw_B\right)w_B =$$
$$= \frac{1+w_A+w_B+w_Aw_B}{1+w_B}w_B = (1+w_A)w_B. \tag{2.21}$$

3. CLASSIC PROBLEM OF QUANTUM MECHANICS: DOUBLE-WELL POTENTIAL

We consider the Schrödinger equation with the quartic potential

$$\left[\hbar^2\partial_z^2 - (z-x_1)(z-x_2)(z-x_3)(z-x_4)\right]\Psi(z) = 0. \tag{3.1}$$

Depending on the choice of zeroes x_k, the structure of levels is rather different.

At a first glance, this may seem somewhat controversial. According to the well-known theorem in ODE theory, solutions of the Cauchy problem are continuous functions of parameters if the coefficients of the equation are continuous. Nevertheless, the problem of finding eigenvalues of self-adjoint operators (Sturm–Liouville problem) is quite different. Once found integrable in the usual Hilbert norm, the eigenfunctions at some chosen values of parameters are not expected to keep integrability at another choice of the parameters.

The integrability of a function depends on its asymptotic behavior. In this particular case, there are two asymptotic forms $e^{\pm\frac{z^3}{\hbar}}$. We choose two linearly independent solutions $\Psi_{1,2}$ with the asymptotics behavior

$$\Psi_1(z) \underset{z\to\pm\infty}{\sim} c_{1+}(\pm\infty)e^{z^3/\hbar} + c_{1-}(\pm\infty)e^{-z^3/\hbar}, \tag{3.2}$$

$$\Psi_2(z) \underset{z\to\pm\infty}{\sim} c_{2+}(\pm\infty)e^{z^3/\hbar} + c_{2-}(\pm\infty)e^{-z^3/\hbar}. \tag{3.3}$$

We define an "$\mathcal{S}$-matrix" as

$$\mathcal{S} = \begin{pmatrix} \sigma_{++} & \sigma_{-+} \\ \sigma_{+-} & \sigma_{--}\end{pmatrix} = \begin{pmatrix} c_{1+}(+\infty) & c_{1-}(+\infty) \\ c_{2+}(+\infty) & c_{2-}(+\infty)\end{pmatrix}^{-1}\begin{pmatrix} c_{1+}(-\infty) & c_{1-}(-\infty) \\ c_{2+}(-\infty) & c_{2-}(-\infty)\end{pmatrix}. \tag{3.4}$$

We note that this matrix is independent of the choice of the basis in solutions. For a generic choice of parameters, this $\mathcal{S}$-matrix is unphysical.

To define an eigenfunction, we require it to be integrable for real $\hbar$, i.e., to behave as

$$\Psi(z) \underset{z\to\pm\infty}{\sim} e^{\mp z^3/\hbar}. \tag{3.5}$$

This imposes a condition on the $\mathcal{S}$-matrix entries

$$\sigma_{--}(x_1,x_2,x_3,x_4) = \sigma_{++}(x_1,x_2,x_3,x_4) = 0. \tag{3.6}$$

The crucial point is that the $\mathcal{S}$-matrix is discontinuous on the moduli space $(x_k,\hbar)$ (see also [15]). To observe this, we consider two well-known physical situations:

I. The ground energy level is below the level of the wall between the wells: all the zeroes x_k are real. The problem can be described by a particle localized either at the left well or at the right well, and hence there are two almost degenerate levels (with the wavefunctions symmetric and antisymmetric with regard to interchanging the wells) that differ only due to instanton jumps between the wells.

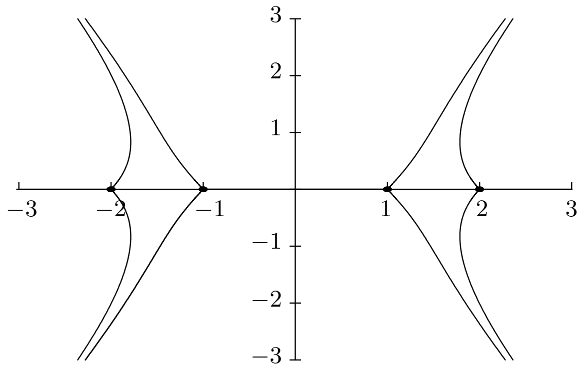


Fig. 4. Topology of the WKB lines when all zeroes are real

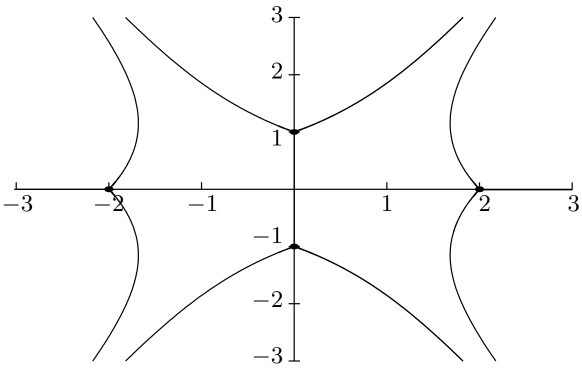


Fig. 5. Topology of the WKB lines when two zeroes are real and two zeroes have opposite imaginary parts

The topology of the WKB lines is depicted in Fig. 4.

The first WKB approximation gives the following expression for σ_{--}:

$$\sigma_{--}(x_k) \sim -4 \underbrace{\operatorname{sh}\left(\frac{1}{\hbar}\int\limits_{x_1}^{x_2}\lambda\right)}_{\text{first well level}} \underbrace{\operatorname{sh}\left(\frac{1}{\hbar}\int\limits_{x_3}^{x_4}\lambda\right)}_{\text{second well level}} + \underbrace{\exp\left\{-\frac{2}{\hbar}\int\limits_{x_2}^{x_3}\lambda\right\}}_{\text{instanton}} \exp\left\{-\frac{1}{\hbar}\int\limits_{x_1}^{x_2}\lambda\right\}\exp\left\{-\frac{1}{\hbar}\int\limits_{x_3}^{x_4}\lambda\right\}. \tag{3.7}$$

II. The ground energy level is above the level of the wall between the wells: two zeroes are real and two zeroes have opposite imaginary parts. In this case, there are no pairs of almost degenerate energy levels.

The topology of the WKB lines is depicted in Fig. 5.

The first WKB approximation gives the following expression for σ_{--}:

$$\sigma_{--}(x_k) \sim -\exp\left\{\frac{1}{2\hbar}\oint\limits_{\infty}\lambda\right\} - 2\sin\left(\frac{1}{\hbar}\int\limits_{x_1}^{x_4}\lambda\right). \tag{3.8}$$

$\mathcal{S}$ jumps discontinuously, the jump being described by an operator $\hat{K}$ that is a counterpart of the KS operator in ordinary quantum mechanics:

$$\hat{K}(\mathcal{S}^{(\mathrm{I})}) = \mathcal{S}^{(\mathrm{II})}. \tag{3.9}$$

However, $\mathcal{S}$ depends only on the point x_k not belonging to the path in the moduli space. Hence, all the jumps along a closed contractible loop should cancel:

$$\prod_{\text{loop}} \hat{K} = 1. \tag{3.10}$$

4. CHECK-OPERATOR "QUANTUM, REFINED"

4.1. Intuitive remarks

In many applications, matrix elements of the $N \times N$ Lax form are themselves operators. As we saw, such are the Seiberg–Witten differentials $\lambda^{(i)}$ and the Abelianized monodromies w_Γ. In such cases, the Kontsevich–Soibelman relations include the Fock–Goncharov coordinates, which take values in operators rather than in numbers.

Here, we first give some heuristic remarks about a possible "refinement" of the Abelianization map construction, and then give a more solid description using "check"-operators [16–18].

Speculations are quite "hand-waving" so far, but we nevertheless try to draw several important conclusions:

1. Gauge covariance: The commutation relations inherited from the natural Poisson structure on Lax operators [19]

$$\varpi = \int d^2 z \ \mathrm{Tr}\ \delta\mathcal{L} \wedge \overline{\delta\mathcal{L}} \tag{4.1}$$

are not gauge invariant. Similarly, the same-time commutator in the Chern–Simons theory is known to be induced by Eq. (4.1) only in the temporal gauge. There is no problem to define an invariant commutator

$$\left[\oint_\gamma \hat{\lambda}, \oint_{\gamma'} \hat{\lambda}\right] = \hbar' \left\langle \gamma, \gamma' \right\rangle, \tag{4.2}$$

where the γ are paths on the spectral curve, and $\langle \star, \star \rangle$ is the cycle pairing on the curve.

2. "Anomaly": Before integration, the commutator $\left[\hat{\lambda}^{(i)}(z), \hat{\lambda}^{(j)}(z')\right]$ is kind of "anomalous": one can smoothly modify the paths as long as the intersection points where the commutator contributes are avoided. This breaks the initial holomorphicity of the problem, such that the expression $\oint_\gamma \hat{\lambda}$ now depends on the regular homotopy class of γ rather than on the homology class of γ. We suppose that a representative $[\gamma]$ in the homology class of γ without self-intersections can be chosen; then

$$\mathcal{P}\mathrm{exp} \oint_\gamma \hat{\lambda} = q^{\mathfrak{wr}\,\gamma} \exp\left\{\oint_{[\gamma]} \hat{\lambda}\right\}, \tag{4.3}$$

where $\mathfrak{wr}$ is a writhe, a signed sum over self-intersections.

Similarly, we define the coordinates depending only on the homology class $[\gamma]$

$$w_{[\gamma]} = \exp\left(\frac{ie^{i\phi}}{\hbar} \int_{[\gamma]} \lambda\right). \tag{4.4}$$

They form a noncommutative algebra

$$w_{[\gamma]} w_{[\gamma']} = q^{\langle [\gamma], [\gamma'] \rangle} w_{[\gamma]+[\gamma']}, \tag{4.5}$$

where $q = \exp\left(ie^{i\phi}\hbar'/2\hbar\right)$ (note that $\hbar$ and $\hbar'$ are generally two independent constants). To derive this, we consider the product of two exponentials

$$\mathcal{P}\mathrm{exp} \oint_\gamma \hat{\lambda}\ \mathcal{P}\mathrm{exp} \oint_{\gamma'} \hat{\lambda} = \mathcal{P}\mathrm{exp} \oint_{\gamma \circ \gamma'} \hat{\lambda}, \tag{4.6}$$

where $\circ$ denotes a consequential concatenation of two paths. Equivalently, this relation can be rewritten in terms of w-variables:

$$q^{\mathfrak{wr}\,\gamma} w_{[\gamma]}\, q^{\mathfrak{wr}\,\gamma'} w_{[\gamma']} = q^{\mathfrak{wr}(\gamma\circ\gamma')} w_{[\gamma\circ\gamma']}. \tag{4.7}$$

Using the relations

$$[\gamma \circ \gamma'] = [\gamma] + [\gamma'], \tag{4.8}$$

$$\mathfrak{wr}(\gamma \circ \gamma') = \mathfrak{wr}\,\gamma + \mathfrak{wr}\,\gamma' + \langle [\gamma], [\gamma'] \rangle, \tag{4.9}$$

we reproduce algebraic relation (4.5). The second relation says that the writhe function is a quadratic refinement of the intersection form (see [20, Appendix C] for the details).

As in the preceding section, the Wilson lines are polynomials in the w variables, although now over $\mathbb{Z}[q, q^{-1}]$:

$$\mathrm{Tr}\,\mathcal{P}\mathrm{exp} \oint L \sim \sum_{\gamma} q^{\mathfrak{wr}\,\gamma} w_{[\gamma]}. \tag{4.10}$$

To conclude this section, we mention that this quite heuristic consideration can be applied to physical problems [20]. $\hat{K}$-jumps of expansion (4.10), similarly to the jumps discussed in the preceding section, allow calculating characteristics of the BPS spectra in $\mathcal{N} = 2$ SYM theories. These invariants are now refined with a deformation parameter q and take the spin of BPS multiplets into account.

4.2. Beta-ensemble construction

Beta-ensembles naturally extend matrix models and inherit their basic properties. The model is given by the partition function of a 2D Coulomb gas (we here consider an example where the gas is placed on a sphere)

$$Z = \prod_i \oint_{\gamma_i} dz_i \prod_{i<j} (z_i - z_j)^{2\beta} \exp\left\{ -\frac{1}{g} \sum_i V(T|z_i) \right\}, \tag{4.11}$$

where g and β are two parameters similar to $\hbar$ and $\hbar'$, and the potential $V(T|z)$ determines the moduli space of the partition function: it is parameterized by the parameters T_k of the potential and by the choice of the integration contours. For definiteness, we choose the potential to be a polynomial and T_k to be coefficients of this polynomial:

$$V(z) = \sum_{k=0}^{n} T_k z^k. \tag{4.12}$$

We consider only closed contours, and hence changing the variables as

$$z_i \to z_i + \frac{\epsilon}{\zeta - z_i} \tag{4.13}$$

does not change the integral, which leads to the following Ward identity in the first order in ϵ:

$$\left\langle \sum_i \frac{1}{(\zeta - z_i)^2} + \beta \sum_{i \neq j} \frac{1}{(\zeta - z_i)(\zeta - z_j)} - \frac{1}{g} \sum_i V'(z_i) \right\rangle = 0. \tag{4.14}$$

After some algebra, this equation can be represented in the form

$$(\beta - 1)\partial_\zeta \left\langle \sum_i \frac{1}{\zeta - z_i} \right\rangle + \beta \left\langle \left(\sum_i \frac{1}{\zeta - z_i} \right)^2 \right\rangle - \frac{V'(\zeta)}{g} \left\langle \sum_i \frac{1}{\zeta - z_i} \right\rangle + \frac{1}{g} \left\langle \sum_i \frac{V'(\zeta) - V'(z_i)}{\zeta - z_i} \right\rangle = 0. \tag{4.15}$$

We define the resolvent as

$$\rho(\zeta) := \frac{1}{Z}\hat{\nabla}(\zeta) Z := g\sqrt{\beta}\left\langle \sum_i \frac{1}{\zeta - z_i} \right\rangle, \tag{4.16}$$

where the operator $\hat{\nabla}(\zeta)$ can be described by the action of $g\sqrt{\beta}\sum \frac{1}{z^{k+1}}\frac{\partial}{\partial t_k}$ on the partition function with the modified potential $V(z) \to V(z) - g\sum_k t_k z^k$, treated as a formal (perturbative) series in the variables t_k:

$$\mathcal{Z} = \prod_i \oint_{\gamma_i} dz_i \prod_{i<j} (z_i - z_j)^{2\beta} \exp\left\{ -\frac{1}{g}\sum_i V(T|z_i) + \sum_{i,k} t_k z_i^k \right\}. \tag{4.17}$$

We use Z for the partition function $\mathcal{Z}$ restricted to all $t_k = 0$.

