

SOME EXACT ANISOTROPIC SOLUTIONS VIA NOETHER SYMMETRY IN $f(R)$ GRAVITY

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We attempt to find exact solutions of the Bianchi I model in $f(R)$ gravity using the Noether symmetry approach. For this purpose, we take a perfect fluid and formulate conserved quantities for the power-law $f(R)$ model. We discuss some cosmological parameters for the resulting solution which are responsible for expanding behavior of the universe. We also explore Noether gauge symmetry and the corresponding conserved quantity. It is concluded that symmetry generators as well as conserved quantities exist in all cases and the behavior of cosmological parameters shows consistency with recent observational data.

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1. INTRODUCTION

There is interest in investigating nonlinear higher-order differential equations whose exact solutions cannot be determined from well-known methods. This problem is resolved by Lie's theory, which helps not only to find exact solutions but also to explore new solutions by applying different transformations. The most interesting aspect of this theory is Noether symmetry, which is used to obtain analytical solutions as well as the corresponding conserved quantities. Different methods have been introduced to establish conservation laws, like the multiplier approach [1] and the partial Noether theorem for variational and non-variational problems [2]. Some authors [3] proposed computer packages to construct conserved quantities. Cheviakov [4] introduced Maple code to formulate conservation laws by using the multiplier approach.

The accelerated expansion of the universe is widely discussed by modified theories of gravity such as the $f(R)$ gravity, $f(\mathcal{T})$ gravity (\mathcal{T} is the torsion), modified Gauss-Bonnet gravity, $f(R, T)$ gravity (T is trace of the energy-momentum tensor), scalar-tensor theories, etc. The $f(R)$ gravity is the simplest modification of general relativity, where the Ricci scalar R is re-

placed by an arbitrary function $f(R)$. The field equations of $f(R)$ gravity are fourth-order nonlinear partial differential equations whose exact solution can be found via the Noether symmetry approach. Different authors [5] elaborate a comprehensive review of $f(R)$ gravity. Starobinsky [6] discussed the stability criteria of some $f(R)$ models.

Observations of the CMBR and experimental data such as WMAP and Planck satellites indicate that the present universe is isotropic and largely homogeneous. This stage of the universe is described by the FRW model, which ignores all structure of the universe and observed anisotropy in the CMB temperature. However, the early stages of the universe are found to be spatially homogeneous as well as anisotropic. The anisotropy is still found in the present universe as the CMB temperature and to discuss this anisotropy, we consider the simplest anisotropic model, i. e., Bianchi type cosmological homogeneous models. These models describe the anisotropy effect in the early universe on present-day observations. Many authors have discussed these models from different standpoints. Akarsu and Kilinc [7] explored the Bianchi type I (BI) model which yields de Sitter volumetric expansion due to a constant effective energy density for anisotropic fluid along with an anisotropic equation of state (EoS) parameter. Yadav and Saha [8] investigated a locally rotationally symmetric (LRS) BI anisotropic cosmological model with dominance of dark energy for the condition $A = B^m$. They found that the anisotropic distribution

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of dark energy leads to the present accelerated expansion of the universe.

The symmetry approximation is extensively used to study exact solutions under different scenarios, which are then used to discuss cosmic aspects. Sharif and Waheed studied the Bardeen model [9] and stringy charged black holes [10] via approximate symmetry. The same authors [11] introduced curvature correction terms in a scalar–tensor theory to explore Noether symmetries for FRW and LRS BI universe models. Aslam et al. [12] found maximum Noether symmetries for the BI universe in $f(\mathcal{T})$ gravity. Kucukakca [13] formulated exact solutions for a flat FRW universe in the scalar–tensor theory nonminimally coupled to torsion scalar via Noether symmetry. Aslam et al. [14] investigated Noether gauge symmetry in $f(\mathcal{T})$ gravity minimally coupled to a scalar field. Jamil et al. [15] explored Noether gauge symmetry in the Saez Ballester scalar–tensor theory for the BI model. Kucukakca et al. [16] found exact solutions by using the Noether symmetry approach for the LRS BI universe.

Many authors explored Noether symmetry in $f(R)$ gravity. Capozziello et al. [17] investigated some exact spherically symmetric solutions with the help of Noether symmetry in $f(R)$ gravity. Vakili [18] studied Noether symmetry for a flat FRW universe and discussed the effective EoS parameter for the quintessence phase. Jamil et al. [19] found Noether symmetry of the tachyon model for a flat FRW metric and discussed cosmic evolution via a power-law model. Hussain et al. [20] explored Noether gauge symmetry for a flat FRW spacetime which yields zero gauge term. They also checked the stability criteria for the power-law $f(R)$ model. Shamir et al. [21] calculated a nonvanishing gauge term for the same model and also discussed Noether gauge symmetry for the static spherically symmetric model. Kucukakca and Camci [22] explored Noether gauge symmetry for the FRW universe in the Palatini $f(R)$ gravity.

In this paper, we explore Noether and Noether gauge symmetries for an LRS BI universe model in $f(R)$ gravity. We discuss some cosmological parameters to elaborate accelerated expansion of the universe. The paper is organized as follows. In Sec. 2, we provide a general formalism of Noether and Noether gauge symmetries and field equations of $f(R)$ gravity. Section 3 is devoted to exact solutions, the Noether symmetry generator and corresponding conserved quantities, while Sec. 4 formulates symmetry generator via Noether gauge symmetry. In the last section, we summarize the results.

