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UNFOLDED EQUATIONS FOR CURRENT INTERACTIONS OF $4d$ MASSLESS FIELDS AS A FREE SYSTEM IN MIXED DIMENSIONS

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Interactions of massless fields of all spins in four dimensions with currents of any spin are shown to result from a solution of the linear problem that describes a gluing between a rank-one (massless) system and a rank-two (current) system in the unfolded dynamics approach. Since the rank-two system is dual to a free rank-one higher-dimensional system that effectively describes conformal fields in six space–time dimensions, the constructed system can be interpreted as describing a mixture between linear conformal fields in four and six dimensions. An interpretation of the obtained results in the spirit of the AdS/CFT correspondence is discussed.

Contribution for the JETP special issue in honor of V. A. Rubakov's 60th birthday

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1. INTRODUCTION

Valery Rubakov has a remarkably broad area of scientific interests, ranging from the theory of fundamental interactions to cosmology. To the volume in honor of Valery's 60th birthday, we contribute a paper that gives hints on a possible unification of such seemingly different concepts of quantum field theory (QFT) as conserved currents in lower dimension and free fields in higher dimension. Although such an identification now sounds natural in the context of the AdS/CFT correspondence [1–3], the particular realization suggested in this paper goes beyond the standard setup that allows interpreting current interactions of $4d$ fields of all spins, including the usual fields of spins $0 \leq s \leq 2$, in terms of a linear system mixing free conformal fields in four and six dimensions. In fact, part of this work has been presented some time ago at a seminar headed by Rubakov, after which we had a stimulating discussion with Valery on whether it is possible to make fields in space–times of different dimensions directly interact in relativistic field theory. Since then, we have obtained more evidence, including that presented in this paper, that this is not only possible but also can eventually drive us to a better understanding of fundamental concepts of QFT, including the very concept of space-time. Hence, we believe that this paper is appropriate for the volume in honor of Valery Rubakov.

Specifically, we consider field equations for massless fields of all spins in a four-dimensional anti-de Sitter space in the lowest order in interactions accounting for the contribution of conserved currents built from bilinears of the same set of massless fields. The problem is analyzed in the framework of the covariant first-order unfolded formulation underlying the known formulation of nonlinear massless field equations [4, 5] (see also [6] for more details and references). Our goal is to clarify the structure of current interactions in the nonlinear higher-spin (HS) theory that describes interactions of massless fields of all spins in four dimensions.

Technically, our approach is based on the correspondence between fields and currents elaborated in [7], where $\mathrm{Sp}(2M)$-invariant field equations corresponding

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to rank-r tensor products of the Fock (singleton) representation of $\mathrm{Sp}(2M)$ were studied. These equations were shown to describe localization on "branes" of different dimensions embedded into the generalized space–time $\mathcal{M}_M$ with matrix coordinates

$$X^{AB} = X^{BA}, \quad A, B = 1, \dots, M$$

(see [8–11]). For $M = 4$, the indices $A, B = 1, \dots, 4$ can be interpreted as Majorana spinor indices of the four-dimensional Minkowski space, while the space $\mathcal{M}_4$ is ten dimensional. Minkowski space is a subspace of $\mathcal{M}_4$ with local coordinates $x^{\alpha\alpha'}$ in the two-component spinor notation[1)]. The relation to the tensor notation is

$$x^{\alpha\beta'} = x^n \sigma_n^{\alpha\beta'},$$

where $\sigma_n^{\alpha\beta'}$ $(n = 0, 1, 2, 3)$ are four independent Hermitian 2×2 matrices.

The conserved currents built from bilinears of the rank-one fields in $\mathcal{M}_M$ were shown in [7] to obey the field equations of rank-two fields in $\mathcal{M}_M$. More generally, it was shown that products of r rank-one fields obey the rank-r field equations. On the other hand, a rank-r field in $\mathcal{M}_M$ was interpreted as a "compactification" of an "elementary" rank-one field in M_{rM}. This correspondence is in the spirit of the AdC/CFT correspondence [1–3], with a field in the higher-dimensional (bulk) space–time identified with a current in a lower-dimensional (boundary) space–time. We believe that this phenomenon has far-reaching consequences, partially discussed already in [10]. In particular, from this perspective, the very notion of the space–time dimension acquires dynamical origin [11, 12].

Genuine massless fields in $d = 4$ are rank-one fields in the ten-dimensional space $\mathcal{M}_4$ [10]. It was shown in [7, 13] that for $M = 4$, the realization of a rank-two field in terms of bilinears of rank-one fields gives rise to the full list of conformal gauge-invariant conserved currents of all spins in four dimensions [14], which generalize the so-called generalized Bell–Robinson currents constructed by Berends, Burgers, and van Dam [15].