The last term in Eq. (4.15) can be reproduced by the action of the differential operator in the T_k:

$$\check{P}(\zeta) = -g^2 \sum_{k=2}^{n} k T_k \sum_{n=0}^{k-2} \zeta^n \partial_{T_{k-2-n}}. \tag{4.18}$$

We call such operators check-operators since they act on the moduli T_k (in contrast to $\hat{\nabla}(\zeta)$). We note that the Miwa transform of the T_k-moduli

$$T_k = \sum_j \frac{\alpha_j \lambda_j^k}{k} \tag{4.19}$$

transforms the check-operator $\check{P}(\zeta)$ into

$$\check{P}(\zeta) \sim \sum_j \frac{\partial_{\lambda_j}}{\zeta - \lambda_j}. \tag{4.20}$$

We also can introduce the check-operator that generates the resolvents:

$$\check{\nabla}\mathcal{Z} = \hat{\nabla}\mathcal{Z}. \tag{4.21}$$

This operator can be constructed recursively from $\check{y} := \sqrt{V'(T|z)^2 - 4\beta\check{P}}$, its derivatives, and $V'(T|z)$ [17]. The recursion is provided by the g^2-expansion, and, in the leading order, $\check{\nabla}^{(0)} = g\sqrt{\beta}\check{y}$.

Then the Ward identity can be rewritten in the form

$$\left[gQ\partial_\zeta \check{\nabla}(\zeta) + :\check{\nabla}(\zeta)^2: - \frac{V'(\zeta)}{\sqrt{\beta}}\check{\nabla}(\zeta) + \check{P}(\zeta) \right] Z = 0, \tag{4.22}$$

where $Q := \sqrt{\beta} - \frac{1}{\sqrt{\beta}}$ and the normal ordering means that the operator $\check{\nabla}(\zeta)$ acts only on Z, but not on itself.

4.3. Determinant check-operator: quantizing the spectral curve

In the leading order of the WKB approximation (i. e., as $g^2 \to 0$), Ward identity (4.22) becomes an algebraic equation for resolvent (4.16):

$$\rho^{(0)}(\zeta)^2 - \frac{V'(\zeta)}{\sqrt{\beta}}\rho^{(0)}(\zeta) + f(\zeta) = 0 \tag{4.23}$$

with the polynomial $f(\zeta) := \check{P}\log Z$. This algebraic equation defines a spectral curve. We note that the monodromies of these check-operators along A- and B-periods of these spectral curve form the Heisenberg algebra [17]:

$$\left[\oint_{A_i} \check{\nabla}(\zeta)\, d\zeta, \oint_{B_j} \check{\nabla}(\zeta)\, d\zeta \right] = \delta_{ij}. \tag{4.24}$$

One may ask what are the ways of quantizing the spectral curve equation (4.23) (which has to become the Baxter equation after quantization). There are two possibilities. One is to consider the limit of Eq. (4.11) as $g^2/\beta = \hbar'^2 \to 0$ with $g^2\beta = \hbar^2$ kept fixed. In this limit (called the Nekrasov–Shatashvili limit [21]), Ward identity (4.22) becomes a Ricatti (or Schrödinger) equation equivalent to the Baxter equation [22] and corresponding to a quantum integrable system [21, 23–26].

However, this system depends only on one parameter $g\sqrt{\beta} = \hbar$. But it is possible to quante the spectral curve so as to preserve the β-ensemble representation for the wavefunction and the dependence on two parameters β and g. For this, it was suggested in Ref. [24] to consider the equation for the β-ensemble average of the would-be determinant in a matrix model: the average $\Psi(\zeta) = \langle \prod_i (\zeta - z_i)\rangle$. To deal with this average, we introduce another check-operator:

$$\frac{1}{\mathcal{Z}}\check{\mathcal{D}}_{[1]}(\zeta)\mathcal{Z} := \left\langle \prod_i (\zeta - z_i) \right\rangle. \tag{4.25}$$

In order to understand the meaning of $\Psi(\zeta)$, we rewrite it as (the number of integrations in the β-ensemble partition function is denoted by N)

$$\Psi(\zeta) = \zeta^N \left\langle \prod_i \left(1 - \frac{z_i}{\zeta}\right) \right\rangle = \zeta^N \left\langle \exp\left\{ \sum_i \left(1 - \frac{z_i}{\zeta}\right) \right\} \right\rangle = \zeta^N \left\langle \exp\left(-\sum_k \frac{\sum_i z_i^k}{k\zeta^k} \right) \right\rangle. \tag{4.26}$$

This expression is equal to both

$$\Psi(\zeta) = \zeta^N \frac{1}{\mathcal{Z}} \exp\left(\int^{\zeta} dz \sum_k \frac{1}{z^{k+1}} \frac{\partial}{\partial t_k} \right) \mathcal{Z} = \frac{\exp\left(\dfrac{1}{\hbar} \displaystyle\int^{\zeta} dz \hat{\nabla}(z) \right) \mathcal{Z}}{\mathcal{Z}}, \tag{4.27}$$

where N is treated as the zeroth time t_0, and

$$\Psi(\zeta) = \frac{\mathcal{Z}\left(t_k - \dfrac{1}{k\zeta^k}\right)}{\mathcal{Z}(t_k)}. \tag{4.28}$$

We consider the case $\beta = 1$, when the β-ensemble reduces to the Hermitean matrix model. Within the AGT conjecture, this case corresponds to the conformal field theory with central charge 1 [27, 28]. In this case, $\mathcal{Z}$ is a τ-function of the Toda chain hierarchy [29] in the time variables t_k, and $\Psi(\zeta)$ given by formula (4.28) is a Baker–Akhiezer function. It corresponds to an insertion of the fermion $\psi(\zeta)$ at the point ζ (and a fermion ψ at infinity). This can be easily understood because in the $c = 1$ theory (free fields), a fermion is described in terms of a free field $\phi(\zeta)$ by the exponent $:\exp(i\phi(\zeta)):$, inserting which into the conformal correlator representation of the matrix model [30] gives exactly the determinant [24]. This fermion describes the simplest fundamental representation of the $SL(N)$ group, which can be understood from the realization of its N-plet as [31]

$$\psi_i|0\rangle = T_-^{i-1}|0\rangle, \tag{4.29}$$

where the fermion modes are defined by $\psi(\zeta) = \sum_i^N \psi_i \zeta^i$ and $T_- = \sum_i^{N-1} T_{-\alpha_i}$ is the sum of all raising operators associated with the negative simple roots of $SL(N)$. Then all other fundamental representations are defined as products of fermions. This connection with the fundamental representations can also be made manifest by expansing the determinant in the fundamental representations:

$$\prod_i \left(1 - \frac{z_i}{\zeta}\right) = \sum_k \chi_{[1^k]}(z_i) \left(-\frac{1}{\zeta}\right)^k, \tag{4.30}$$

where $\chi_{[1^k]}(z_i)$ is the Schur function, i. e., the character of the $SL(N)$ group associated with the fundamental representations.

We can now convert Ward identity (4.22) into an equation for $\Psi(\zeta)$. For this, we need to rewrite the Ward identity in terms of shifted time variables and of operators acting on $\mathcal{Z}$:

$$\left[gQ\partial_\zeta\hat{\nabla}(\zeta)+\hat{\nabla}(\zeta)^2-\frac{1}{\sqrt{\beta}}\left(V'\left(t_k-\frac{1}{k\zeta^k}\Big|\zeta\right)\hat{\nabla}(\zeta)\right)_-\right]\mathcal{Z}\left(t_k-\frac{1}{k\zeta^k}\right)=0, \tag{4.31}$$

where the subscript "$-$" refers to negative powers of ζ and $\hat{\nabla}(\zeta)$ does not need to be normal ordered, since it does not act on itself. We next use

$$\left(V'\left(t_k-\frac{1}{k\zeta^k}\Big|\zeta\right)\hat{\nabla}(\zeta)\right)_-=g\sqrt{\beta}\sum_{n\geq-1}\frac{1}{\zeta^{n+2}}\sum_{k\geq1}k\left(t_k-\frac{1}{k\zeta^k}\right)\frac{\partial}{\partial t_{k+n}}=$$

$$=\left(V'(t|\zeta)\hat{\nabla}(\zeta)\right)_--g\sqrt{\beta}\sum_{k,n}\frac{1}{\zeta^{k+n+2}}\frac{\partial}{\partial t_{k+n}}=\left(V'(t|\zeta)\hat{\nabla}(\zeta)\right)_--\partial_\zeta\hat{\nabla}(\zeta), \tag{4.32}$$

where in the last step, we compared the coefficient $k+1$ in

$$\partial_\zeta\hat{\nabla}(\zeta)=g\sqrt{\beta}\sum_{m\geq0}\frac{m+1}{z^{m+2}}\frac{\partial}{\partial t_m}$$

with

$$\sum_{\substack{k\geq1\\ n\geq-1}}\delta_{k+n,m}=m+1.$$

For the "nonnormalized" $\tilde{\Psi}(\zeta):=\mathcal{Z}\left(t_k-\frac{1}{k\zeta^k}\right)$, we can directly check that

$$g\sqrt{\beta}\partial_\zeta\tilde{\Psi}(\zeta)=\hat{\nabla}(\zeta)\tilde{\Psi}(\zeta)$$

and

$$g^2\beta\partial_\zeta^2\tilde{\Psi}=\left(\hat{\nabla}^2(\zeta)+g\sqrt{\beta}\partial_\zeta\hat{\nabla}(\zeta)\right)\tilde{\Psi}(\zeta),$$

whence we finally obtain the differential equation

$$\boxed{\left[g^2\beta\partial_\zeta^2-V'(\zeta)\partial_\zeta+\check{P}(\zeta,T_k)\right]\Psi(\zeta)=0}, \tag{4.33}$$

which is a quantization of the algebraic spectral curve equation depending on both deformation parameters g and β. Rescaling the wave function as $\Psi(\zeta)=\exp\left(V(\zeta)/(g^2\beta)\right)\tilde{\Psi}(\zeta)$, we can rewrite this equation in the form

$$\left[g^2\beta\partial_\zeta^2+\frac{1}{2}\left(V''(\zeta)-\frac{V'^2(\zeta)}{2g^2\beta}+\frac{1}{g^2\beta}\left[\check{P}(\zeta,T_k)V(\zeta)\right]\right)+\check{P}(\zeta,T_k)\right]\tilde{\Psi}(\zeta)=0. \tag{4.34}$$

Now we can easily construct the action of the determinant check-operator on the partition function using (4.27):

$$\left\langle\prod_i(\zeta-z_i)\right\rangle\mathcal{Z}=\exp\left\{\frac{1}{\hbar}\int^\zeta dz\hat{\nabla}(z)\right\}\mathcal{Z}=:\exp\left\{\frac{1}{\hbar}\int^\zeta dz\check{\nabla}(z)\right\}:\mathcal{Z}. \tag{4.35}$$

It is certainly clear how to construct the ordinary operator itself in terms of time variables t_k [32]:

$$\hat{\mathcal{D}}_{[1]}(\zeta)=:\exp\left(\int^\zeta dz\hat{J}(z)\right):, \tag{4.36}$$

$$\hat{J}(z)=\sum_k\left(\frac{1}{2}kt_kz^{k-1}+\frac{1}{z^{k+1}}\frac{\partial}{\partial t_k}\right), \tag{4.37}$$

since this is nothing but the exponential of a free field realized via its action on functions of time variables t_k that corresponds to the fermion (see above). The normal ordering here means that all the t_k- derivatives are put to the right.

The equation for the determinant operator is of the second order in ζ, and as we move along some closed contour γ, the two solutions might have some monodromy. We define a gauge-invariant operator $\check{\mathcal{O}}_{[1]}$ as the trace of this monodromy matrix:

$$\check{\mathcal{O}}_{[1]}(\gamma) = \operatorname{Tr}\operatorname{Mon}(\gamma,\zeta)\check{\mathcal{D}}_{[1]}(\zeta). \tag{4.38}$$

If we take both branches and corrections from “the measure anomaly” into account, extra conjugation factors are [38]

$$\boxed{\check{\mathcal{O}}_{[1]}(\gamma) = \sum_{r=\pm} Z^{-1}(V\to 0, \check{\nabla}_{(r)}) e^{\frac{1}{g\sqrt{\beta}}\oint\limits_{\gamma} dx\check{\nabla}_{(r)}(x)} Z(V\to 0,\check{\nabla}_{(r)})}\,. \tag{4.39}$$

Thus we establish the following dictionary between integrable models and beta-ensembles:

$$\begin{array}{ccc} \lambda & \leftrightsquigarrow & \check{\nabla}, \\ \hbar & \leftrightsquigarrow & g\sqrt{\beta}, \\ \hbar' & \leftrightsquigarrow & g/\sqrt{\beta}, \\ \operatorname{Tr}_R \mathcal{P}\exp \oint\limits_{\gamma} \mathcal{L} & \leftrightsquigarrow & \check{\mathcal{O}}_R(\gamma), \\ w_\gamma & \leftrightsquigarrow & \exp\left(\dfrac{1}{g\sqrt{\beta}}\oint\limits_{\gamma}\check{\nabla}\right). \end{array} \tag{4.40}$$

4.4. Higher weight operators and spectral covers

We used the subscript [1] in the notation for the determinant operator to stress that the operator defined in this way represents a Wilson line in the fundamental representation. Hence, we expect a natural generalization

$$\operatorname{Tr}_R \mathcal{P}\exp\oint\limits_{\gamma} L \leftrightsquigarrow \check{\mathcal{O}}_R(\gamma). \tag{4.41}$$

A naive expression for this operator is expected to be the trace of monodromy of the determinant operator $\check{\mathcal{D}}_R(\zeta)$ that inserts something like $\det_R(\zeta - M)$ into the beta-ensemble averaging:

$$\check{\mathcal{O}}_R(\gamma) = \operatorname{Tr}\operatorname{Mon}(\gamma,\zeta)\check{\mathcal{D}}_R(\zeta). \tag{4.42}$$

We expect that these operators should satisfy the Wilson-loop OPE algebra

$$\check{\mathcal{O}}_R \otimes \check{\mathcal{O}}_{R'} = \sum_{Q\vdash|R|+|R'|} C^Q_{R,R'}\check{\mathcal{O}}_Q, \tag{4.43}$$

where $C^Q_{R,R'}$ are the corresponding Clebsch–Gordan coefficients.