2. NOETHER SYMMETRY AND $f(R)$ GRAVITY

In this section, we briefly discuss Noether and Noether gauge symmetry and $f(R)$ field equations for the LRS BI universe model. Noether symmetry is obtained when a Lie derivative of the Lagrangian vanishes, while Noether gauge symmetry is its generalized form with a nonzero gauge term. The Noether theorem describes a strong connection between symmetries and conservation laws. We consider a point transformation that depends on an infinitesimal parameter λ , i. e., $Q^i = Q^i(q^j, \lambda)$ and generates a one-parameter Lie group. The vector field

$$X = \beta^i(q^i) \frac{\partial}{\partial q^i} + \left[\frac{d}{dt}(\beta^i(q^i)) \right] \frac{\partial}{\partial \dot{q}^i},$$

is referred to as Noether symmetry if the Lagrangian remains invariant, i. e., $L_X \mathcal{L} = 0$. For Noether gauge symmetry, the vector field is defined as

$$X = \alpha(t, q^i) \frac{\partial}{\partial t} + \gamma^j(t, q^i) \frac{\partial}{\partial q^j},$$

where α , β^i , and γ^j are unknown functions and the dot represents the time derivative. This field yields Noether gauge symmetry if the Lagrangian satisfies the equation

$$X^{[1]} \mathcal{L} + (D\alpha) \mathcal{L} = DG(t, q^i).$$

Here, $G(t, q^i)$ represents the gauge term, D and $X^{[1]}$ are the total derivative operator and the first-order prolongation given by

$$D = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i},$$

$$X^{[1]} = X + (\gamma^j{}_{,t} + \gamma^j{}_{,i} \dot{q}^i - \alpha_{,t} \dot{q}^j - \alpha_{,i} \dot{q}^i \dot{q}^j) \frac{\partial}{\partial \dot{q}^j}.$$

The conserved quantity corresponding to X is defined as

$$I = G - \alpha \mathcal{L} - (\gamma^j - \dot{q}^j \alpha) \frac{\partial \mathcal{L}}{\partial \dot{q}^j}.$$

For Noether symmetry, the gauge term vanishes and the conserved quantity takes the form

$$I = -\gamma^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j}.$$

For a dynamical system, the Euler–Lagrange equation and the associated energy function are defined as

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = 0, \quad \sum_i \dot{q}^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L} = E_{\mathcal{L}}.$$

The action of $f(R)$ gravity is given by

$$A = \frac{1}{2\kappa} \int d^4x \sqrt{-g} (f(R) + \mathcal{L}_m(g_{\alpha\beta}, \psi)), \quad (1)$$

where f is an arbitrary differentiable function of the scalar curvature and \mathcal{L}_m is the matter Lagrangian. Using metric variation of this action, we obtain the corresponding field equations as

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R + g_{\mu\nu} \square f_R = \kappa T_{\mu\nu}. \quad (2)$$

Here, $f_R = df/dR$, ∇_μ represents the covariant derivative, and $\square = \nabla_\mu \nabla^\mu$. In terms of the Einstein tensor, Eq. (2) takes the form

$$G_{\alpha\beta} = \frac{\kappa}{f_R} (T_{\alpha\beta} + T_{\alpha\beta}^{(D)}), \quad (3)$$

where the effective stress-energy tensor is

$$T_{\alpha\beta}^{(D)} = \frac{1}{\kappa} \left(\frac{f - R f_R}{2} g_{\alpha\beta} + \nabla_\alpha \nabla_\beta f_R - \square f_R g_{\alpha\beta} \right). \quad (4)$$

This contains such ingredients that are required to deal with accelerated expansion of the universe, referred to as dark source terms.

The LRS BI universe model is given as

$$ds^2 = dt^2 - A^2(t) dx^2 - B^2(t)(dy^2 + dz^2), \quad (5)$$

where the scale factors A and B are functions of time. In order to calculate the Lagrangian, we can write the action as

$$A = \int [AB^2 \dot{f} - \chi(R - \bar{R}) + \mathcal{L}_m] dt, \quad (6)$$

where the dynamical constraint \bar{R} , the Lagrangian multiplier χ , and the matter part of the Lagrangian \mathcal{L}_m (taking perfect fluid in a matter-dominated universe) are

$$\bar{R} = -\frac{2}{AB^2} (\ddot{A}B^2 + 2A\dot{B}\ddot{B} + 2B\dot{A}\dot{B} + A\dot{B}^2), \quad (7)$$

$$\chi = AB^2 \dot{f}', \quad \mathcal{L}_m = \rho_0 (AB^2)^{-1}. \quad (8)$$

The Lagrangian corresponding to the action becomes

$$\mathcal{L}(A, B, R, \dot{A}, \dot{B}, \dot{R}) = AB^2(f - Rf') + 4B\dot{A}\dot{B}f' + 2A\dot{B}^2f' + 2B^2\dot{A}\dot{R}f'' + 4AB\dot{B}\dot{R}f'' + \rho_0(AB^2)^{-1}, \quad (9)$$

where the prime represents the derivative with respect to R .