On the other hand, a rank-two field in $\mathcal{M}_4$ can be identified with an elementary rank-one field in $\mathcal{M}_8$ that gives rise to usual conformal fields in six dimensions [9, 11, 16], which, in accordance with the general results in [17, 18], are the mixed-symmetry fields described by various two-row rectangular Young diagrams. We note that the idea that currents realized as bilinears of elementary fields behave as fields in higher dimension is not new and was discussed, for example, in [19, 20] (also see the references therein). However, in the framework of HS theories that describe infinite towers of massless fields of all spins, this idea can be given a particularly neat realization.

This correspondence suggests the idea that the current interaction in four dimensions can be interpreted as a mixture between linear rank-one and rank-two fields in $\mathcal{M}_4$, where the latter field is only assumed to satisfy the rank-two unfolded field equations. This implies that the seemingly nonlinear interaction of massless fields in four dimensions with the currents (that can be constructed from the same fields) results from a solution of the linear problem that describes a gluing between rank-one and rank-two fields in the unfolded dynamic approach. As mentioned above, an interesting interpretation of this system is that it mixes massless fields in four space–time dimensions with conformal fields in six space–time dimensions interpreted as currents in the four-dimensional space.

In this paper, we show how this works in practice. Namely, we present a linear unfolded system of equations that glues the unfolded equations of rank-one and rank-two fields in such a way that, after realizing the rank-two fields in terms of bilinears of the rank-one fields, the usual field equations for massless fields receive corrections that just describe the contribution of currents to the field equations. It is interesting to note that the same mechanism brings Yukawa interactions to the field equations of massless fields of spins 0 and 1/2.

The rest of the paper is organized as follows. In Sec. 2, we recall the unfolded form of $4d$ free HS field equations in AdS_4 proposed in [21, 22] and their flat limit. In Sec. 3, the constructions in [7, 13] of conserved currents in the flat space is recalled and its generalization to AdS_4 is given. The nontrivial current deformation of the rank-one unfolded system with the rank-two unfolded system is presented in Sec. 4. In Sec. 5, it is shown in detail how the deformed unfolded equations affect the form of dynamical equations for massless fields, bringing currents to their right-hand sides. Section 6 contains a summary of the obtained results and a discussion of further research directions. Appendices A, B, C, and D collect technical details of the calculations.

1) (Un)primed indices from the beginning of the Greek alphabet take two values $\alpha, \beta = 1, 2$ and $\alpha', \beta' = 1', 2'$. The two-component indices are raised and lowered as follows: $A^\alpha = \varepsilon^{\alpha\beta} A_\beta$, $A_\alpha = \varepsilon_{\beta\alpha} A^\beta$, where $\varepsilon_{\beta\alpha} = -\varepsilon_{\alpha\beta}$, $\varepsilon_{12} = 1$, and analogously for primed indices.

2. PRELIMINARIES

2.1. Higher-spin gauge fields in AdS_4

In this section, we recall the unfolded form of $4d$ free HS field equations proposed in [21, 22]. It is based on the frame-like approach to HS gauge fields [23, 24], where a spin-s HS gauge field is described by the set of one-forms

$$\omega_{\alpha_1\ldots\alpha_k,\alpha'_1\ldots\alpha'_l} = dx^n \omega_{n\,\alpha_1\ldots\alpha_k,\alpha'_1\ldots\alpha'_l}, \quad k+l = 2(s-1),$$

and the set of zero-forms $C_{\alpha_1\ldots\alpha_n,\beta'_1\ldots\beta'_m}(x)$ with $n-m = 2s$ along with their conjugates $\overline{C}_{\alpha_1\ldots\alpha_n,\beta'_1\ldots\beta'_m}(x)$ with $m-n=2s$. The HS gauge fields are self-conjugate

$$\overline{\omega_{\alpha_1\ldots\alpha_k,\beta'_1\ldots\beta'_l}} = \omega_{\beta_1\ldots\beta_l,\alpha'_1\ldots\alpha'_k}.$$

This set is equivalent to the real one-form $\omega_{A_1\ldots A_{2(s-1)}}$, symmetric in the Majorana spinor indices $A = 1,\ldots 4$, that carries an irreducible module of the AdS_4 symmetry algebra $\mathfrak{sp}(4,\mathbb{R}) \sim \mathfrak{o}(3,2)$.