In the case of $SU(2)$, these operators are usually associated with degenerate operators in the Liouville conformal field theory (see Sec. 5.2 below)

$$\check{\mathcal{D}}_{[j]}(\zeta) \sim \Phi_{(j+1,1)}(\zeta) \tag{4.44}$$

with the fusion algebra

$$\Phi_{(j+1,1)} \otimes \Phi_{(j'+1,1)} = \bigoplus_{s=|j-j'|}^{j+j'} \Phi_{(s+1,1)}. \tag{4.45}$$

Translating this remark back to the beta-ensemble framework, we define

$$\check{\mathcal{D}}_{[j]}(\zeta)Z := \left\langle \prod_i (\zeta - z_i)^j \right\rangle Z. \tag{4.46}$$

Nevertheless, the naive form of the R-dependence reveals itself in the form of the spectral curve. We can define a generic spectral cover as

$$\mathrm{Det}_R(\lambda - L(z)) = 0. \tag{4.47}$$

According to this remark, for a symmetric representation $[r]$, we expect the spectral curve to be polynomial in λ of degree $r+1$, the same as the order of the differential equation.

We present an explicit example of the differential equation satisfied by $\check{D}_{[2]}$.

We seek a variation such that the variation of the measure in partition function (4.11) can be rewritten as a derivative acting on $\check{\mathcal{D}}_{[2]}(\zeta)$. With variation (4.13) denoted by $\delta(\zeta)$, we expect some linear combination of variations $\delta(\zeta)\partial_\zeta$ and $\partial_\zeta\delta(\zeta)$ to give the desired result. Indeed,

$$\left\{ \left(1+\frac{2}{\beta}\right)\delta(\zeta)\partial_\zeta + \left(1-\frac{2}{\beta}\right)\partial_\zeta\delta(\zeta) \right\} \check{\mathcal{D}}_{[2]}(\zeta)Z = \left\{ \frac{\beta}{2}\partial_\zeta^3 + \frac{1}{g^2}\check{T}_{[2]}(\zeta) \right\} \check{\mathcal{D}}_{[2]}(\zeta)Z = 0, \tag{4.48}$$

where $\check{T}_{[2]}(\zeta)$ is a contribution from the potential V:

$$\check{T}_{[2]}(\zeta) = \left(1-\frac{2}{\beta}\right)\left(-\partial_\zeta\check{P}(\zeta) + \frac{V'(\zeta)}{2}\partial_\zeta^2\right) - \left(1+\frac{2}{\beta}\right)\check{P}(\zeta)\partial_\zeta + \\ + \frac{2}{\beta}V'(\zeta)\left(\frac{1+\beta}{2}\partial_\zeta^2 + \frac{1}{g^2}\check{P}(\zeta) + \frac{V'(\zeta)}{2g^2}\partial_\zeta\right). \tag{4.49}$$

The operators $\mathcal{O}_R$ are counterparts of the linear group (Schur) characters for the corresponding representations R. To clarify this point, we first assume that the flat connection A is not quantized. Then the naive asymptotic form is

$$\mathrm{Tr}_R\,\mathcal{P}\exp\,\frac{1}{\hbar}\oint_\gamma A \sim P_R\left(\exp\left\{\frac{1}{\hbar}\oint_\gamma \lambda^{(i)}\right\}\right), \tag{4.50}$$

where $\lambda^{(i)}$ are solutions of the equation

$$\det_R(\lambda - A) = 0 \tag{4.51}$$

and the polynomial P_R depends only on the representation R. We can use this last fact and assume that A is a constant field. Hence, we can substitute the ordered exponential by the ordinary exponential and omit the integral, rewriting this equation as

$$\mathrm{Tr}_R\exp\left\{\frac{\alpha}{\hbar}A\right\} = P_R\exp\left\{\frac{\alpha}{\hbar}\lambda^{(i)}\right\}, \tag{4.52}$$

where α is a constant, the "length" of the integration contour γ. We note that this relation is nothing but the Weyl determinant formula relating characters to the Schur polynomials:

$$P_R\exp\left\{\frac{\alpha}{\hbar}\lambda^{(i)}\right\} = \chi_R\left(t_k = \frac{1}{k}\sum_i \exp\left\{\frac{k\alpha}{\hbar}\lambda^{(i)}\right\}\right), \tag{4.53}$$

$$\chi_R(t) = \det_{ij} s_{\lambda_i - i + j}(t), \quad \exp\left\{\sum_k t_k z^k\right\} = \sum_k s_k(t)z^k. \tag{4.54}$$

The integrals over $\gamma \in \Sigma_0$ of the eigenvalues $\lambda^{(i)}$ are thought of as integrals over different pre-image contours $\tilde{\gamma}$ on the spectral cover Σ of the SW differential λ

$$\oint_{\gamma} \lambda^{(i)} := \oint_{\tilde{\gamma}^{(i)}} \lambda, \quad \pi(\tilde{\gamma}^{(i)}) = \gamma. \tag{4.55}$$

As regards the exponential of the last expression, we define it as the Fock–Goncharov coordinate (cluster variable):

$$w_{\tilde{\gamma}} := \exp\left\{ \frac{1}{\hbar} \oint_{\tilde{\gamma}} \lambda \right\}. \tag{4.56}$$

Thus, by analogy, we write a naive asymptotic form as

$$\mathrm{Tr}_R \, \mathcal{P}\mathrm{exp}\, \frac{1}{\hbar} \oint_{\gamma} A \sim \chi_R \left[t_k = \frac{1}{k} \sum_{\tilde{\gamma} | \pi(\tilde{\gamma}) = \gamma} w_{\tilde{\gamma}}^k \right]. \tag{4.57}$$

The basic example is a line in the fundamental representation of $SU(2)$. In this case, the cover is 2-fold, and the contour γ has two pre-images $\tilde{\gamma}_1 = \tilde{\gamma}$ and $\tilde{\gamma}_2 = -\tilde{\gamma}$. The fundamental character is then given by

$$\mathrm{Tr}_{[1]} \, \mathcal{P}\mathrm{exp}\, \frac{1}{\hbar} \oint_{\gamma} A = \chi_{[1]}(t) = t_1 = w_{\tilde{\gamma}} + w_{-\tilde{\gamma}} = w_{\tilde{\gamma}} + \frac{1}{w_{\tilde{\gamma}}}. \tag{4.58}$$

This expression should be compared with the usual sl_2 character

$$\mathrm{Tr}_{[1]} \, y^{\sigma_3} = y^{1/2} + \frac{1}{y^{1/2}}. \tag{4.59}$$

We note that this asymptotic form holds when we consider the quantized connection and the differential on the spectral curve, although it misses two important effects: the Stokes phenomenon and the measure contributions, respectively described in Sec. 2 and Sec. 4.3.

A similar approach with the instanton corrections from the quantum mechanics description of integrable systems taken into account is developed in Ref. [26]. Unfortunately, calculations are done in the Nekrasov–Shatashvili limit ($\hbar' = 0$ in our language), although nonperturbative Stokes corrections are applied to construct an extra nonperturbative contribution to the prepotential in this limit. It is natural to assume consequent nonperturbative corrections to the Nekrasov partition function for $\mathcal{N} = 2$ SUSY gauge theory.

A similar deformation of characters can be encountered under similar circumstances in Ref. [33] (so called qq-characters).

To consider not only symmetric representations, we need to introduce multiple covers representing higher-rank groups or multi-matrix models.

In such models, eigenvalues $z_i^{(j)}$ acquire an extra “flavor” index (j) ranging from 1 to $n-1$ for $SU(n)$. We then expect the following kind of expression for the determinant operator:

$$\check{\mathcal{D}}_{[r_1,\ldots,r_{n-1}]}(\zeta) \sim \left\langle \prod_{j=1}^{n-1} \prod_{i} (\zeta - z_i^{(j)})^{r_j} \right\rangle. \tag{4.60}$$

5. KNOT INVARIANTS FROM WKB MORPHISMS

5.1. Reidemeister invariants from quantum field theory

We consider the Chern–Simons theory with a gauge group G [34], and the Wilson averages in this theory, which are knot polynomials that can be associated with conformal blocks of two-dimensional conformal theory with positions of points changing in time [2]. Since the theory is topological, we can consider just monodromies of the conformal blocks. The conformal theory that corresponds to this gauge theory is the Wess–Zumino–Witten–Novikov (WZWN) model [35], its correlators satisfy the Knizhnik–Zamolodchikov equation [36], and they can be considered a wave function in the Chern–Simons theory [58].

This picture of Wilson averages allows connecting knots with the Knizhnik–Zamolodchikov equation. Indeed, we consider a knot on a 3-manifold $M_3 = \mathcal{C} \times [0,1]$ in a braid representation. The braid is given by n trajectories $\gamma_i:\ z_i(t) \in \mathcal{C}$, $t \in [0,1]$. The wave function on a time slice t depends on the positions of the strands in the braid z_i. We consider a $\bigotimes_j Q_j \bigotimes_j \bar{Q}_j$ bundle over the configuration space $\mathcal{C}^n$ with the connection

$$\mathcal{D}_j = \partial_{z_j} - A(z_j). \tag{5.1}$$

If the connection is flat,

$$[\mathcal{D}_i, \mathcal{D}_j] = 0, \tag{5.2}$$

we can construct the wave function as its flat section:

$$\mathcal{D}_i \Psi = 0. \tag{5.3}$$

The evolution operators can be interpreted as open Wilson lines in the ambient 3D theory:

$$I = \mathcal{P}\exp \int\limits_0^1 dt \bigoplus_j A\left(z_j(t)\right) \dot{z}_j(t). \tag{5.4}$$

They are braid invariants (due to the flatness condition):

$$\frac{\delta}{\delta\gamma_j(t)} I = 0. \tag{5.5}$$

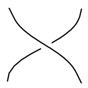
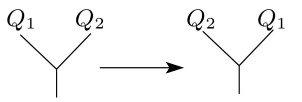
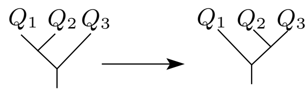
The Wilson operators have a natural structure of a Hopf algebra. Correspondingly, the space of wave functions can be endowed with the structure of a tensor category:

$$\bigotimes_j R_j = \bigoplus_{Q \vdash \sum\limits_j |R_j|} \mathcal{M}_Q \otimes Q. \tag{5.6}$$

The Wilson operators diagonalize under this decomposition:

$$I(\bigotimes_j R_j) = \bigoplus_{Q \vdash \sum\limits_j |R_j|} I(\mathcal{M}_Q) \otimes \mathbf{1}(Q). \tag{5.7}$$

There are two generating elements of the braid group that represent the following two cobordisms:

Cobordism	Trajectory	Representations	Diagram
$T_{i,i+1}$		$(V_1(z_1) \otimes V_2(z_2)) \to (V_1(z_2) \otimes V_2(z_1))$	
$S_{i,i+1,i+2}$		$(V_1(z_1) \otimes V_2(z_2)) \otimes V_3(z_3) \to V_1(z_1) \otimes (V_2(z_2) \otimes V_3(z_3))$	

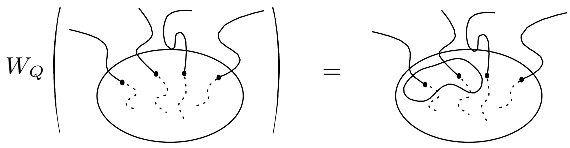


Fig. 6

Now, in order to construct a knot invariant colored by a representation Q from the braid, we have to construct a projector onto Q:

$$\mathcal{P}_Q\left(\bigoplus_R \mathcal{M}_R\otimes R\right)=\mathcal{M}_Q\otimes Q \tag{5.8}$$

and then to "remove" the free ends of the braid either by taking trace or by gluing "caps" to its pairwise ends: the wave functions with vacuum quantum numbers $\Psi_Q(z_1,z_2)\rightsquigarrow\mathcal{P}_\varnothing(Q\otimes Q)$.

There is another set of important operators inserting a Wilson line into a fixed time slice. Since the lines are completely within the slice, these operators are colorless and act on the wave functions as

$$W_Q=\mathrm{Tr}_Q\left[\mathcal{P}\mathrm{exp}\oint A(z)\,dz\;Q\otimes\right]=\mathrm{Unknot}_Q\;\mathcal{P}_\varnothing\left(Q\otimes\mathcal{P}\mathrm{exp}\oint A(z)\,dz\;Q\right)\otimes. \tag{5.9}$$

By the trace, we here mean only the trace over the representation of the gauge group, and hence the operator is a well-defined scalar. Decorating knots with operations of this kind was considered in detail in Ref. [37]. In fact, it is easy to understand that for the one-point conformal block (or for the operator W_Q that inserts a loop surrounding only one point),

$$W_Q\Psi_{Q'}=\frac{H_{Q,Q'}}{\dim_q(Q')}\Psi_{Q'}, \tag{5.10}$$

where $H_{Q,Q'}$ is a Hopf link HOMFLY polynomial and $\dim_q(Q)$ is the quantum dimension of Q, which is the same as the (unreduced = nonnormalized) HOMFLY polynomial of the unknot. If we now apply such an operator to a few strands, it can be expanded into the cluster coordinates, which are a kind of basis in the Hilbert space. Then, generalizing [24, 38], we can expect that W_Q is a polynomial in these coordinates and is a character, in full analogy with Sec. 4.4,

$$W_Q=\chi_Q(Y_1,\ldots,Y_k), \tag{5.11}$$

and Y_k are the Darboux coordinates on the moduli space of flat connections.