3. EXACT SOLUTIONS AND CONSERVED QUANTITIES

Here, we attempt to find exact solutions through Noether symmetry and the corresponding conserved quantities. We also check the behavior of some cosmological parameters for the resulting model to study the accelerated expansion of the universe. We assume $A = B^m$, $m \neq 0, 1$, which is obtained from the constant ratio of shear and expansion scalars [23]. With this condition, Lagrangian (9) takes the form

$$\mathcal{L}(B, R, \dot{B}, \dot{R}) = B^{m+2}(f - Rf') + 2(2m+1)B^m \dot{B}^2 f' + 2(m+2)B^{m+1} \dot{B} \dot{R} f'' + \rho_0. \quad (10)$$

The corresponding vector field for Noether symmetry is

$$X = \eta \frac{\partial}{\partial B} + \xi \frac{\partial}{\partial R} + \dot{\eta} \frac{\partial}{\partial \dot{B}} + \dot{\xi} \frac{\partial}{\partial \dot{R}}, \quad (11)$$

where η and ξ are unknown functions that depend on the canonical variables B and R . The derivatives of these unknowns with respect to time are

$$\dot{\eta} = \dot{B} \frac{\partial \eta}{\partial B} + \dot{R} \frac{\partial \eta}{\partial R}, \quad \dot{\xi} = \dot{B} \frac{\partial \xi}{\partial B} + \dot{R} \frac{\partial \xi}{\partial R}. \quad (12)$$

Using Eqs. (10)–(12) in the condition for the existence of symmetry ($L_X \mathcal{L} = 0$), we obtain the set of equations

$$2(m+2)B^{m+1} f'' \eta_{,R} = 0, \quad (13)$$

$$(m+2)B^{m+1} (f - Rf') \eta - B^{m+2} R f'' \xi = 0, \quad (14)$$

$$m(2m+1)B^{m-1} f' \eta + (2m+1)B^m f'' \xi + 2(2m+1)B^m f' \eta_{,B} + (m+2)B^{m+1} f'' \xi_{,B} = 0, \quad (15)$$

$$(m+1)(m+2)B^m f'' \eta + (m+2)B^{m+1} f''' \xi + 2(2m+1)B^m f' \eta_{,R} + (m+2)B^{m+1} f'' \eta_{,B} + (m+2)B^{m+1} f'' \xi_{,R} = 0. \quad (16)$$

Equation (13) implies that either $f'' = 0$ with $\eta_{,R} \neq 0$ or vice versa. The first case leads to a trivial solution. Thus, we consider $f'' \neq 0$ and $\eta_{,R} = 0$ and choose the power-law $f(R)$ model, i. e., $f(R) = f_0 R^n$ ($n \neq 0, 1$), where f_0 and n are constants. Inserting this model in Eqs. (13)–(16), we have

$$2n(m+2)(n-1)B^{m+1} R^{n-2} \eta_{,R} = 0, \quad (17)$$

$$(m+2)B^{m+1} \eta + nB^{m+2} R^{-1} \xi = 0, \quad (18)$$

$$m(2m + 1)B^{m-1}\eta + (n - 1)(2m + 1)B^m R^{-1}\xi + 2(2m + 1)B^m \eta_{,B} + (n - 1)(m + 2)B^{m+1}R^{-1}\xi_{,B} = 0, \quad (19)$$

$$(n - 1)(m + 1)(m + 2)B^m R^{-1}\eta + (n - 1)(n - 2)(m + 2)B^{m+1}R^{-2}\xi + 2(2m + 1)B^m \eta_{,R} + (n - 1)(m + 2)B^{m+1}R^{-1}\eta_{,B} + (m + 2)(n - 1)B^{m+1}R^{-1}\xi_{,R} = 0. \quad (20)$$

We solve this system of equations by assuming a power-law form of (η, ξ) and separation of variables.

3.1. Power-law form

We consider unknowns in a power-law form as

$$\eta = \delta_0 B^{\delta_1} R^{\delta_2}, \quad \xi = \beta_0 B^{\beta_1} R^{\beta_2},$$

where the powers $\delta_1, \delta_2, \beta_1$ and β_2 are arbitrary constants. Inserting these values in Eqs. (17)–(20), we obtain

$$\eta = \delta_0 B^{\frac{n(n-2) \pm \sqrt{3n-2n^2}}{n(n-1)}}, \quad \xi = -\delta_0 \left(\frac{n \pm \sqrt{3n-2n^2}}{n(n-1)} \right) B^{\frac{n \pm \sqrt{3n-2n^2}}{n(n-1)}} R, \quad (21)$$

where $n \in [1.1, 1.5]$. The corresponding Lagrangian and symmetry generator become

$$\mathcal{L} = -(n - 1)B^{\frac{n \pm \sqrt{3n-2n^2}}{n(n-1)}} R^n + 2n(2m + 1)B^{\frac{2-n \pm \sqrt{3n-2n^2}}{n-1}} \dot{B}^2 R^{n-1} + 2n(n - 1)(m + 2)B^{\frac{1 \pm \sqrt{3n-2n^2}}{n-1}} R^{n-2} \dot{B} \dot{R} + \rho_0,$$

$$X = \delta_0 \left(B^{\frac{n(n-2) \pm \sqrt{3n-2n^2}}{n(n-1)}} \frac{\partial}{\partial B} - \left(\frac{n \pm \sqrt{3n-2n^2}}{n(n-1)} \right) \times B^{\frac{n \pm \sqrt{3n-2n^2}}{n(n-1)}} R \frac{\partial}{\partial R} \right).$$