The AdS_4 space is described by the Lorentz connection $w^{\alpha\beta}$, $\overline{w}^{\alpha'\beta'}$ and vierbein $e^{\alpha\alpha'}$. Together, they form an $\mathfrak{sp}(4,\mathbb{R})$ connection $w^{AB} = w^{BA}$ that satisfies the $\mathfrak{sp}(4,\mathbb{R})$ zero-curvature conditions

$$R^{AB} = 0, \quad R^{AB} = dw^{AB} + w^{AC}\wedge w_C{}^B, \tag{2.1}$$

where the indices are raised and lowered by an $\mathfrak{sp}(4,\mathbb{R})$ invariant form $\mathrm{C}_{AB} = -\mathrm{C}_{BA}$:

$$A_B = A^A \mathrm{C}_{AB}, \quad A^A = \mathrm{C}^{AB}A_B, \quad \mathrm{C}_{AC}\mathrm{C}^{BC} = \delta^B_A. \tag{2.2}$$

In terms of Lorentz components

$$w^{AB} = (w^{\alpha\beta}, \overline{w}^{\alpha'\beta'}, \lambda e^{\alpha\beta'}),$$

where λ^{-1} is the AdS_4 radius, the AdS_4 equations (2.1) take the form

$$R_{\alpha\beta} = 0, \quad \overline{R}_{\alpha'\beta'} = 0, \quad R_{\alpha\alpha'} = 0, \tag{2.3}$$

where

$$R_{\alpha\beta} = dw_{\alpha\beta} + w_\alpha{}^\gamma \wedge w_{\beta\gamma} + \lambda^2\, e_\alpha{}^{\delta'} \wedge e_{\beta\delta'}, \tag{2.4}$$

$$\overline{R}_{\alpha'\beta'} = d\overline{w}_{\alpha'\beta'} + \overline{w}_{\alpha'}{}^{\gamma'} \wedge \overline{w}_{\beta'\gamma'} + \lambda^2\, e^\gamma_{\alpha'} \wedge e_{\gamma\beta'},$$

$$R_{\alpha\beta'} = de_{\alpha\beta'} + w_\alpha{}^\gamma \wedge e_{\gamma\beta'} + \overline{w}_{\beta'}{}^{\delta'} \wedge e_{\alpha\delta'}. \tag{2.5}$$

The unfolded equations of motion of a spin-s massless field are [22]

$$D^{ad}\omega(y,\bar y|x) = \overline{H}^{\alpha'\beta'} \frac{\partial^2}{\partial\overline{y}^{\alpha'}\partial\overline{y}^{\beta'}}\overline{C}(0,\bar y|x) + H^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(y,0|x), \tag{2.6}$$

$$D^{tw}C(y,\bar y|x) = 0, \tag{2.7}$$

where

$$H^{\alpha\beta} = e^\alpha_{\alpha'} \wedge e^{\beta\alpha'}, \quad \overline{H}^{\alpha'\beta'} = e_\alpha{}^{\alpha'} \wedge e^{\alpha\beta'}, \tag{2.8}$$

y^α and $\bar y^{\beta'}$ are auxiliary commuting conjugate two-component spinor variables, the 1-form $\omega(y,\bar y|x)$ has the form

$$\omega(y,\bar y|x) = \sum_{m,n\ge 0} \omega_{\alpha_1\ldots\alpha_n,\beta'_1\ldots\beta'_m}(x) y^{\alpha_1}\ldots y^{\alpha_n} \times \bar y^{\beta'_1}\ldots \bar y^{\beta'_m}$$

with $n+m = 2(s-1)$ (for $s\ge 1$). The 0-form $C(y,\bar y|x)$ has the form

$$C(y,\bar y|x) = \sum_{m,n\ge 0} C_{\alpha_1\ldots\alpha_n,\beta'_1\ldots\beta'_m}(x) y^{\alpha_1}\ldots y^{\alpha_n} \times \bar y^{\beta'_1}\ldots \bar y^{\beta'_m}$$

with $n-m = 2s$; $\overline{C}(y,\bar y|x)$ is complex conjugate to $C(y,\bar y|x)$, and

$$D^{ad}\omega(y,\bar y|x) = D^L\omega(y,\bar y|x) - \lambda e^{\alpha\beta'}\left(y_\alpha \frac{\partial}{\partial \bar y^{\beta'}} + \frac{\partial}{\partial y^\alpha}\bar y_{\beta'}\right)\omega(y,\bar y|x), \quad (D^{ad})^2 = 0, \tag{2.9}$$

$$D^{tw}C(y,\bar y|x) = D^L C(y,\bar y|x) + \lambda e^{\alpha\beta'}\left(y_\alpha \bar y_{\beta'} + \frac{\partial^2}{\partial y^\alpha \partial \bar y^{\beta'}}\right) C(y,\bar y|x), \quad (D^{tw})^2 = 0, \tag{2.10}$$

where the Lorentz covariant derivative D^L is

$$D^L A(y,\bar y|x) = dA(y,\bar y|x) - \left(w^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \overline{w}^{\alpha'\beta'}\bar y_{\alpha'}\frac{\partial}{\partial \bar y^{\beta'}}\right) A(y,\bar y|x). \tag{2.11}$$

Here, $x^{\alpha\beta'} = x^n\sigma_n^{\alpha\beta'}$ are Minkowski coordinates where $\sigma_n^{\alpha\beta'}$ are four Hermitian 2×2 matrices.