In the next two subsections, we present two different realizations of the described scheme, which are basically related to two different realizations of $\mathcal{R}$-matrices.

5.2. Knot invariants from the RTW representation via duality kernels

5.2.1. The basic idea

One of the possibilities to realize this general construction due to Witten [2] is the Reshetikhin–Turaev scheme [3], which was realized in detail in Ref. [39] and in Refs. [6, 40] for different braid representations of knots/links. The approach is based on assigning any cross of the braid the $\mathcal{R}$-matrix of $U_q(G)$. This $\mathcal{R}$-matrix can either come as a monodromy (modular) matrix of the WZWN theory [39] or be treated as just a numeric $\mathcal{R}$-matrix from representation theory [6]. Here, we propose a third possibility: to reproduce the $\mathcal{R}$-matrix by the modular kernel from conformal field theory. Since this case is described by the Virasoro algebra, the obtained $\mathcal{R}$-matrix is associated with $SU_q(2)$ and the corresponding knot invariants are the Jones polynomials.

We explain how to apply modular transformations to the evaluation of knot polynomials. The idea is that, if there are three strands, we can describe the crossing of the first two strands and the second and the third strands respectively as

$$\boxed{\mathcal{R}\otimes I = T, \quad I\otimes\mathcal{R} = STS^\dagger}, \tag{5.12}$$

i.e., the modular matrix T plays the role of an $\mathcal{R}$-matrix acting in the space of intertwining operators $\mathcal{M}_Q$ in (5.6) and S plays the role of the mixing matrix in the RTW formalism (see details in Ref. [6]). These transformations S and T are known to form a Moore–Seiberg grouppoid [42, 58].

Now we have to check the Reidemeister moves:

• 3rd Reidemeister move = YB relation

$$\Big(I\otimes R\Big)\Big(R\otimes I\Big)\Big(I\otimes R\Big) = \Big(R\otimes I\Big)\Big(I\otimes R\Big)\Big(R\otimes I\Big), \tag{5.13}$$

i.e.,

$$STS^\dagger TSTS^\dagger = TSTS^\dagger T \tag{5.14}$$

is solved by the anzatz

$$\begin{gathered} SS^\dagger = 1,\\ STS^\dagger TST = I \end{gathered} \tag{5.15}$$

because it can be rewritten as

$$\Big(STS^\dagger TST\Big)S^\dagger = T\Big(STS^\dagger TST\Big)T^{-1}S^{-1}. \tag{5.16}$$

In the simplest situation (the four-point spherical conformal block), we additionally have $S^\dagger = S$, and therefore Eqs. (5.15) reduce to

$$S^2 = 1 \quad\text{and}\quad (ST)^3 = 1. \tag{5.17}$$

• 2nd Reidemeister move: $TT^{-1} = 1$

• 1st Reidemeister move:

$$T^{ij}_{kl}P^k_i = P^j_l, \tag{5.18}$$

where P is the cap projector.

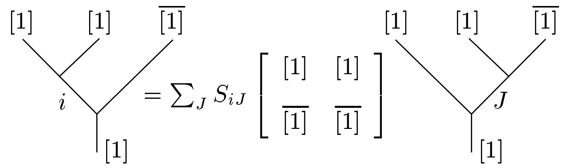
The simplest nontrivial solution of (5.17) is in 2×2 matrices. If T is diagonalized, then S is the elementary mixing matrix of [6]:

$$T = \begin{pmatrix} q & 0\\ 0 & -\dfrac{1}{q}\end{pmatrix}, \quad S = \begin{pmatrix} \dfrac{1}{[2]} & \dfrac{\sqrt{[3]}}{[2]}\\ \dfrac{\sqrt{[3]}}{[2]} & -\dfrac{1}{[2]}\end{pmatrix} \tag{5.19}$$

with the quantum integers $[2] = q + q^{-1}$ and $[3] = q^2 + 1 + q^{-2}$.

5.2.2. S in (5.19) as the Racah matrix

We consider the representation-product diagrams from Ref. [6] for a particular choice of external legs (see top of the next page). Both sets of intermediate states are 2-dimensional, but different: $i = [2], [11]$ and $J = 0, Adj$, i.e., $J = [0]$, $[21^{N-1}]$. We also note that the conjugate fundamental representation is $\overline{[1]} = [1^{N-1}]$. For $N = 2$, there are coincidences: $\overline{[1]} = [1]$, $Adj = [2]$, $[11] = [0]$, and therefore the two diagrams are the same; moreover, the matrix S_{iJ} coincides with that for the Racah matrix for $[1]^{\otimes 3}\to[1]$, which is known from Ref. [6] to be exactly (5.19).



5.2.3. *S* and *T* matrices from conformal theory

Instead of trying to find solutions of Eqs. (5.17) within the group theory framework, one can use another possibility: the same equations are solved by the modular kernels that control modular transformations of conformal blocks. In the generic case, these transformations are given by integral kernels. However, in the case of degenerate fields, they become matrices. Since the Virasoro algebra is associated with $SU(2)$, one expects the S- and T-matrices obtained in this way to generate the colored Jones polynomials, while going further to $SU(N)$ with $N > 2$ would require modular transformations of conformal blocks of the corresponding W_N-algebras.

Therefore, we consider conformal blocks with the fields $\Phi_{(m,n)}$ degenerate at a level $m \cdot n$, with the conformal dimensions [41]

$$\begin{aligned} \Delta_{(m,n)} &= \alpha_{(m,n)} \left(\alpha_{(m,n)} - b + \frac{1}{b} \right), \\ \alpha_{(m,n)} &= \frac{1}{2} \left(\frac{m-1}{b} - (n-1)b \right), \end{aligned} \tag{5.20}$$

where b parameterizes the central charge of the conformal theory: $c = 1 - 6(b - 1/b)^2$. At the same time, choosing different n changes the spin of the representation (of the colored Jones polynomial).

We now read off the matrix S from the modular transformation

$$B_{j_s} \left[\begin{array}{cc} j_2 & j_3 \\ j_1 & j_4 \end{array} \right] (x) = \sum_{j_t} S_{j_s\, j_t} \left[\begin{array}{cc} j_2 & j_3 \\ j_1 & j_4 \end{array} \right] B_{j_t} \left[\begin{array}{cc} j_2 & j_1 \\ j_3 & j_4 \end{array} \right] (1-x). \tag{5.21}$$

The fundamental representation. We consider the simplest example, the fundamental representation of $SU(2)$. In this case, we can expect that the end of the Wilson line in the fundamental representation [1] behaves as $\Phi_{(1,2)}$ in the conformal theory:

$$\Phi_{(1,2)} \otimes \Phi_{(1,2)} = \Phi_{(1,3)} \oplus \Phi_{(1,1)}. \tag{5.22}$$

Then we have (here, the index means the projection on the corresponding state in the intermediate channel)

$$B_{[11]}(x) = \left\langle \Phi_{(1,2)}(0)\Phi_{(1,2)}(x)\big|_{(1,1)} \Phi_{(1,2)}(1)\Phi_{(1,2)}(\infty) \right\rangle = x^{\delta}(1-x)^{\delta}{}_2F_1 \left[\begin{array}{c} \alpha,\ \beta \\ \gamma \end{array} \right] (x), \tag{5.23}$$

$$B_{[2]}(x) = \left\langle \Phi_{(1,2)}(0)\Phi_{(1,2)}(x)\big|_{(1,3)} \Phi_{(1,2)}(1)\Phi_{(1,2)}(\infty) \right\rangle = x^{\bar\delta}(1-x)^{\delta}{}_2F_1 \left[\begin{array}{c} \alpha-\gamma+1,\ \beta-\gamma+1 \\ 2-\gamma \end{array} \right] (x), \tag{5.24}$$

where

$$\begin{aligned} &\alpha = \frac{3}{2b^2} + \frac{7}{2}, \quad \beta = \frac{1}{2b^2} + \frac{3}{2}, \quad \gamma = \frac{1}{b^2} + 3, \\ &\delta = \frac{3b^2}{4} + \frac{1}{4b^2} + \frac{3}{2}, \quad \bar\delta = \frac{3b^2}{4} - \frac{3}{4b^2} - \frac{1}{2}. \end{aligned} \tag{5.25}$$

Then the matrix S (with $S^2 = 1$) is given by

$$S = \left(\begin{array}{cc} \dfrac{\Gamma\left(2+\dfrac{2}{b^2}\right)\Gamma\left(-\dfrac{2}{b^2}-1\right)}{\Gamma\left(1+\dfrac{1}{b^2}\right)\Gamma\left(-\dfrac{1}{b^2}\right)} & \dfrac{\Gamma\left(2+\dfrac{2}{b^2}\right)\Gamma\left(\dfrac{2}{b^2}+1\right)}{\Gamma\left(1+\dfrac{1}{b^2}\right)\Gamma\left(\dfrac{3}{b^2}+2\right)} \\ \dfrac{\Gamma\left(-\dfrac{2}{b^2}\right)\Gamma\left(-\dfrac{2}{b^2}-1\right)}{\Gamma\left(-\dfrac{1}{b^2}\right)\Gamma\left(-\dfrac{3}{b^2}-1\right)} & \dfrac{\Gamma\left(-\dfrac{2}{b^2}\right)\Gamma\left(\dfrac{2}{b^2}+1\right)}{\Gamma\left(-\dfrac{1}{b^2}\right)\Gamma\left(\dfrac{1}{b^2}+1\right)} \end{array}\right) =$$

$$= e^{i\pi}\left(\begin{array}{cc} \dfrac{1}{[2]} & \dfrac{\sqrt{[3]}}{[2]}\gamma^2 \\ \dfrac{\sqrt{[3]}}{[2]}\gamma^{-2} & -\dfrac{1}{[2]} \end{array}\right) = U\cdot\left(\begin{array}{cc} \dfrac{1}{[2]} & \dfrac{\sqrt{[3]}}{[2]} \\ \dfrac{\sqrt{[3]}}{[2]} & -\dfrac{1}{[2]} \end{array}\right)\cdot U^{-1}, \qquad (5.26)$$

where

$$\gamma^2 = (2+b^2)\frac{[2]}{\sqrt{[3]}}\frac{\Gamma^2\left(1+\dfrac{2}{b^2}\right)}{\Gamma\left(\dfrac{1}{b^2}\right)\Gamma\left(2+\dfrac{3}{b^2}\right)}, \qquad (5.27)$$

and

$$U = e^{\frac{i\pi}{2}}\left(\begin{array}{cc} \gamma & 0 \\ 0 & \gamma^{-1} \end{array}\right) \qquad (5.28)$$

while the matrix T is[2] ($(ST)^3 \sim 1$)

$$T \sim \left(\begin{array}{cc} q & 0 \\ 0 & -\dfrac{1}{q} \end{array}\right). \qquad (5.29)$$

The overall normalization of the matrix T is an inessential overall state space phase and can be fixed from the requirement $(ST)^3 = 1$. When verifying various relations here, we used equations

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \quad \cos(\pi\alpha b^{-2}) = \frac{q^\alpha + q^{-\alpha}}{2}, \quad \sin(\pi\alpha b^{-2}) = \frac{q^\alpha - q^{-\alpha}}{2i}. \qquad (5.30)$$

We note that this matrix S differs from that in Eq. (5.19) by the additional U-conjugation (5.28). However, this conjugation influences neither the relation $(ST)^3 = 1$ nor $S^2 = 1$, and does not change the answers for the knot polynomials.

Higher spin representations. So far, we considered only we fundamental representation of $SU(2)$. Similarly, we can consider representations of higher spins. For this, we have to use $\Phi_{(1,2)}\otimes\Phi_{(1,2)}\otimes\Phi_{(1,s+1)}\otimes\Phi_{(1,s+1)}$ with the fusion matrices

[2] In this section, $q = e^{\pi i b^{-2}}$.

$$S = \begin{pmatrix} \dfrac{\Gamma\left(2+\dfrac{2}{b^2}\right)\Gamma\left(-\dfrac{s+2}{b^2}+\dfrac{1}{b^2}-1\right)}{\Gamma\left(1+\dfrac{1}{b^2}\right)\Gamma\left(-\dfrac{s+2}{b^2}+\dfrac{2}{b^2}\right)} & \dfrac{\Gamma\left(2+\dfrac{2}{b^2}\right)\Gamma\left(\dfrac{s+2}{b^2}-\dfrac{1}{b^2}+1\right)}{\Gamma\left(1+\dfrac{1}{b^2}\right)\Gamma\left(\dfrac{s+2}{b^2}+2\right)} \\ \dfrac{\Gamma\left(-\dfrac{2}{b^2}\right)\Gamma\left(-\dfrac{s+2}{b^2}+\dfrac{1}{b^2}-1\right)}{\Gamma\left(-\dfrac{1}{b^2}\right)\Gamma\left(-\dfrac{s+2}{b^2}-1\right)} & \dfrac{\Gamma\left(-\dfrac{2}{b^2}\right)\Gamma\left(\dfrac{s+2}{b^2}-\dfrac{1}{b^2}+1\right)}{\Gamma\left(-\dfrac{1}{b^2}\right)\Gamma\left(\dfrac{s+2}{b^2}-\dfrac{2}{b^2}+1\right)} \end{pmatrix}, \tag{5.31}$$

$$T = \begin{pmatrix} q^{\frac{s+1}{2}}e^{\frac{i\pi}{2}} & 0 \\ 0 & q^{-\frac{s+1}{2}}e^{-\frac{i\pi}{2}} \end{pmatrix}. \tag{5.32}$$

These matrices can be obtained either directly from the equations for the degenerate conformal fields, or from the general expression for the modular kernel due to Ponsot and Teschner [42]. This latter procedure is discussed in the Appendix.