The conserved quantity associated to this symmetry generator is

$$I = \delta_0 \left[4n(2m + 1)B^{\frac{m(n-1) + n(n-2) \pm \sqrt{3n-2n^2}}{n(n-1)}} R^{n-1} \dot{B} + 2n(n - 1)(m + 2) \times R^{n-2} \left(B^{\frac{(m+1)(n-1) + n(n-2) \pm \sqrt{3n-2n^2}}{n(n-1)}} \dot{R} + B^{\frac{(m+1)(n-1) + n \pm \sqrt{3n-2n^2}}{n(n-1)}} \dot{B} \right) \right].$$

3.2. Separation of variables

In this approach, we consider functions in the form

$$\eta = \eta_1(B)\eta_2(R), \quad \xi = \xi_1(B)\xi_2(R),$$

where $\eta_1, \eta_2, \xi_1,$ and ξ_2 are unknown functions to be determined. Using these unknowns in Eqs. (17)–(20), we obtain

$$\eta = \alpha_0 B^{1 - \frac{m+2}{n}}, \quad \xi = -\alpha_0 \left(\frac{m+2}{n} \right) B^{-\frac{m+2}{n}} R, \quad (22)$$

where α_0 is an integration constant. It is found that the above solutions are satisfied for the constraint $m = -1$, which gives

$$\eta = \alpha_0 B^{-1}, \quad \xi = -2\alpha_0 B^{-2} R. \quad (23)$$

Thus, the corresponding Lagrangian (10) and symmetry generator (11) become

$$\mathcal{L} = \frac{BR^{\frac{1}{2}}}{2} f_0 - (BR^{\frac{1}{2}})^{-1} \dot{B}^2 f_0 - \frac{R^{-\frac{3}{2}}}{2} \dot{B} \dot{R} f_0 + \rho_0, \quad X = \alpha_0 \left(B^{-1} \frac{\partial}{\partial B} - 2B^{-2} R \frac{\partial}{\partial R} \right). \quad (24)$$

The corresponding conserved quantity is given by

$$I = \alpha_0 (B^2 R^{1/2})^{-1} \dot{B}. \quad (25)$$

We see that the symmetry generator and the corresponding conserved quantity obtained by the power-law approach are more complicated than the separation of variables technique. Thus, we proceed with the exact solution with the symmetry generator found through separation of variables.

We introduce cyclic variables to solve the field equations. The existence of a Noether symmetry generator ensures the presence of cyclic variables, which are found by using a point transformation, $\varphi: (B, R) \rightarrow (u, z)$ such that $\varphi_X du = 0$ and $\varphi_X dz = 1$. We consider z to be a cyclic variable; the corresponding Lagrangian becomes independent of this variable. Using this transformation, the complexity of the system is reduced by

$$u = B^2 R, \quad z = \frac{a_1}{2} B^2, \quad (26)$$

where a_1 is an arbitrary constant. The corresponding inverse transformation is

$$R = \frac{u}{2a_1 z}, \quad B = (2a_1 z)^{1/2}. \quad (27)$$

Using Eq. (27) in Lagrangian (10), we have

$$\mathcal{L} = \frac{1}{2} \left(u^{1/2} - \frac{u^{-3/2} \dot{u} \dot{z}}{2} \right) f_0 + \rho_0. \quad (28)$$

The Euler–Lagrange equations take the form

$$\frac{m\dot{B}^2}{B^2} + \frac{2\ddot{B}}{B} = \frac{2(m+2)}{(2m+1)f'} \times \left(\frac{f-Rf'}{2} - \frac{2(2m+1)\dot{B}\dot{R}f''}{B} - \ddot{R}f'' - \dot{R}^2 f''' \right), \quad (29)$$

$$Rf'' + 2(m^2+m+2)\frac{\dot{B}^2}{B^2}f'' + 2(m+2)\frac{\ddot{B}}{B}f'' = 0. \quad (30)$$

The associated energy function is

$$\frac{\dot{B}^2}{B^2} = \frac{1}{2f'(2m+1)} \times \left(f - Rf' - 2(m+2)\frac{\dot{B}\dot{R}}{B}f'' + \rho_0 \right). \quad (31)$$

We obtain the pressure and energy density from Eqs. (29) and (31) as

$$p = \frac{2(m+2)}{(2m+1)f'} \times \left(\frac{f-Rf'}{2} - \ddot{R}f'' - \dot{R}^2 f''' - 2(2m+1)\frac{\dot{B}\dot{R}f''}{B} \right), \quad (32)$$

$$\rho = \frac{1}{(2m+1)f'} \times \left(\frac{f-Rf'}{2} - (m+2)\frac{\dot{B}\dot{R}f''}{B} + \frac{\rho_0}{2} \right). \quad (33)$$

In terms of cyclic variables, it follows that

$$u + \ddot{z} = 0, \quad \dot{u} = \frac{4c_0}{f_0}u^{3/2}, \quad \frac{4\rho_0}{f_0}u^{3/2} - 2u^2 = \dot{u}\dot{z}. \quad (34)$$