As explained in [22, 25, 26], the dynamical massless fields are

- $C(x)$ and $\overline{C}(x)$ for two spin-zero fields,
- $C_\alpha(x)$ and $\overline{C}_{\alpha'}(x)$ for a massless spin-1/2 field,
- $\omega_{\alpha_1\ldots\alpha_{s-1},\alpha'_1\ldots\alpha'_{s-1}}(x)$ for an integer spin-$s \ge 1$ massless field,
- $\omega_{\alpha_1\ldots\alpha_{s-3/2},\alpha'_1\ldots\alpha'_{s-1/2}}(x)$ and its complex conjugate $\omega_{\alpha_1\ldots\alpha_{s-1/2},\alpha'_1\ldots\alpha'_{s-3/2}}(x)$ for a half-integer spin-$s \ge 3/2$ massless field.

All other fields are auxiliary, being expressed in terms of derivatives of the dynamical massless fields by Eqs. (2.6) and (2.7).

Equations (2.7) are independent of (2.6) for spins $s = 0$ and $s = 1/2$ and partially independent for $s = 1$, but become consequences of (2.6) for $s > 1$. Equations (2.6) express the holomorphic and antiholomorphic components of the spin-$s \geq 1$ zero-forms $C(y, \bar{y}|x)$ via derivatives of the massless field gauge one-forms described by $\omega(y, \bar{y}|x)$. This identifies the spin-$s \geq 1$ holomorphic and antiholomorphic components of the zero-forms $C(y, \bar{y}|x)$ with the Maxwell tensor, the on-shell Rarita–Schwinger curvature, the Weyl tensor, and their HS generalizations. In addition, Eqs. (2.6) impose the standard field equations on the spin-$s > 1$ massless gauge fields. The dynamical equations for $s \leq 1$ are contained in Eqs. (2.7).

2.2. σ_--cohomology

In the unfolded dynamics approach, dynamical fields, their differential gauge symmetries (i. e., those that are not Stueckelberg (i. e., shift) symmetries), and differential field equations (i. e., those that are not constraints) are characterized by the so-called σ_--cohomology.

We briefly recall the σ_--cohomology analysis following [26]. A space V_0 where zero-forms C and $\overline{C}$ are valued is endowed with the grading G_0

$$G_0 = \frac{1}{2}(n + \bar{n}), \quad n = y^\beta \frac{\partial}{\partial y^\beta}, \quad \bar{n} = \bar{y}^{\beta'} \frac{\partial}{\partial \bar{y}^{\beta'}}. \tag{2.12}$$

This gives

$$D^{tw} = D^L + \lambda\sigma_-^{tw} + \lambda\sigma_+^{tw}, \tag{2.13}$$

where

$$\sigma_-^{tw} = e^{\alpha\alpha'} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\alpha'}}, \quad \sigma_+^{tw} = e^{\alpha\alpha'} y_\alpha \bar{y}_{\alpha'}.$$

We have

$$[G_0, \sigma^{tw}{}_\pm] = \pm\sigma^{tw}{}_\pm, \quad [G_0, \mathcal{D}^L] = 0, \quad (\sigma^{tw}{}_\pm)^2 = 0.$$

A space V_1 where one-forms ω are valued, is endowed with the grading G_1,

$$G_1 = \frac{1}{2}|n - \bar{n}|. \tag{2.14}$$

This gives

$$D^{ad} = D^L - \lambda\sigma_-^{ad} - \lambda\sigma_+^{ad}, \tag{2.15}$$

where

$$\sigma_-^{ad} = \rho_-\,\theta(n - \bar{n} - 2) + \overline{\rho}_-\,\theta(\bar{n} - n - 2), \quad \sigma_+^{ad} = \rho_-\theta(\bar{n} - n) + \overline{\rho}_-\theta(n - \bar{n}), \tag{2.16}$$

$$\rho_- = e^{\alpha\beta'} \frac{\partial}{\partial y^\alpha} \bar{y}_{\beta'}, \quad \overline{\rho}_- = e^{\alpha\beta'} \frac{\partial}{\partial \bar{y}^{\beta'}} y_\alpha, \quad \theta(m) = 1\,(0), \quad m \geq 0\ (m < 0). \tag{2.17}$$