Using these S and T matrices, we can easily generate the Jones polynomials as it was explained above. We note that it is easy to construct the most generic modular kernel S, when only one field is degenerate at the second level: the general degenerate conformal block with $j_2 = 1$ is described by the hypergeometric function

$$B(x) \sim {}_2F_1\left[\begin{matrix} 2+\dfrac{b^{-2}}{2}(3+j_1+j_3+j_4) & 1+\dfrac{b^{-2}}{2}(1+j_1+j_3-j_4) \\ \multicolumn{2}{c}{2+b^{-2}(1+j_1)} \end{matrix}\right](x). \tag{5.33}$$

The corresponding monodromy matrix is[3)]

$$S(j_1,1,j_3,j_4) = \\ = \begin{pmatrix} \dfrac{\Gamma\left(\frac{j_1+1}{b^2}+2\right)\Gamma\left(-\frac{b^2+j_3+1}{b^2}\right)}{\Gamma\left(\frac{2b^2+j_1-j_3+j_4+1}{2b^2}\right)\Gamma\left(-\frac{1-j_1+j_3+j_4}{2b^2}\right)} & \dfrac{\Gamma\left(\frac{j_1+1}{b^2}+2\right)\Gamma\left(\frac{b^2+j_3+1}{b^2}\right)}{\Gamma\left(\frac{2b^2+j_1+j_3-j_4+1}{2b^2}\right)\Gamma\left(\frac{4b^2+j_1+j_3+j_4+3}{2b^2}\right)} \\ \dfrac{\Gamma\left(-\frac{j_1+1}{b^2}\right)\Gamma\left(-\frac{b^2+j_3+1}{b^2}\right)}{\Gamma\left(-\frac{j_1+j_3-j_4+1}{2b^2}\right)\Gamma\left(-\frac{2b^2+j_1+j_3+j_4+3}{2b^2}\right)} & \dfrac{\Gamma\left(-\frac{j_1+1}{b^2}\right)\Gamma\left(\frac{b^2+j_3+1}{b^2}\right)}{\Gamma\left(-\frac{j_1-j_3+j_4+1}{2b^2}\right)\Gamma\left(\frac{2b^2-j_1+j_3+j_4+1}{2b^2}\right)} \end{pmatrix}. \tag{5.35}$$

The higher spin matrix is generated from the recursion formula derived in Ref. [43] from the "cabling" procedure ($[r]\otimes[1] = [r+1]\oplus[r-1]$):

$$S_{q,q'}\begin{bmatrix} r+1 & j_3 \\ j_1 & j_4 \end{bmatrix} = \sum_{s,p} S_{r+1,s}\begin{bmatrix} 1 & q \\ r & j_1 \end{bmatrix} S_{q,p}\begin{bmatrix} 1 & j_3 \\ s & j_4 \end{bmatrix} S_{s,q'}\begin{bmatrix} r & p \\ j_1 & j_4 \end{bmatrix} S_{p,r+1}\begin{bmatrix} r & 1 \\ q' & j_3 \end{bmatrix}. \tag{5.36}$$

5.2.4. Plat representation of link diagrams (spherical conformal block)

Since the operators S and T satisfying (5.15) naturally arise as modular transformations of conformal blocks, we can associate them with link diagrams in the plat representation in the following way.

---

3) To be precise, this matrix is related to the modular kernel as

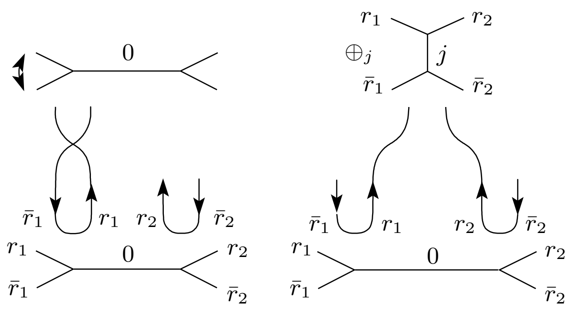
$$S_{j_s\,j_t}\begin{bmatrix} 1 & j_3 \\ j_1 & j_4 \end{bmatrix} = \sum_{h,h'=\pm1} \delta(j_s - j_1 - h)\delta(j_t - j_3 - h')S(j_1,1,j_3,j_4)_{h,h'}. \tag{5.34}$$

We begin with examples.

1 cap: There is nothing to consider in the case of one cap: independently of the number of interweavings between two strands, it is always the unknot:

$$(T^n)_{22} = (-q)^{-n}. \tag{5.37}$$

2 caps: Our notation should be clear from the picture, where the bottom pictures present the conformal block we start with, the middle pictures present the monodromy of points in the conformal block, and the top pictures present the resulting conformal block:



Expressions for the two operations are respectively $T_0(\bar r_1, r_1)$ and $S_{j0}\begin{pmatrix} r_1 & \bar r_2 \\ \bar r_1 & r_2 \end{pmatrix}$.

A generic knot/link in this sector is a sequence of T-twists between parallel strands in the channel 23 and antiparallel strands in the channel 12 (the numbers $1, 2, 3, 4$ here label the vertical lines in the picture). This family includes 2-strand links and knots, twist knots, antiparallel 2-strand links, double braids from Ref. [44], and is known in general as the family of 2-bridge links.

Two unknots

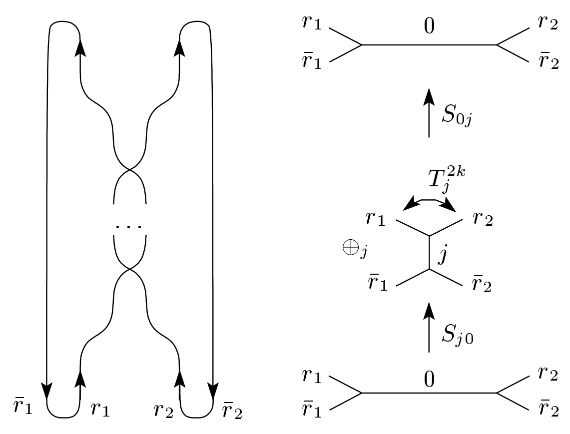
If no S operators are applied, we obtain two disconnected unknots. The answer for two fundamental representations of $SU(2)$ is obtained with the help of Eq. (5.19):

$$\left(T^{n_1}\right)_{22}\left(T^{n_2}\right)_{22} = (-q)^{-n_1-n_2}. \tag{5.38}$$

Up to the framing factor, this is the fully reduced knot polynomial (i. e., unreduced expression $[2]^2$ is divided by the square of the quantum dimension [2]).

2-strand torus links

The plat diagram and the sequence of modular transformations in this case are



The corresponding analytic expression is

$$\sum_j S_{0j} \begin{bmatrix} \bar r_1 & \bar r_2 \\ r_1 & r_2 \end{bmatrix} T_j[r_1, r_2]^{2k} S_{j0} \begin{bmatrix} r_1 & r_2 \\ \bar r_1 & \bar r_2 \end{bmatrix}. \tag{5.39}$$

In the case of two fundamental representations of $SU(2)$, $r_1 = r_2 = [1]$, we can use matrices (5.19) and obtain

$$ST^{2k}S \overset{(5.19)}{=} \frac{1}{[2]^2} \begin{pmatrix} q^{2k} + q^{-2k}[3] & \left(q^{2k} - q^{-2k}\right)\sqrt{[3]} \\ \left(q^{2k} - q^{-2k}\right)\sqrt{[3]} & q^{2k}[3] + q^{-2k} \end{pmatrix}. \tag{5.40}$$

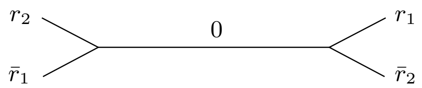
The expression in the lower right corner (the matrix element 22) is exactly the reduced Jones polynomial

$$J^{[2,2k]}_{[1],[1]} = \frac{1}{[N]^2}\left(q^{2k}\frac{[N][N+1]}{[2]} + q^{-2k}\frac{[N][N-1]}{[2]}\right)\Bigg|_{N=2} = \frac{1}{[2]^2}\Big([3]q^{2k} + q^{-2k}\Big) \tag{5.41}$$

for the 2-strand torus links (in the Rosso–Jones framing [45]).

2-strand torus knots

The only difference in this case is that the even power $2k$ is substituted by the odd one $2k+1$, but this is only possible for two coincident representations $r_1 = r_2$. This restriction is obvious from the plat diagram on the left-hand side of the above picture; on the right-hand side, we obtain the top picture in the form of the diagram



and again this is possible (the singlet can run in the intermediate line) only if $r_1 = r_2$.

As regards formula (5.40), it remains just the same, with the obvious change $2k \to 2k+1$, and the 22th element of the matrix reproduces the reduced Jones polynomial

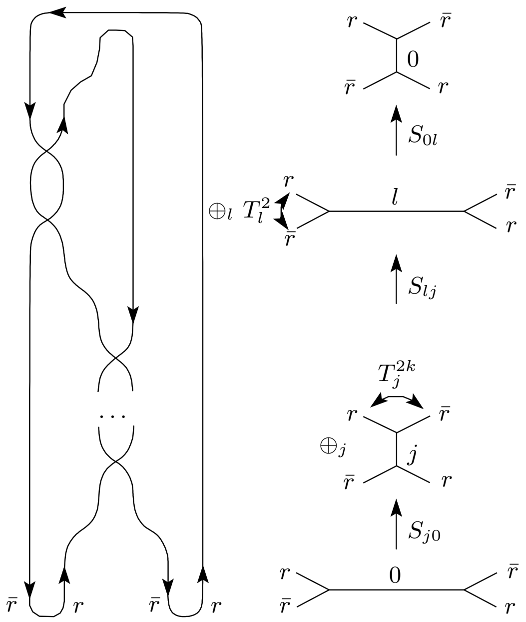
$$ST^{2k+1}S \overset{(5.19)}{=} \frac{1}{[2]^2} \begin{pmatrix} q^{2k+1} - q^{-2k-1}[3] & \left(q^{2k+1} - q^{-2k-1}\right)\sqrt{[3]} \\ \left(q^{2k+1} - q^{-2k-1}\right)\sqrt{[3]} & q^{2k+1}[3] - q^{-2k-1} \end{pmatrix} = \begin{pmatrix} \ldots & \ldots \\ \ldots & \frac{1}{[2]}J^{[2,2k+1]}_{[1]} \end{pmatrix}, \tag{5.42}$$

where

$$J^{[2,2k+1]}_{[1]} = \frac{1}{[N]}\left(q^{2k+1}\frac{[N][N+1]}{[2]} - q^{-2k-1}\frac{[N][N-1]}{[2]}\right)\Bigg|_{N=2} = \frac{1}{[2]}\Big([3]q^{2k+1} - q^{-2k-1}\Big). \tag{5.43}$$

We note that in contrast to links, only one of the two factors [2] is eliminated by expressing the answer through the reduced knot polynomial. Also, like in Eq. (5.41), the Jones polynomial appeared in the Rosso–Jones framing rather than in the topological one.

Twist knots differ by an insertion of two additional twists in the channel 12:



We note that in order to have a closed oriented line, we should not change the order in which the representation and its conjugate appear in the last two vertical lines.

The analytic expression is now

$$\sum_{l,j} S_{0l}\begin{bmatrix} r & \bar r\\ \bar r & r\end{bmatrix}\ T_l\left[r,r\right]^2\ S_{lj}\begin{bmatrix} r & \bar r\\ \bar r & r\end{bmatrix}\ T_j\left[r,\bar r\right]^{2k}\ S_{j0}\begin{bmatrix} r & \bar r\\ \bar r & r\end{bmatrix}. \tag{5.44}$$

In the case of the fundamental representation of $SU(2)$, $r=[1]=\overline{[1]}$, we can use (5.19) to obtain

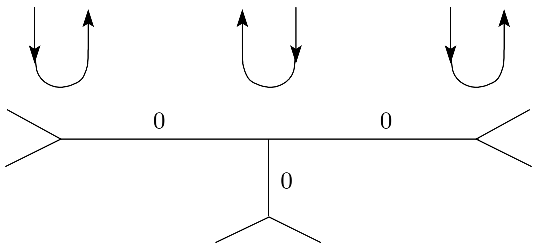
$$ST^2ST^{2k}S \overset{(5.19)}{=} \frac{1}{[2]^2}\begin{pmatrix} q^{2k-1}(q^2+q^{-2})+q^{-2k}\{q^3\} & \Big(q^{2k-1}(q^2+q^{-2})-q^{-2k}\{q\}\Big)\sqrt{[3]}\\ \Big(q^{2k}\{q\}+q^{1-2k}(q^2+q^{-2})\Big)\sqrt{[3]} & q^{2k}\{q^3\}-q^{1-2k}(q^2+q^{-2})\end{pmatrix} =$$

$$= \begin{pmatrix} \cdots & \cdots \\ \cdots & -\frac{q^{-2k-2}}{[2]}J_{[1]}^{Tw(k)}\end{pmatrix}, \tag{5.45}$$

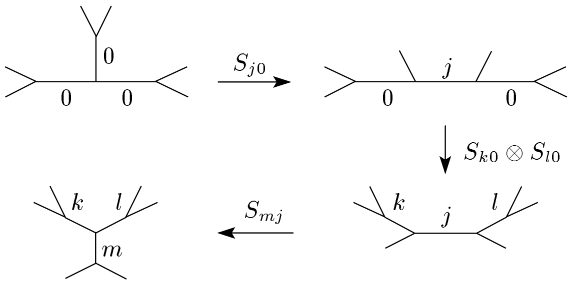
$$J_{[1]}^{Tw(k)} = 1+\frac{A^{k+1}\{A^{-k}\}}{\{A\}}\{Aq\}\{A/q\}\Big|_{A=q^2} = \frac{1}{[2]}\Big(-q^{4k+2}\{q^3\}+q^3(q^2+q^{-2})\Big) =$$

$$= -\frac{q^{2k+2}}{[2]}\Big(q^{2k}\{q^3\}-q^{1-2k}(q^2+q^{-2})\Big). \tag{5.46}$$

3 caps: The initial state in this case can be represented as



This picture shows one of the possible mutual orientations of the three lines, which is suitable for the following example. We can now apply a chain of modular transformations (it should be clear from the above 2-cap examples what the associated link diagram is, but analytic expressions can be read off from the chain of modular transformations):



Now we can apply T transformations in any of the channels, moving back and forth along this chain.