Solving these equations, we obtain

$$u = -(b_1t + b_2)^{-2}, \quad z = \ln(b_1t + b_2)^{-1/b_1^2} + b_3t + b_4, \quad (35)$$

where $b_1 = 4c_1/f_0$, $b_2 = c_2/2$, b_3 , and b_4 are integration constants. Finally, inserting these values in Eq. (27), we obtain

$$B = (\ln(b_1t + b_2)^{-1/b_1^2} + b_3t + b_4)^{1/2}, \quad (36)$$

$$A = B^{-1} = \{(\ln(b_1t + b_2)^{-1/b_1^2} + b_3t + b_4)^{-1}\}^{-1/2}, \quad (37)$$

$$R = -(b_1t + b_2)^{-2} \{(\ln(b_1t + b_2)^{-1/b_1^2} + b_3t + b_4)^{-1}\}. \quad (38)$$

Now, we discuss some cosmological parameters, i. e., the Hubble, deceleration and EoS parameters for this model. The Hubble parameter is given by

$$H = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} \right).$$

Using Eqs. (36) and (37), we obtain

$$H = \frac{1}{6}(b_3 - (b_1(b_1t + b_2))^{-1}) \times \{ \ln(b_1t + b_2)^{-1/b_1^2} + b_3t + b_4 \}^{-1}. \quad (39)$$

The deceleration parameter q is an important factor in cosmology because it measures cosmic acceleration of the expanding universe. The positive sign of this parameter corresponds to deceleration, whereas negative behavior indicates the accelerated expansion, but $q = 0$ describes expansion with a constant velocity. The deceleration parameter $q = -\dot{H}/H^2 - 1$ takes the form

$$q = -6 \{ \ln(b_1t + b_2)^{-1/b_1^2} + b_3t + b_4 \}^{-1} \times \{ \{ (b_1t + b_2)(b_3 - (b_1(b_1t + b_2))^{-1}) \}^{-2} - (\ln(b_1t + b_2)^{-1/b_1^2} + b_3t + b_4)^{-1} \} - 1. \quad (40)$$

The EoS parameter ($\omega = p/\rho$) is used to distinguish different phases of the universe and further divides the DE phase into different eras. The DE phase is divided by the quintessence era for $-1 < \omega \leq -1/3$, whereas $\omega = -1$ and $\omega < -1$ correspond to the cosmological constant and phantom eras, respectively. With Eqs. (32) and (33), the EoS parameter becomes

$$\omega = \frac{R^3 + \ddot{R}R - 2B^{-1}R\dot{B}\dot{R} - 3/2\dot{R}^2}{R^3 + 3B^{-1}R\dot{B}\dot{R} + R^4\rho_0/2f_0}.$$

With Eqs. (36) and (38) used in above equation, it takes the form

$$\omega = d_0 \left[d_0^{-1}(b_1t + b_2)^2 + \frac{1}{2} \{ 2b_1(b_1 + b_2)^{-1} + d_0^{-1}(b_3 - (b_1(b_1t + b_2))^{-1}) \}^2 + \{ (b_1t + b_2)^{-2}(d_0^{-1} - 2b_1^2) - d_0^{-2}(b_3 - (b_1(b_1t + b_2))^{-1})^2 \} - d_0^{-1}(b_3 - (b_1(b_1t + b_2))^{-1}) \{ 2b_1(b_1 + b_2)^{-1} + d_0^{-1}(- (b_1(b_1t + b_2) + b_3)^{-1}) \} \right] \left[(b_1t + b_2)^{-2} + \frac{3}{2}(b_3 - (b_1(b_1t + b_2))^{-1}) \{ 2b_1(b_1 + b_2)^{-1} + d_0^{-1}(b_3 - (b_1(b_1t + b_2))^{-1}) \} + \frac{\rho_0}{2f_0} d_0^{-1}(b_1t + b_2)^{-4} \right]^{-1}, \quad (41)$$

where $d_0 = \{ \ln(b_1t + b_2)^{-1/b_1^2} + b_3t + b_4 \}$.

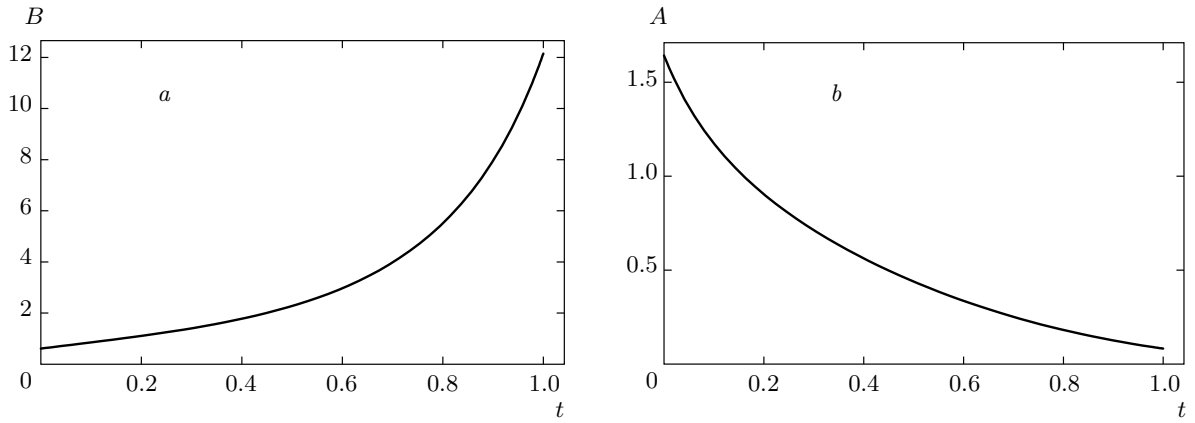


Fig. 1. Plots of scale factors versus time: (a) B versus t ; (b) A versus t for $b_1 = 0.5, b_2 = 0.25, b_3 = 1.5, b_4 = 0.1$