We have

$$[G_1, \sigma^{ad}{}_\pm] = \pm\sigma^{ad}{}_\pm, \quad [G_1, \mathcal{D}^L] = 0.$$

Although ρ_- and $\overline{\rho}_-$ do not anticommute,

$$(\sigma_-^{ad})^2 = 0$$

because

$$(\rho_-)^2 = (\overline{\rho}_-)^2 = 0$$

and the step functions guarantee that the parts of σ_- associated with ρ_- and $\overline{\rho}_-$ act in different spaces.

We set

$$\sigma_- = \sigma_-^{tw} + \sigma_-^{ad}, \tag{2.18}$$

where σ_-^{tw} acts on zero-forms while σ_-^{ad} acts on one-forms. Then the cohomology of σ_- determines the dynamical content of the relevant dynamical system. Namely, from the level-by-level analysis of Eqs. (2.6) and (2.7), it follows that all fields that do not belong to $\operatorname{Ker}\sigma_-$ are auxiliary, being expressed by (2.6) and (2.7) via derivatives of the lower-grade fields. (For more details, see, e. g., [6, 26].) In the case of massless fields, the nontrivial cohomology of σ_- is concentrated in the subspaces with $G_j = 0$ and $\pm 1/2$ [26]. In particular, the nontrivial cohomology of $H^0(\sigma_-)$ appears in the subspaces of grades $G_1 = 0$ or $1/2$, where σ_- acts trivially because of the step functions in (2.16).

Field equations contained in the sector of $(p+1)$-form curvatures are characterized by $H^{p+1}(\sigma_-)$, which describes those parts of the generalized curvatures that contain nontrivial gauge-invariant combinations of derivatives of dynamical fields. Since massless equations for bosons and fermions are respectively of the second and first order, the respective cohomologies have levels two and one. As anticipated, there are as many nontrivial field equations as components of the Fronsdal fields. In particular, in the bosonic case, dynamical equations for a spin-s field are described by traceless symmetric tensors of ranks s and $s-2$ (for $s \geq 2$). For example, in the case of gravity, these include the traceless part of the Ricci tensor and the scalar curvature. In this paper, we only consider conformal HS currents that are generated by generalized HS stress tensors that in the tensor notation are described by traceless tensors. This means that we here study only those current deformations of the massless field equations that contribute to the rank-s traceless part of the HS field equations.

2.3. Flat limit

To take the flat limit, it is necessary to perform certain rescalings. For this, it is useful to introduce the notation [26] $A_\pm$ and A_0 such that the spectrum of the operator

$$\left(y^\alpha \frac{\partial}{\partial y^\alpha} - \overline{y}^{\alpha'} \frac{\partial}{\partial \overline{y}^{\alpha'}}\right)$$

is positive on $A_+(y,\overline{y}|x)$, negative on $A_-(y,\overline{y}|x)$, and zero on $A_0(y,\overline{y}|x)$. With the decomposition

$$A(y,\overline{y}|x) = A_+(y,\overline{y}|x) + A_-(y,\overline{y}|x) + A_0(y,\overline{y}|x), \quad (2.19)$$

the rescaled fields are introduced as follows:

$$\tilde{A}(y,\overline{y}|x) = A_+(\lambda^{\frac{1}{2}}y, \lambda^{-\frac{1}{2}}\overline{y}|x) + A_-(\lambda^{-\frac{1}{2}}y, \lambda^{\frac{1}{2}}\overline{y}|x) + A_0(\lambda^{\frac{1}{2}}y, \lambda^{-\frac{1}{2}}\overline{y}|x), \quad (2.20)$$

$$\tilde{\tilde{A}}(y,\overline{y}|x) = A_+(\lambda^{\frac{1}{2}}y, \lambda^{\frac{1}{2}}\overline{y}|x) + A_-(\lambda^{\frac{1}{2}}y, \lambda^{\frac{1}{2}}\overline{y}|x) + A_0(\lambda^{\frac{1}{2}}y, \lambda^{\frac{1}{2}}|x).$$