The typical analytic expression begins from

$$\ldots\ T_m\big[r,\bar r\big]^a\ S_{mj}\begin{bmatrix} k & l \\ \bar t & r\end{bmatrix}\ T_k\big[r,\bar r\big]^b\ T_l\big[r,\bar r\big]^c\ S_{l0}\begin{bmatrix} \bar r & \bar r \\ \bar j & \bar r\end{bmatrix} S_{k0}\begin{bmatrix} r & r \\ \bar r & \bar j\end{bmatrix} S_{j0}\begin{bmatrix} r & \bar r \\ 0 & 0\end{bmatrix}. \tag{5.47}$$

As usual, it is read from right to left, and we can add arbitrarily many S transformations and their conjugates of the same type to the left.

We note that the obvious selection rule dictates that $j = r$, and hence, actually, there is no sum involving arguments (not just indices) of the matrix S. However, such sums can appear after additional applications of S.

5.3. Hikami knot invariants from check-operators

There is another, alternative construction of $\mathcal{R}$-matrices, which has a geometric origin and is associated with the tetrahedron volume [46–49]. It is basically associated with Chern–Simons theory with a complex gauge group $G_{\mathbb{C}}$ [50].

5.3.1. Quantum spectral curve in Chern–Simons theory

The Chern–Simons theory on a 3d manifold $\mathcal{M}$ with a complex gauge group $\mathfrak{SL}(2,\mathbb{C})$ is defined by the action [51]

$$S(\mathcal{A},\tilde{\mathcal{A}}) = \frac{t_+}{8\pi}\int\limits_{\mathcal{M}} \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \frac{t_-}{8\pi}\int\limits_{\mathcal{M}} \mathrm{Tr}\left(\tilde{\mathcal{A}}\wedge d\tilde{\mathcal{A}} + \frac{2}{3}\tilde{\mathcal{A}}\wedge\tilde{\mathcal{A}}\wedge\tilde{\mathcal{A}}\right), \tag{5.48}$$

where

$$t_\pm = k \pm is, \quad k\in\mathbb{Z}, \quad s\in\mathbb{R} \tag{5.49}$$

and the path integral is taken with respect to both $\mathcal{A}$ and $\bar{\mathcal{A}}$:

$$\int D\mathcal{A}D\bar{\mathcal{A}}\, e^{iS(\mathcal{A},\bar{\mathcal{A}})}. \tag{5.50}$$

Here, we deal with correlation functions of only fields $\mathcal{A}$. Since we are interested in constructing knot polynomials in this theory, i. e., the Wilson averages, we follow the logic in Sec. 5.1: consider a monodromy of the wave function as a time evolution. We consider the case where $\mathcal{C}$ is a sphere and the gauge group is $\mathfrak{SL}(2,\mathbb{C})$. Fixing the representations along the Wilson lines, we can represent the average of the Wilson lines that begin at points v_i and end at points x_i as the tensor

$$\rho\left[\left\langle\bigotimes_i \mathcal{P}\mathrm{exp}\int_{v_i}^{x_i}\mathcal{A}\right\rangle\right] = \bigotimes_i \rho_i(g(x_i))\bigotimes_j \rho_j(g^{-1}(x_j))\Psi(x_1,\ldots,x_n,v_1,\ldots,v_n). \tag{5.51}$$

The tensor-valued wave function Ψ here is a conformal block of the WZWN theory, which satisfies the Knizhnik–Zamolodchikov equation with respect to both sets of variables v_i and x_i:

$$(t_+ - 2)\partial_{x_i}\Psi = \sum_{j\neq i}\frac{\rho_i(\tau^a)\otimes\rho_j(\tau^a)}{x_i - x_j}\,\Psi. \tag{5.52}$$

We let one of the representations be fundamental and let the corresponding end-point be denoted by z, and the remaining ones by x_i. Then,

$$(t_+ - 2)\partial_z\Psi = \Phi(z)\Psi, \quad \Phi(z) := \sum_i \frac{\sigma^a\otimes\rho_i(\tau^a)}{z - x_i}, \tag{5.53}$$

where σ^a are the Pauli matrices. The equation for the first component Ψ_1 is

$$\hbar\partial_z^2\Psi_1 = \Phi_{11}\partial_z\log\left(\frac{\Phi_{11}}{\Phi_{12}}\right)\cdot\Psi_1 + \hbar\partial_z\log\Phi_{12}\cdot\partial_z\Psi_1 + \frac{1}{\hbar}\left(\Phi^2\right)_{11}\cdot\Psi_1 \tag{5.54}$$

with $\hbar = t_+ - 2$. In the semiclassical limit, only the last term in the r.h.s. of this equation survives, and hence we finally obtain

$$\hbar\partial_z^2\Psi_1 = \sum_i \frac{c_2(\rho_i)}{(z-x_i)^2}\Psi_1 + \sum_i\frac{1}{z-x_i}\underbrace{\sum_{j\neq i}\frac{\rho_i(\tau^a)\otimes\rho_j(\tau^a)}{x_i-x_j}}_{\hbar\partial_{x_i}}\Psi_1, \tag{5.55}$$

where $c_2(\rho)$ is the second Casimir element. This is the spectral curve equation of form (4.34), with the potential parameterized by z_i, and hence the check-operator acts on z_i. The potential itself can be restored from comparing this equation and (4.34). This similarity of (5.55) and (4.34) allows "β-ensemble interpretations" in Chern–Simons theory. The derivative ∂_z is definitely replaced in the semiclassical approximation with the spectral parameter λ.

5.3.2. Verlinde operators

We consider an operator that acts on the Hilbert space in Chern–Simons theory,

$$O_\gamma^{(R)} :\ \mathcal{H}_{CS}\to\mathcal{H}_{CS} \tag{5.56}$$

and we use Knizhnik–Zamolodchikov equation (5.52) to define an "evolution" operator that moves points of the wave function, or of the conformal block, from their initial positions z_i (the initial instant of the evolution, $t=0$) to the final positions v_i after the monodromy transformation (the final instant of the evolution, $t=T$):

$$\mathcal{U}(\mathrm{Knot}) := \bigotimes_i\mathcal{P}\mathrm{exp}\int_{i\text{th strand}} A_i d\zeta_i = \bigotimes_i\mathcal{P}\mathrm{exp}\int_{z_i}^{v_i} d\zeta_i\,\frac{1}{\hbar}\sum_{j\neq i}\frac{\rho_i(\tau^a)\otimes\rho_j(\tau^a)}{\zeta_i-\zeta_j}. \tag{5.57}$$

We can then generate knots invariants by evaluating either the trace of this operator in $\bigotimes_i R_i$ or its projection. In the Heisenberg representation,

$$\mathcal{U}(\mathrm{Knot})^{-1} O_\gamma^{(R)}(0)\, \mathcal{U}(\mathrm{Knot}) = O_\gamma^{(R)}(T), \tag{5.58}$$

and if we know both $O_\gamma^{(R)}(0)$ and $O_\gamma^{(R)}(T)$, it is possible to evaluate $\mathcal{U}(\mathrm{Knot})$ from the equation

$$\boxed{O_\gamma^{(R)}(0)\, \mathcal{U}(\mathrm{Knot}) = \mathcal{U}(\mathrm{Knot})\, O_\gamma^{(R)}(T)}. \tag{5.59}$$

We introduce a set of "Verlinde operators" acting on the space of conformal blocks $\mathcal{H}_{CS}$ as the monodromy trace (direct analogs of W_R in Sec. 5.1):

$$O_\gamma^{(R)} := \mathrm{Tr}_R\, \mathcal{P}\exp \oint_\gamma d\zeta\, \frac{1}{\hbar} \sum_i \frac{\boldsymbol{\tau} \otimes \boldsymbol{\tau}_i}{\zeta - z_i}. \tag{5.60}$$

They can be rewritten following "the β-ensemble interpretation" in terms of check-operators as the trace of monodromy:

$$O_\gamma^{(R)} = \mathrm{Tr}_R \exp\left\{ \frac{1}{\hbar} \oint_\gamma \check{\nabla} \right\} \tag{5.61}$$

while the counterparts of monodromy itself are the (Fock–Goncharov) cluster coordinates

$$w_\gamma := \exp\left\{ \frac{1}{\hbar} \oint_\gamma \check{\nabla} \right\}, \tag{5.62}$$

which form a Heisenberg algebra:

$$w_\gamma w_{\gamma'} = q^{\langle \gamma, \gamma' \rangle} w_{\gamma+\gamma'}. \tag{5.63}$$

We can then solve Eq. (5.59) in terms of this algebra:

$$\boxed{\mathcal{U}(\mathrm{Knot}) = f(w_\gamma)}. \tag{5.64}$$

This algebra admits a realization in the space of conformal blocks [38] with the manifest realization $w_A = e^a$, $w_B = q^{\partial_a}$, whence the evolution operator is realized as $\mathcal{U}(\mathrm{Knot}) = f(e^a, q^{\partial_a})$ and reduces to a modular transformation of the conformal block in terms of S- and T-matrices/kernels.

5.3.3. Knots and flips

Semiclassical limit. As we described, with each WZWN conformal block, we can associate the corresponding Knizhnik–Zamolodchikov equation (and its derivative (5.55)), and with this latter, a WKB network. When the points of the conformal block are subject to monodromy transformations, this Chern–Simons evolution can be described by reconstructions of the WKB network by a series of flips (mutations). In terms of Heisenberg algebra (5.63) associated with the Stokes lines γ, we reinterpret flips as the action of some evolution operators u on w_γ, and the discretized smooth evolution is now

$$\mathcal{U}(\mathrm{Knot}) \rightsquigarrow \prod_{\gamma \in \mathrm{flips}} u_\gamma(X). \tag{5.65}$$

Therefore, we can consider the spectral curve that emerges in the "semiclassical" limit $\hbar \to 0$, Eq. (5.55): $\check{\nabla}^2(z) - T(z) = 0$ and the Stokes lines $\boxed{\mathrm{Im}\, \hbar^{-1} \check{\nabla} = 0}$ (see (2.2)), such that we are able to present semiclassical expressions for the operators:

$$O_\gamma^{(R)} \sim \sum_{\mathrm{sheets}} \exp\left\{ \oint_\gamma \check{\nabla} \right\} + \text{Stokes detours}. \tag{5.66}$$

A single flip along the edge γ is calculated as in Sec. 2.2 and is equal to (see Eq. (2.12))

$$\mathrm{Flip}_\gamma(w_{\gamma'}) = \begin{cases} w_{\gamma'}, & \langle\gamma,\gamma'\rangle = 0, \\ w_{\gamma'}(1+w_\gamma), & \langle\gamma,\gamma'\rangle = 1. \end{cases} \tag{5.67}$$

Quantization. Similarly to Sec. 4.1, using the technique presented in Ref. [20], we can calculate the quantum flip. The result is

$$\mathrm{Flip}_\gamma(w_{\gamma'}) = \begin{cases} w_{\gamma'}, & \langle\gamma,\gamma'\rangle = 0, \\ w_{\gamma'} \prod\limits_{a=1}^{|\langle\gamma,\gamma'\rangle|} \left(1+q^{2a-1} w_\gamma^{\langle\gamma,\gamma'\rangle}\right)^{\langle\gamma,\gamma'\rangle}, & |\langle\gamma,\gamma'\rangle| = 1. \end{cases} \tag{5.68}$$

This flip is described by the adjoint action of u_γ:

$$w_{\gamma'} u_\gamma(w) = u_\gamma(w)\mathrm{Flip}_\gamma(w_{\gamma'}) \tag{5.69}$$

which means that

$$\boxed{u_\gamma(w) \sim \Phi(\log w_\gamma)}, \tag{5.70}$$

where the quantum dilogarithm is defined as [52]

$$\Phi(z|\tau) = \exp\left(-\frac{1}{4}\int\limits_{\mathbf{R}+i0} \frac{dw}{w}\,\frac{e^{-2iwz}}{\mathrm{sh}\, b^{-1}w\cdot \mathrm{sh}\, bw}\right). \tag{5.71}$$

5.3.4. Hikami invariant as a KS invariant

Thus, the evolution operator $\mathcal{U}(\mathrm{Knot})$ can be rewritten as a product of $\mathcal{R}$-matrices, each of which, in its turn, is a product of mutations:

$$\mathcal{R} =: \prod_i u_{\gamma_i} : . \tag{5.72}$$

Having the manifest expression for u_γ in terms of quantum dilogarithm (5.71), we can construct a manifest representation for the $\mathcal{R}$-matrix. This can be done either by using our manifest realization of cluster coordinates (5.62), or in a more formal way [49], with the answer for the $\mathcal{R}$-matrix being a product of ratios of quantum dilogarithms. To obtain the knot polynomial, we still have to calculate the trace of a product of $\mathcal{R}$-matrices.

Afterwards, the R-matrix can be rewritten in terms of tensor categories after the substitution of values for cluster coordinates in terms of tensor categories. From this standpoint, $\mathcal{V}ir_c$ and $\mathcal{U}_q(sl_2)$ are equivalent tensor categories [42][4)].