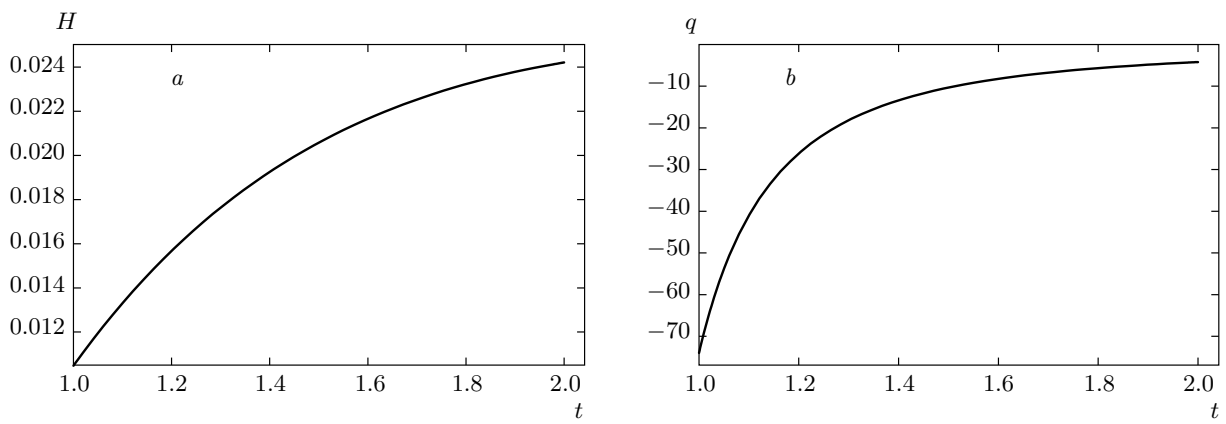


Fig. 2. Plots of (a) the Hubble parameter and (b) the deceleration parameter versus t for $b_1 = 1.5, b_2 = 0.25, b_3 = 0.5, b_4 = 0.1$

The graphical behavior of scale factors B and A is shown in Fig. 1. The plot in 1a represents increasing behavior of B indicating expansion in y and z directions, while A evolves in decreasing manner as shown in Fig. 1b. This shows that expansion is observed in opposite direction (x -direction) for A . Figure 2a indicates that the Hubble parameter increases with the passage of time, whereas Fig. 2b shows negative behavior of the deceleration parameter representing accelerated expansion of the universe. The behavior of ω is shown in Fig. 3, which indicates the quintessence phase initially. This parameter crosses the phantom dividing line and meets the phantom phase with the passage of time. Thus, all the parameters indicate the accelerated expansion of the universe for the LRS BI model in $f(R)$ gravity.

We now investigate Noether symmetry without imposing the condition ($A = B^m$) for the LRS BI model.

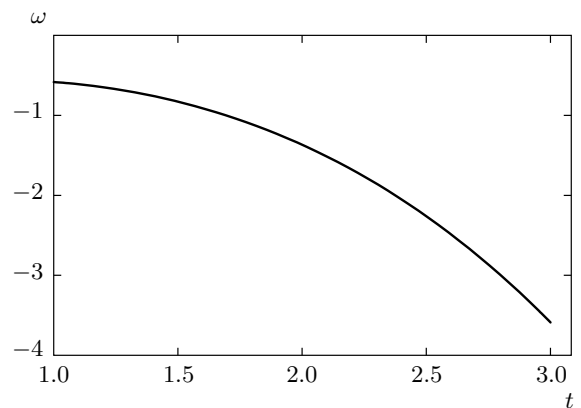


Fig. 3. Plot of the EoS parameter versus time for $b_1 = 1.5, b_2 = 0.2, b_3 = -4.5, b_4 = -0.5, \rho_0 = -1.5, f_0 = 0.5$

We define the vector field as

$$X = \phi \frac{\partial}{\partial A} + \eta \frac{\partial}{\partial B} + \xi \frac{\partial}{\partial R} + \dot{\phi} \frac{\partial}{\partial \dot{A}} + \dot{\eta} \frac{\partial}{\partial \dot{B}} + \dot{R} \frac{\partial}{\partial \dot{R}}, \quad (42)$$

where ϕ , η and ξ are unknown functions depending on A , B , and R . The derivatives of these functions are defined as

$$\begin{aligned} \dot{\phi} &= \dot{A} \frac{\partial \phi}{\partial A} + \dot{B} \frac{\partial \phi}{\partial B} + \dot{R} \frac{\partial \phi}{\partial R}, & \dot{\eta} &= \dot{A} \frac{\partial \eta}{\partial A} + \dot{B} \frac{\partial \eta}{\partial B} + \dot{R} \frac{\partial \eta}{\partial R}, \\ \dot{\xi} &= \dot{A} \frac{\partial \xi}{\partial A} + \dot{B} \frac{\partial \xi}{\partial B} + \dot{R} \frac{\partial \xi}{\partial R}. \end{aligned}$$