We note that

$$A_0(\lambda y,\overline{y}|x) = A_0(y,\lambda\overline{y}|x).$$

For the rescaled variables, the flat limit $\lambda \to 0$ of the adjoint and twisted adjoint covariant derivatives in (2.9) and (2.10) gives

$$D^{ad}_{fl}\tilde{A}(y,\bar{y}|x) = D^L\tilde{A}(y,\bar{y}|x) - e^{\alpha\beta'}\left(y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}}\tilde{A}_-(y,\bar{y}|x) + \frac{\partial}{\partial y^\alpha}\bar{y}_{\beta'}\tilde{A}_+(y,\bar{y}|x)\right), \quad (2.21)$$

$$D^{tw}_{fl}\tilde{\tilde{A}}(y,\bar{y}|x) = D^L\tilde{\tilde{A}}(y,\bar{y}|x) + e^{\alpha\beta'}\frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta'}}\tilde{\tilde{A}}(y,\bar{y}|x). \quad (2.22)$$

The flat limit of the unfolded massless equations follows from (2.6) and (2.7) via the substitution of D^L and $e^{\alpha\alpha'}$ of Minkowski space and the replacement of D^{ad} and D^{tw} with D^{ad}_{fl} and D^{tw}_{fl}. The resulting field equations describe free HS fields in Minkowski space. We stress that the flat limit prescription in (2.20), which may look somewhat unnatural in the two-component spinor notation, is designed just to give rise to the theory of Fronsdal [27] and Fang and Fronsdal [28] (for more details, see [26]).

We note that although the contraction $\lambda \to 0$ with rescaling (2.20) is consistent with the free HS field equations, it turns out to be inconsistent in the nonlinear HS theory because negative powers of λ survive in the full nonlinear equations upon rescaling (2.20), not allowing the flat limit in the nonlinear theory. This is why the Minkowski background is unreachable in the nonlinear HS gauge theories in [4, 5, 29].

2.4. Unfolded equations in matrix spaces $\mathcal{M}_M$

As observed in [10], massless equations (2.7) can be promoted to a larger space $\mathcal{M}_4$ with matrix coordinates $X^{AB} = X^{BA}$ by extending system (2.7) to

$$dX^{AB}\left(\frac{\partial}{\partial X^{AB}} \pm \frac{\partial^2}{\partial Y^A \partial Y^B}\right) C_\pm(Y|X) = 0, \quad (2.23)$$

where the "±" sign is introduced for the future convenience. This extension makes the Sp(8) symmetry of the tower of massless fields of all spins, observed originally by Fronsdal [8], geometrically realized on a Lagrangian Grassmannian, which was shown in [8] to be a minimal Sp(8) invariant space that contains Minkowski space as a subspace. (We note that it was also observed in [9] that the tower of $4d$ massless fields of all spins is naturally realized in $\mathcal{M}_4$.)

That Sp(8) is a symmetry of both systems (2.7) and (2.23) follows from the general property of unfolded equations that any subalgebra in End V, where V is the module where zero-forms C are valued, forms a symmetry of the free system (for more details, see, e. g., [26] and the references therein). The Lie algebra $\mathfrak{sp}(8)$ is the algebra of various bilinears of Y^A and $\partial/\partial Y^A$ that act on the space V of functions $C(Y)$. The conformal algebra $\mathfrak{su}(2,2)$ is the subalgebra of $\mathfrak{sp}(8)$ spanned by those bilinears that commute to the helicity operator

$$H = y^\alpha \frac{\partial}{\partial y^\alpha} - \bar{y}^{\gamma'} \frac{\partial}{\partial \bar{y}^{\gamma'}} \in \mathfrak{sp}(8), \quad (2.24)$$

which associates helicities of fields to its eigenvalues. More precisely, the centralizer of H in $\mathfrak{sp}(8)$ is

$$\mathfrak{su}(2,2) \oplus \mathfrak{u}(1),$$

where $\mathfrak{u}(1)$ is generated by H while $\mathfrak{su}(2,2)$ is the conformal algebra. Thus, in the zero-form sector, massless equations of fields of different spins are conformal.

System (2.23) extends the $4d$ massless equations in Minkowski background formulated in Cartesian coordinates to $\mathcal{M}_4$. Its extension to an AdS-like version of $\mathcal{M}_4$, which is the group manifold Sp(4) [10], is also available [30] in any coordinate system. We note that more recently, the one-form sector of HS equations (2.6) was also extended to $\mathcal{M}_4$ in [26]. By general properties of unfolded equations, Eqs. (2.23) are equivalent to the flat limit of $4d$ HS equations (2.7). Interesting details of this correspondence were worked out in [11, 16].

In Ref. [7], Eq. (2.23) was extended to so-called rank-r systems of the form

$$dX^{AB}\left(\frac{\partial}{\partial X^{AB}} \pm \eta^{ij}\frac{\partial^2}{\partial Y^{iA}\partial Y^{jB}}\right)\times \times C^r_{\pm}(Y|X) = 0, \quad (2.25)$$

where $i,j = 1,\ldots,r$ and $\eta^{ij} = \eta^{ji}$ is some nondegenerate metric. The following comments on the properties of higher-rank systems are most relevant to the analysis in this paper.