Semiclassical limit [48]. As was demonstrated in Refs. [48, 49], the $\mathcal{R}$-matrix can be associated with an ideal hyperbolic octahedron. This is not surprising because quantum dilogarithm (5.71) in the semiclassical limit ($q \to 1$) is related to the hyperbolic volume of an ideal tetrahedron Δ, and hence the $\mathcal{R}$-matrix has the asymptotic form

$$\mathcal{R} \sim \exp\left\{\frac{1}{\hbar^2}\sum_i \Delta_i\right\}. \tag{5.73}$$

4) Indeed, this equivalence can be explicitly demonstrated [38], and on general grounds it is a consequence of a mythical mirror symmetry in a mythical $\mathcal{V}ir_{q,t}$-tensor category.

5.4. Stokes phenomenon in conformal blocks

5.4.1. WKB approximation

As we have seen, the spectral curve for a braid of n strands placed at x_i is given by (5.55)

$$\lambda^2 = \sum_{i=1}^{n} \left(\frac{c_2(\rho_i)}{(z-x_i)^2} + \frac{u_i}{z-x_i} \right) dz^2. \tag{5.74}$$

The web of WKB lines is constructed as trajectories of solutions of Eq. (2.2)

$$\mathrm{Im}\ \hbar^{-1}\lambda = 0. \tag{5.75}$$

Here, we present the evolution of WKB lines for six strands associated with the premutation of two middle strands in order to mimic the action of the $\mathcal{R}$-matrix. The blue dots mark positions of zeroes of the discriminant, while the red ones mark positions of the strands (singularities of the curve).

In Fig. 8, it is easy to observe four simple flips associated with the cycles γ_1, γ_2, γ_3, γ_4 in the order they are mentioned.

Hence, the action of the $\mathcal{R}$-matrix is mimicked by the following operator (cf. [49, Eq. (3.15)])

$$\boxed{\mathcal{R} \sim \Phi(w_{\gamma_4})\Phi(w_{\gamma_3})\Phi(w_{\gamma_2})\Phi(w_{\gamma_1}).} \tag{5.76}$$

To clarify the relation to what is discussed in Ref. [49], we mention the relation between the Fock–Goncharov and Kashaev coordinates on triangulations. We consider a triangulation of a punctured Riemann surface depicted in Fig. 9. Here, the singularities are marked with red dots, branching points are marked with purple crosses, and the WKB lines are dashed lines. We then restore the triangulation edges, as shown by the blue lines. We can associate the Kashaev coordinates with these edges. The Fock–Goncharov coordinates are associated with the cycles that are projected to the green lines connecting the branching points, the centers of triangles. In other words, the Fock–Goncharov coordinates are associated with the graph dual to the triangulation and are dual to the Kashaev coordinates correspondingly.

The antisymmetric matrix B_{ij} associated with a quiver (see [49, Eq. (2.1)] represents an intersection pairing matrix associated with the corresponding cycles.

In Fig. 10, we present the WKB triangulation of the spectral curve under consideration and flips of its edges associated with γ_1, γ_2, γ_3, and γ_4 respectively marked by the corresponding colors (the initial edge is marked by a solid line, the mutated edge is marked by a dashed line.) Comparing this triangulation with that presented in [49, Fig. 2], we note that all the horizontal edges are flipped from the very beginning in that paper as compared with the present ones, and the top and bottom tips are not glued together.

6. CONCLUSION AND DISCUSSION

In this review, we tried to give a simple intuitive description for various new interesting phenomena recently described in the literature. The story includes such issues as quantum spectral curves, Teichmüller theory, cluster varieties, moduli space of flat connections, different theories produced by M5-brane compactification, and so on.

Many of these subjects are related to a 2D Coulomb gas system (β-ensemble), maybe via different chains of dualities or correspondences. The usual technique to derive correlators in the β-ensemble system, the topological recursion, gives rise to an infinite chain of linked equations: the loop equations. We argued that the loop equations arising within this approach are similar to those arising within the WKB approach to solving differential equations. The major modification is that in β-ensembles, there are two deformation coefficients: g, a string coupling constant, and β. This can be reinterpreted such that, to the usual Planck constant $\hbar$ controlling the WKB expansion of, say, the Schrödinger equation, one should add another deformation parameter $\hbar'$ that controls commutation relations of eikonals. Thus, the eikonals become noncommutative operators acting on some modified Hilbert space, where the β-ensemble partition function behaves as a wave function on the moduli space.

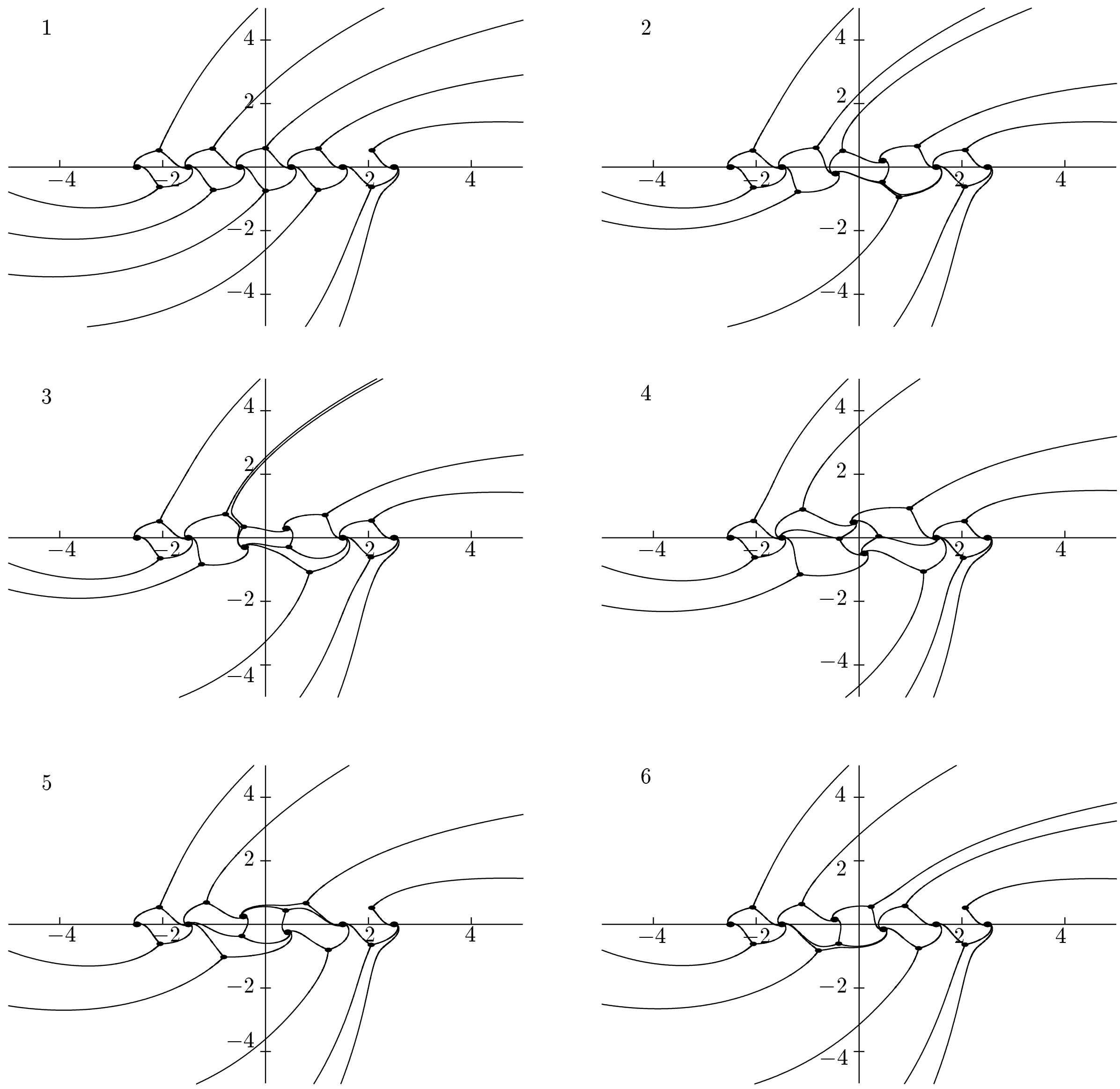


Fig. 7. Evolution of WKB lines (see color online at arXiv:1410.8482v1[hep-th])

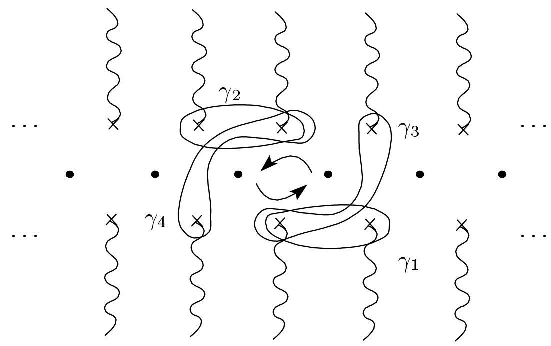


Fig. 8. Cycles associated with the flips corresponding to the permutation of two strands (see color online at arXiv:1410.8482v1[hep-th])

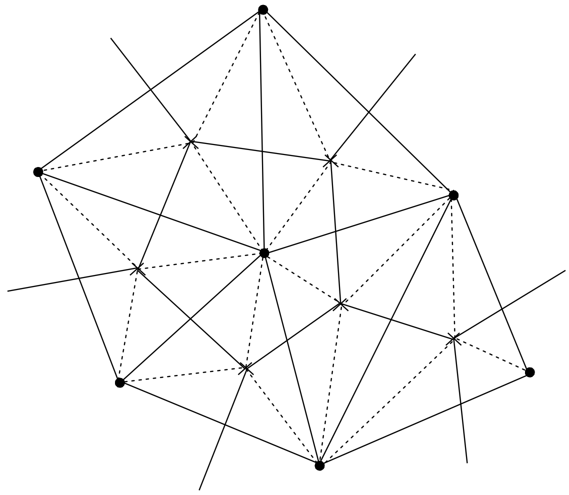


Fig. 9. Triangulation

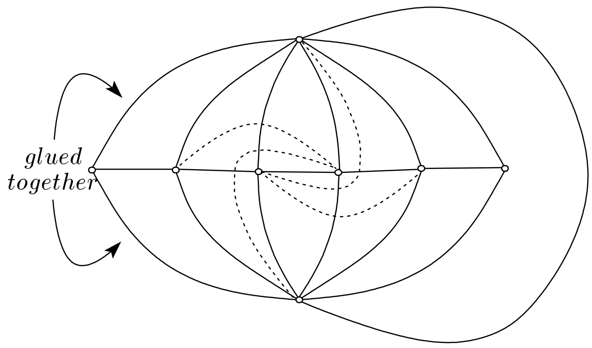


Fig. 10. Flips of the triangulation associated with the conformal block spectral curve

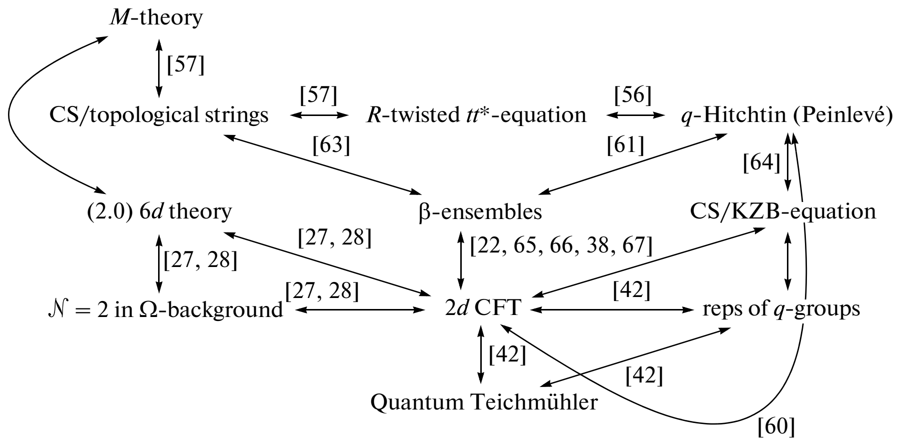
As we discussed, the main characters of this construction are as follows: a flat connection on Riemann surface, the holonomy of this connection, the spectral parameter or eikonal, the cluster coordinates arising as holonomies of this eikonal, and the Hilbert space associated with the partition faction playing the role of a wave function. Also, many theories are tightly related to some dualities. Here, we present various avatars of the same notions emerging in different theories. Some of the avatars have not yet been discussed in the literature, and we therefore leave the corresponding boxes blank:

To complete this table with references, we present the following list and web of correspondences between different theories and references:

1. 2D CFTs: [27, 28].
2. CS/topological strings: [57].
3. CS/KZB equations: [58].
4. β-ensembles: [59].
5. Quantum integrable systems (q-Hitchin, Painlevé): [60, 61].
6. R-twisted tt^*-equations: [19, 53, 57].
7. Quantum Teichmühler theory: [62].
8. $\mathcal{N} = 2$ 4D theory in the Ω-background: [27, 28].
9. Representations of quantum groups: [42].

In this review, we only tried to touch the tip of this iceberg by applying the described framework, in particular, to constructing the Hikami invariants for knots. Nevertheless, the framework seems to be general

Theory	Hilbert space	Flat connection	Holonomy	Spectral parameter	Cluster coordinate	Duality
2d CFT	conf. block	$T(z)$	Verlinde operator [43, 54]	chiral current [55]	Y-functions	fusion rules
β-ensembles	part. func.			resolvent		S-duality
top. strings	part. func.			$U(1)$-Chern–Simons	Wilson loops	
Knizhnik–Zamolodchikov–Bernard	conf. block, CS wave function	$\mathfrak{g}$-connection	Wilson loops			Reidemeister moves
Quantum Hitchin	Landau–Ginzburg wave func.	Lax	Lax holonomies			spectral dualities MMZZ
q-Teichmühler	wave func.	spin connection	spin conn. holonomies	geodesics	Fock–Goncharov, Kashaev	Moore–Seiberg groupoid
Nekrasov	part. func.	2d defect UV	1d defect UV	2d defect IR	1d defect IR	S-duality



enough to be extended to include a third deformation allowing quantum W-algebras, 5D SYM theory, and desirably superpolynomial invariants for knots.