The symmetry existence condition for Lagrangian (9) yields

$$2f'\eta_{,A} + Bf''\xi_{,A} = 0, \quad (43)$$

$$Bf''\phi_{,R} + 2Af''\eta_{,R} = 0, \quad (44)$$

$$(B^2\phi + 2AB\eta)(f - Rf') - AB^2R\xi = 0, \quad (45)$$

$$f'\phi + Af''\xi + 2Bf'\phi_{,B} + 2Af'\eta_{,B} + 2ABf''\xi_{,B} = 0, \quad (46)$$

$$2f'\eta + 2Bf''\xi + 2Bf'\phi_{,A} + 2Af'\eta_{,A} + 2Bf'\eta_{,B} + 2ABf''\xi_{,A} + B^2f''\xi_{,B} = 0, \quad (47)$$

$$2f''\eta + Bf''' \xi + Bf''\phi_{,A} + 2Af''\eta_{,A} + 2f'\eta_{,R} + Bf''\xi_{,R} = 0, \quad (48)$$

$$2Bf''\phi + 2Af''\eta + 2ABf''' \xi + B^2f''\phi_{,B} + 2Bf'\phi_{,R} + 2ABf''\eta_{,B} + 2Af'\eta_{,R} + 2ABf''\xi_{,R} = 0. \quad (49)$$

To solve this system of equations, we consider the same power-law $f(R)$ model for separation of variables as in the preceding case, which leads to $n = 1$. Thus, we proceed further using the power-law technique with unknowns in the form

$$\begin{aligned} \phi &= \phi_0 A^{\phi_1} B^{\phi_2} R^{\phi_3}, & \eta &= \eta_0 A^{\eta_1} B^{\eta_2} R^{\eta_3}, \\ \xi &= \xi_0 A^{\xi_1} B^{\xi_2} R^{\xi_3}. \end{aligned}$$

Substituting these values in the above equations, we obtain

$$\phi = -2\eta_0 A, \quad \eta = \eta_0 B, \quad \xi = 0. \quad (50)$$

The symmetry generator becomes

$$X = -2\eta_0 A \frac{\partial}{\partial A} + \eta_0 B \frac{\partial}{\partial B},$$

which corresponds to scaling symmetry. The corresponding conserved quantity is

$$I = \eta_0(4nABR^{n-1}\dot{B} - 4nB^2R^{n-1}\dot{A}).$$

4. NOETHER GAUGE SYMMETRY

In this section, we explore Noether gauge symmetry for $A = B^m$. For this symmetry, we define the vector field as

$$Y = \tau(t, B, R) \frac{\partial}{\partial t} + \phi(t, B, R) \frac{\partial}{\partial B} + \psi(t, B, R) \frac{\partial}{\partial R}, \quad (51)$$

where τ , ϕ and ψ are unknown functions to be determined. The first-order prolongation of this vector field is given by

$$Y^{[1]} = \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial B} + \psi \frac{\partial}{\partial R} + \dot{\phi} \frac{\partial}{\partial \dot{B}} + \dot{\psi} \frac{\partial}{\partial \dot{R}}, \quad (52)$$

where

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \dot{B} \frac{\partial \phi}{\partial B} + \dot{R} \frac{\partial \phi}{\partial R} - \dot{B} \frac{\partial \tau}{\partial t} - \dot{B}^2 \frac{\partial \tau}{\partial B} - \dot{B} \dot{R} \frac{\partial \tau}{\partial R}, \quad (53)$$

$$\dot{\psi} = \frac{\partial \psi}{\partial t} + \dot{B} \frac{\partial \psi}{\partial B} + \dot{R} \frac{\partial \psi}{\partial R} - \dot{R} \frac{\partial \tau}{\partial t} - \dot{R}^2 \frac{\partial \tau}{\partial R} - \dot{B} \dot{R} \frac{\partial \tau}{\partial B}. \quad (54)$$

This vector field yields a Noether gauge symmetry if the corresponding Lagrangian satisfies the condition

$$Y^{[1]}\mathcal{L} + (D\tau)\mathcal{L} = DG, \quad D = \frac{\partial}{\partial t} + \dot{B} \frac{\partial}{\partial B} + \dot{R} \frac{\partial}{\partial R}. \quad (55)$$

The corresponding conserved quantity takes the form

$$I = G - \tau\mathcal{L} - (\phi - \dot{B}\tau) \frac{\partial \mathcal{L}}{\partial \dot{B}} - (\psi - \dot{R}\tau) \frac{\partial \mathcal{L}}{\partial \dot{R}}. \quad (56)$$

Using Eqs. (51)–(54) in (55) for Lagrangian (10), we obtain

$$\tau_{,B} = 0, \quad (57)$$

$$\tau_{,R} = 0, \quad (58)$$

$$2(m+2)B^{m+1}f''\phi_{,R} = 0, \quad (59)$$

$$2(m+2)B^{m+1}f''\phi_{,t} = G_{,R}, \quad (60)$$

$$4(2m+1)B^m f'\phi_{,t} + 2(m+2)B^{m+1}f''\psi_{,t} = G_{,B}, \quad (61)$$

$$\begin{aligned} &(2m+1)mB^{m-1}f'\phi + (2m+1)B^m f''\psi + \\ &+ 2(2m+1)B^m f'\phi_{,B} + (m+2)B^{m+1}f''\psi_{,B} - \\ &- 2(2m+1)B^m f'\tau_{,t} = 0, \end{aligned} \quad (62)$$

$$(m + 1)(m + 2)B^m f'' \phi + (m + 2)B^{m+1} f''' \psi + (m + 2)B^{m+1} f'' \phi_{,B} + 2(2m + 1)B^m f' \phi_{,R} - (m + 2)B^{m+1} f'' \tau_{,t} + (m + 2)B^m f'' \psi_{,R} = 0, \quad (63)$$

$$(f - Rf')\{(m + 2)B^{m+1} \phi + B^{m+2} \tau_{,t}\} - B^{m+2} R f'' \psi + \tau_{,t} \rho_0 = G_{,t}. \quad (64)$$