Higher-rank systems inherit all symmetries of the lower-rank system from which they are built simply because they correspond to the tensor product of the lower-rank representation of one symmetry or another. In particular, this means that higher-rank systems are conformal once the underlying lower-rank systems are.

In the basis where η^{ij} is diagonal, higher-rank equations (2.25) are satisfied by the products of rank-one fields

$$C^r(Y_i|X) = C_1(Y_1|X)C_2(Y_2|X)\ldots C_r(Y_r|X). \quad (2.26)$$

The rank-r systems in $\mathcal{M}_M$ can further be extended to a rank-one system (2.23) in the larger space $\mathcal{M}_{rM}$ with coordinates X^{AB}_{ij} by reinterpreting the twistor coordinates:

$$Y^A_i \to Y^{\widetilde{A}}, \quad \widetilde{A} = 1,\ldots,rM. \quad (2.27)$$

The diagonal embedding of $\mathcal{M}_M$ into $\mathcal{M}_{rM}$ is

$$X^{AB}_{11} = X^{AB}_{22} = \ldots = X^{AB}_{rr} = X^{AB}. \quad (2.28)$$

On the other hand, as shown in [9, 11, 16], the rank-one fields in $\mathcal{M}_M$ with higher M describe conformal fields in diverse space–time dimensions. In particular, a rank-one field in $\mathcal{M}_8$ describes all conformal fields in the six-dimensional Minkowski space. This implies that conformal currents in four space–time dimensions, which were shown in [13] to be described by rank-two fields in $\mathcal{M}_4$, are equivalent to conformal fields in six space–time dimensions. More precisely, we should say that the $4d$ currents are dual to the $6d$ conformal fields. The reason is that the space of states of higher-dimensional fields is represented by the product of C_- fields in (2.23) while the currents are represented by the product of C_+ and C_-, where C_+ and C_- respectively describe particles and anti-particles, i. e., the space of single-particle states and its dual[2)]. In this paper, we loosely identify the currents with the fields.

Now we are in a position to explain how rank-two equations give rise to conserved currents, considering the reduction of $\mathcal{M}_4$ to the usual Minkowski space for simplicity.

3. CONSERVED CURRENTS

3.1. Minkowski case

The reduction of the rank-two field equations in [13] to Minkowski space gives

$$D^{tw}_{fl\,2}J(y^{\pm},\bar{y}^{\pm}|x) = = \left(D^L + e^{\alpha\beta'}\left(\frac{\partial^2}{\partial y^{+\alpha}\partial\bar{y}^{-\beta'}} + + \frac{\partial^2}{\partial y^{-\alpha}\partial\bar{y}^{+\beta'}}\right)\right)J(y^{\pm},\bar{y}^{\pm}|x) = 0. \quad (3.1)$$

We say that $J(y^{\pm},\bar{y}^{\pm}|x)$ that satisfies Eq. (3.1) is a rank-two current field. Introducing basis three-forms

$$\mathcal{H}^{\alpha\delta'} = -\frac{1}{3}e^{\alpha}{}_{\alpha'}\wedge e^{\beta\alpha'}\wedge e_{\beta}{}^{\delta'} \quad (3.2)$$

and using the relations

$$e^{\gamma\rho'}\wedge\mathcal{H}^{\alpha\delta'} = \frac{1}{4}\epsilon^{\gamma\alpha}\epsilon^{\rho'\delta'}e_{\eta\sigma'}\wedge\mathcal{H}^{\eta\sigma'}, \quad (3.3)$$

it is easy to verify that the three-forms

$$\Omega_-(J) = \mathcal{H}^{\alpha\alpha'}\frac{\partial}{\partial y^{-\alpha}}\times \times\frac{\partial}{\partial\bar{y}^{-\alpha'}}J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0}, \quad (3.4)$$

$$\Omega_+(J) = \mathcal{H}^{\alpha\alpha'}\frac{\partial}{\partial y^{+\alpha}}\times \times\frac{\partial}{\partial\bar{y}^{+\alpha'}}J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0}, \quad (3.5)$$

2) Strictly speaking, this interpretation requires an additional factor of i in the second term in (2.23), omitted in this paper. For more details on these issues, we refer the reader to [13].

$$\Omega_\pm(J) = \mathcal{H}^{\alpha\alpha'}\left(\frac{\partial}{\partial y^{-\alpha}}\frac{\partial}{\partial \bar{y}^{+\alpha'}} - \frac{\partial}{\partial y^{+\alpha}}\frac{\partial}{\partial \bar{y}^{-\alpha'}}\right) J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} \quad (3.6)$$

are closed if $J(y^\pm,\bar{y}^\pm|x)$ satisfies (3.1).