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APPENDIX

In this Appendix, we demonstrate how manifest expressions for the duality matrices can be obtained within the approach due to Ponsot and Teschner [42]. We start with the simpler case where $q = e^{\pi i b^2} \to 1$, which corresponds to $b \to 0$, $c \to \infty$ in conformal theory. The answers for a generic q are obtained by just replacing all hypergeometric functions with q-hypergeometric functions, and the Γ-functions with q-Γ-functions.

A.1. The special case $q \to 1$

Preliminaries

The Ponsot–Teschner approach is manifestly invariant with respect to $b \leftrightarrow b^{-1}$, and hence in this limit we cannot use the already obtained formulas directly. However, we can apply their framework. The crucial point is that in the limit $c \to \infty$, conformal blocks become the "conformal blocks" of sl_2 (see, e. g., [68, 69]),

$$\Psi_\Delta(x) = B_\Delta \left[\begin{array}{cc} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{array}\right] \to x^{\Delta-\Delta_1-\Delta_2}{}_2F_1\left[\begin{array}{c} \Delta+\Delta_2-\Delta_1,\ \Delta+\Delta_3-\Delta_4 \\ 2\Delta \end{array}\right](x). \tag{A.1}$$

This is an eigenfunction of the operator

$$\left[\left(x\frac{d}{dx}+\Delta_1+\Delta_2-\frac{1}{2}\right)^2 - x\left(x\frac{d}{dx}+\Delta_1+\Delta_2+\Delta_3-\Delta_4\right)\left(x\frac{d}{dx}+2\Delta_1\right)\right]\Psi_\Delta(x) = \\ = \left(\Delta-\frac{1}{2}\right)^2\Psi_\Delta(x). \tag{A.2}$$

This operator is self-adjoint with respect to the measure

$$d\mu(x) = x^{2(\Delta_1+\Delta_2-1)}(x-1)^{\Delta_1-\Delta_2+\Delta_3-\Delta_4}. \tag{A.3}$$

Thus, we can calculate the modular kernel as

$$S_{\Delta\Delta'}\left[\begin{array}{cc} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{array}\right] = \int d\mu(x)\ \Psi^{(s)}_\Delta(x)\Psi^{(t)}_{\Delta'}(1-x). \tag{A.4}$$

Finite-dimensional representations

Finite-dimensional representations are described by half-integer spins and correspond to degenerate fields, their dimensions being enumerated by the Kac determinant zeroes, Eq. (5.20):

$$\Delta([s]) = \Delta_{1,s+1} = \frac{b^2s^2}{4} - \frac{s}{2} \to -\frac{s}{2}. \tag{A.5}$$

For the degenerate fields, the conformal blocks are just finite polynomials,

$$\Psi_{j/2}(x) = B_{[j]}\left[\begin{array}{cc} [s] & [s] \\ [s] & [s] \end{array}\right] \to x^{s-\frac{j}{2}}{}_2F_1\left[\begin{array}{c} -\frac{j}{2},\ -\frac{j}{2} \\ -j \end{array}\right](x), \quad j/2 = 0,\ 1,\ 2, \ldots, s. \tag{A.6}$$

In this case, the modular kernel is just a finite matrix, because

$$\Psi_{j/2}(x) = \sum_{j'/2=0}^{s} S_{[2j][2j']}\left[\begin{array}{cc} [s] & [s] \\ [s] & [s] \end{array}\right]\Psi_{j'/2}(1-x). \tag{A.7}$$

For instance,

$$S_{[j/2][j'/2]}\left[\begin{array}{cc} [1] & [1] \\ [1] & [1] \end{array}\right] = \left(\begin{array}{cc} -\dfrac{1}{2} & 1 \\ \dfrac{3}{4} & \dfrac{1}{2} \end{array}\right), \tag{A.8}$$

$$S_{[j/2][j'/2]}\begin{bmatrix}[5] & [5]\\ [5] & [5]\end{bmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{5}{7} & -\frac{25}{14} & \frac{25}{9} & -\frac{5}{2} & 1\\ \frac{7}{60} & -\frac{31}{70} & \frac{23}{28} & -\frac{11}{18} & -\frac{1}{4} & \frac{1}{2}\\ -\frac{7}{90} & \frac{23}{105} & -\frac{1}{12} & -\frac{29}{54} & \frac{7}{12} & \frac{1}{6}\\ \frac{7}{100} & -\frac{33}{350} & -\frac{87}{280} & \frac{79}{180} & \frac{19}{40} & \frac{1}{20}\\ -\frac{1}{10} & -\frac{3}{49} & \frac{15}{28} & \frac{95}{126} & \frac{1}{4} & \frac{1}{70}\\ \frac{11}{36} & \frac{275}{294} & \frac{1375}{1176} & \frac{1375}{2268} & \frac{55}{504} & \frac{1}{252} \end{pmatrix}. \tag{A.9}$$

The other matrix T is then just $T = \mathrm{diag}(1, -1, 1, -1, \ldots)$. In this case, the measure is

$$d\mu(x) = x^{-2(s+1)} \tag{A.10}$$

and the orthogonality condition for Ψ_j is

$$\left\langle \Psi_{j/2}, \Psi_{j'/2}\right\rangle = \int\limits_1^\infty dx\; x^{j/2+j'/2-2}\left({}_2F_1\left[\begin{matrix} -\frac{j}{2},\ -\frac{j}{2}\\ -j\end{matrix}\right](x)\right)\left({}_2F_1\left[\begin{matrix} -\frac{j'}{2},\ -\frac{j'}{2}\\ -j'\end{matrix}\right](x)\right) = \\ = \delta_{j,j'}\frac{(2j)!^4}{(4j)!(4j+1)!}. \tag{A.11}$$

Particular values of the matrix elements are

$$S_{[0][j/2]}\begin{bmatrix}[s] & [s]\\ [s] & [s]\end{bmatrix} = \frac{1}{||\Psi_{j/2}||^2} S_{[2j][0]}\begin{bmatrix}[s] & [s]\\ [s] & [s]\end{bmatrix}, \tag{A.12}$$

$$S_{[j/2][0]}\begin{bmatrix}[s] & [s]\\ [s] & [s]\end{bmatrix} = \int\limits_1^\infty dx\; (1-x)^s x^{-s-j/2-2}{}_2F_1\left[\begin{matrix} -\frac{j}{2},\ -\frac{j}{2}\\ -j\end{matrix}\right](x) = \\ = (-1)^s\sum_{n=0}^{\infty}\frac{\pi(n-j/2)}{\sin\pi(n-j/2)}\frac{\Gamma(1-j/2+n)}{\Gamma(1-j+n)\Gamma(s+j/2+2-n)n!}. \tag{A.13}$$

Explicit calculations

In (A.7), we consider the normalized matrix S

$$\boxed{S_{kk'} = (-1)^{k+k'-s}\frac{\int\limits_1^\infty dx\; x^{k-k'-s-2}\,{}_2F_1\left[\begin{matrix} -k\ \ -k\\ -2k\end{matrix}\right]\left(x^{-1}\right)\,{}_2F_1\left[\begin{matrix} -k'\ \ -k'\\ -2k'\end{matrix}\right](x)}{\int\limits_1^\infty dx\; x^{-2(k'+1)}{}_2F_1\left[\begin{matrix} -k'\ \ -k'\\ -2k'\end{matrix}\right](x)^2},\quad k,k' = 0,\ldots,s.} \tag{A.14}$$

The normalizing multiplier is independent of s, and hence, we can evaluate it, for instance, using the OEIS [70]:

$$\int\limits_1^\infty dx\; x^{-2(k'+1)}{}_2F_1\left[\begin{matrix} -k'\ \ -k'\\ -2k'\end{matrix}\right](x)^2 = \frac{k!^4}{(2k)!(2k+1)!} = \frac{\Gamma(k+1)^4}{\Gamma(2k+1)\Gamma(2k+2)}. \tag{A.15}$$

Now the problem of constructing the matrix S is reduced to evaluating the unnormalized integrals

$$\tilde{S}_{kk'} = \int_1^\infty dx\, x^{k-k'-s-2}\, {}_2F_1\left[\begin{matrix} -k\ -k \\ -2k \end{matrix}\right](x^{-1})\, {}_2F_1\left[\begin{matrix} -k'\ -k' \\ -2k' \end{matrix}\right](x), \quad k,k' = 0,\ldots,s. \tag{A.16}$$

In this case, for instance, at $s = 1$

$$\tilde{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{12} \\ \frac{3}{4} & -\frac{1}{24} \end{pmatrix}. \tag{A.17}$$

We use the Barnes integral representation for the hypergeometric function,

$${}_2F_1\left[\begin{matrix} \alpha\ \beta \\ \gamma \end{matrix}\right](z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\, \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s. \tag{A.18}$$

Thus,

$$\tilde{S}_{kk'} = \frac{\Gamma(-2k)\Gamma(-2k')}{\Gamma(-k)^2\Gamma(-k')^2}\left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{i\infty} dt\, \frac{\Gamma(t-k)^2\Gamma(-t)}{\Gamma(t-2k)} \int_{-i\infty}^{i\infty} dt' \frac{\Gamma(t'-k')^2\Gamma(-t')}{\Gamma(t'-2k')} \times \\ \times \underbrace{\int_1^\infty dx\, x^{k-k'-s-2}\left(-\frac{1}{x}\right)^t (-x)^{t'}}_{\dfrac{(-1)^{t+t'}}{k-k'+t'-t-s-1}}. \tag{A.19}$$

The last term gives a pole, and hence one of the integrals can be calculated as

$$\tilde{S}_{kk'} = (-1)^{k-k'-s+1}\frac{\Gamma(-2k)\Gamma(-2k')}{\Gamma(-k)^2\Gamma(-k')^2}\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\,\frac{\Gamma(t-k'-s-1)^2\Gamma(-t-k+k'+s+1)}{\Gamma(t-k-k'-s-1)} \times \\ \times \frac{\Gamma(t-k')^2\boxed{\Gamma(-t)}}{\Gamma(t-2k')}. \tag{A.20}$$

The boxed term gives the poles contributing to the answer. For example,

$$\tilde{S}_{11} = \frac{\Gamma(-2)^2}{\Gamma(-1)^4}\left(\frac{\Gamma(-3)^2\Gamma(2)}{\Gamma(-4)}\frac{\Gamma(-1)^2}{\Gamma(-2)} - \frac{\Gamma(-2)^2\Gamma(1)}{\Gamma(-3)}\frac{\Gamma(0)^2}{\Gamma(-1)}\right) = -\frac{1}{24}. \tag{A.21}$$

Similarly, we can calculate

$$\tilde{S}_{k0} = (-1)^{k-s}\frac{\Gamma(-2k)}{\Gamma(-k)^2}\frac{\Gamma(-s-1)^2\Gamma(s+1-k)}{\Gamma(-s-1-k)}, \tag{A.22}$$

$$\tilde{S}_{0k} = (-1)^{k+s+1}\frac{\Gamma(-2k)}{\Gamma(-k)^2}\frac{\Gamma(s+1)^2\Gamma(-k-s-1)}{\Gamma(s+1-k)}. \tag{A.23}$$

A.2. Explicit calculations for $q \neq 1$

Similarly to the previous consideration,

$$S_{kk'} = \frac{\int\limits_1^\infty d_q x\, x^{k-k'-s-2}\, {}_2\phi_1 \left[\begin{array}{c} -k \;\; -k \\ -2k \end{array} \right] (x^{-1})\, {}_2\phi_1 \left[\begin{array}{c} -k' \;\; -k' \\ -2k' \end{array} \right] (x)}{\int\limits_1^\infty d_q x\, x^{-2(k'+1)}\, {}_2\phi_1 \left[\begin{array}{c} -k' \;\; -k' \\ -2k' \end{array} \right] (x)^2}, \tag{A.24}$$

where

$${}_2\phi_1 \left[\begin{array}{c} \alpha \; \beta \\ \gamma \end{array} \right] (z) = \sum_{n=0}^{\infty} (-z)^n \frac{\left(\prod\limits_{j=0}^{n-1} [\alpha + j]_q \right) \left(\prod\limits_{j=0}^{n-1} [\beta + j]_q \right)}{\left(\prod\limits_{j=0}^{n-1} [\gamma + j]_q \right) [n]_q!}, \tag{A.25}$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \tag{A.26}$$

$$\int\limits_1^\infty d_q x\, x^{-n} = \frac{(-1)^n}{[n-1]_q}. \tag{A.27}$$

Formulas in the case $q \to 1$ are generalized straightforwardly:

$$\int\limits_1^\infty d_q x\, x^{-2(k'+1)}\, {}_2\phi_1 \left[\begin{array}{c} -k' \;\; -k' \\ -2k' \end{array} \right] (x)^2 = \frac{([k]_q!)^4}{[2k]_q!\,[2k+1]_q!} \tag{A.28}$$

and

$$\tilde{S}_{k0} = (-1)^{k-s+1} \frac{\Gamma_q(-2k)}{\Gamma_q(-k)^2} \frac{\Gamma_q(-s-1)^2 \Gamma_q(s+1-k)}{\Gamma_q(-s-1-k)}, \tag{A.29}$$

$$\tilde{S}_{0k} = (-1)^{k+s+1} \frac{\Gamma_q(-2k)}{\Gamma_q(-k)^2} \frac{\Gamma_q(s+1)^2 \Gamma_q(-k-s-1)}{\Gamma_q(s+1-k)}, \tag{A.30}$$

where

$$\Gamma_q(x+1) = \frac{q^x - q^{-x}}{q - q^{-1}} \Gamma_q(x). \tag{A.31}$$

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