For a nontrivial solution, we choose the power-law form as $f = f_0 R^{3/2}$ and solve the above system by the separation of variables approach, i. e.,

$$\tau = \tau_0 \tau_1(t) \tau_2(B) \tau_3(R), \quad \phi = \phi_0 \phi_1(t) \phi_2(B) \phi_3(R),$$

$$\psi = \psi_0 \psi_1(t) \psi_2(B) \psi_3(R),$$

where $\tau_0, \phi_0,$ and ψ_0 are arbitrary constants while $\tau_i, \phi_i,$ and ψ_i are unknown functions to be found. Also, we assume that $\dot{\tau}_3 = 0,$ which yields $\tau = c_3.$ Thus, we obtain

$$\tau = c_3,$$

$$\phi = B^{-1} \left(c_5 \cos \sqrt{\frac{2R}{3}} t - c_6 \sin \sqrt{\frac{2R}{3}} t \right) c_4, \quad (65)$$

$$\psi = 2RB^{-2} \left(c_5 \cos \sqrt{\frac{2R}{3}} t - c_6 \sin \sqrt{\frac{2R}{3}} t \right) c_4, \quad (66)$$

$$G = -\sqrt{6}B^{-1}R \left(c_5 \cos \sqrt{\frac{2R}{3}} t + c_6 \sin \sqrt{\frac{2R}{3}} t \right) c_4 - c_7, \quad (67)$$

where $c_k, (k = 3, 4, 5, 6, 7)$ are integration constants. These solutions satisfy Eqs. (57)–(64) with

$$\sin \sqrt{\frac{2R}{3}} t = -\frac{c_6}{c_5} \cos \sqrt{\frac{2R}{3}} t.$$

With this condition, it follows that

$$\tau = c_3, \quad G = -c_7, \quad \phi = c_8 B^{-1} \cos \sqrt{\frac{2R}{3}} t,$$

$$\psi = 2c_8 B^{-2} R \cos \sqrt{\frac{2R}{3}} t,$$

where c_8 is a redefined constant. The corresponding symmetry generator is

$$Y = c_4 \frac{\partial}{\partial t} + c_8 \left(B^{-1} \cos \sqrt{\frac{2R}{3}} t \frac{\partial}{\partial B} + 2B^{-2} R \cos \sqrt{\frac{2R}{3}} t \frac{\partial}{\partial R} \right),$$

which can be split into two generators as

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = B^{-1} \cos \sqrt{\frac{2R}{3}} t + 2B^{-2} R \cos \sqrt{\frac{2R}{3}} t,$$

where the first symmetry generator Y_1 corresponds to energy conservation. The conserved quantities associated to the vector field are

$$I_1 = \frac{1}{2} (BR^{3/2} - 6B^{-1}R^{1/2}\dot{B}^2 + 3\dot{B}\dot{R}R^{-1/2}),$$

$$I_2 = 3B^{-2}R^{1/2}\dot{B} \cos \sqrt{\frac{2R}{3}} t - \frac{3}{2}B^{-1}R^{-1/2}\dot{R} \cos \sqrt{\frac{2R}{3}} t.$$

5. SUMMARY

In this paper, we have explored the LRS BI universe model via Noether symmetry in $f(R)$ gravity. Noether symmetry is a powerful tool to find exact solutions of nonlinear partial differential equations, symmetry generators, and the corresponding conserved quantities. We have explored the power-law form and separation of variables to formulate the Noether symmetry generators and the associated conserved quantities by assuming $A = B^m.$ We have considered the power-law $f(R)$ model to avoid trivial solutions. The symmetry generators and the corresponding conserved quantities are obtained through both approaches but the separation of variables approach is simpler. We have also formulated Noether symmetry without assuming the condition $A = B^m,$ which leads to the scaling symmetry.

We have explored the behavior of some cosmological parameters like the Hubble, deceleration, and EoS parameters for this model. These parameters indicate that the results correspond to the accelerated expansion of the universe. The EoS parameter shows the crossing of the phantom dividing line from the quintessence to phantom era, which is consistent with recent cosmological observations [24]. We have also explored Noether gauge symmetry by assuming the condition $A = B^m$ and formulated a symmetry generator associated with the energy conservation for $n = 3/2.$ Sharif and Shamir [25] have found two exact solutions for BI and Bianchi type-V spacetimes by using the variational law of the Hubble parameter in $f(R)$ gravity. They formulated solutions for a singular model with

power-law expansion and a nonsingular model with exponential expansion. The deceleration and Hubble parameters are positive and infinite for power-law expansion, respectively. For exponential expansion, the deceleration parameter is negative, whereas the Hubble parameter is finite for finite values of t . We have found one exact solution via the Noether symmetry approach for the power-law $f(R)$ model. The Hubble and deceleration parameters are finite and negative, respectively. It would be interesting to examine the LRB BI universe model for different choices of $f(R)$ models.

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