To define symmetry parameters that produce more conserved currents, we consider the adjoint covariant derivative

$$D_{fl\,2} = D^L + e^{\alpha\beta'}\left(u_{-\alpha}\frac{\partial}{\partial \bar{y}^{+\beta'}} + \bar{u}_{-\beta'}\frac{\partial}{\partial y^{+\alpha}}\right), \quad (3.7)$$

resulting from $D^{tw}_{fl\,2}$ by the substitution

$$y^{-\alpha} \to -\frac{\partial}{\partial u_{-\alpha}}, \quad \bar{y}^{-\alpha'} \to -\frac{\partial}{\partial \bar{u}_{-\alpha'}}, \qquad \frac{\partial}{\partial y^{-\alpha}} \to u_{-\alpha}, \quad \frac{\partial}{\partial \bar{y}^{-\alpha'}} \to \bar{u}_{-\alpha'}, \quad (3.8)$$

which formally coincides with the "half Fourier transform" in [13]. Since covariant derivative (3.7) is of the first order, the space of regular solutions of the equation

$$D^{tw}_{fl\,2}\eta(y^+,\bar{y}^+,u_-,\bar{u}_-|x) = 0 \quad (3.9)$$

forms a commutative algebra $\mathcal{P}_{fl}$. Evidently, $\mathcal{P}_{fl}$ is generated by the elementary solutions

$$u_{-\beta}, \quad y^{+\alpha} - x^{\alpha\beta'}\bar{u}_{-\beta'}, \qquad \bar{u}_{-\beta'}, \quad \bar{y}^{+\alpha'} - x^{\beta\alpha'}u_{-\beta}. \quad (3.10)$$

By the substitution inverse to (3.8),

$$u_{-\alpha} \to \frac{\partial}{\partial y^{-\alpha}}, \quad \bar{u}_{-\alpha'} \to \frac{\partial}{\partial \bar{y}^{-\alpha'}}, \qquad \frac{\partial}{\partial u_{-\alpha}} \to -y^{-\alpha}, \quad \frac{\partial}{\partial \bar{u}_{-\alpha'}} \to -\bar{y}^{-\alpha'} \quad (3.11)$$

the algebra $\mathcal{P}_{fl}$ is mapped to the algebra $\mathcal{R}_{fl}$ of differential operators $\eta(\xi_{-\beta},\bar{\xi}_{-\beta'},\xi^{+\alpha},\bar{\xi}^{+\alpha'})$ generated by

$$\xi_{-\alpha} = \frac{\partial}{\partial y^{-\alpha}}, \quad \bar{\xi}_{-\beta'} = \frac{\partial}{\partial \bar{y}^{-\beta'}}, \qquad \xi^{+\alpha} = y^{+\alpha} - x^{\alpha\beta'}\frac{\partial}{\partial \bar{y}^{-\beta'}}, \qquad \bar{\xi}^{+\alpha'} = \bar{y}^{+\alpha'} - x^{\beta\alpha'}\frac{\partial}{\partial y^{-\beta}}. \quad (3.12)$$

Since any

$$\eta(\xi_{-\beta},\bar{\xi}_{-\beta'},\xi^{+\alpha},\bar{\xi}^{+\alpha'}) \in \mathcal{R}_{fl}$$

satisfies (3.9), it follows that

$$D^{tw}_{fl\,2} J(y^\pm,\bar{y}^\pm|x) = 0 \Longrightarrow D^{tw}_{fl\,2}\left(\eta(\xi,\bar{\xi})\; J(y^\pm,\bar{y}^\pm|x)\right) = 0. \quad (3.13)$$

Hence, the three-form

$$\Omega(\eta J) = \mathcal{H}^{\alpha\alpha'}\frac{\partial}{\partial y^{-\alpha}}\frac{\partial}{\partial \bar{y}^{-\alpha'}} \times \eta(\xi,\bar{\xi})J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} \quad (3.14)$$

is closed. Thus, any element of $\mathcal{R}_{fl}$ generates a conservation law. As explained in more detail in [13], $\mathcal{R}_{fl}$ matches the space of HS global symmetry parameters in [14].

The relation with the usual currents is based on the fact that Eq. (3.1) is solved by the bilinear expression [7]

$$J(y^\pm\,\bar{y}^\pm|x) = C_+(y^+ + y^-,\bar{y}^+ + \bar{y}^-|x) \times C_-(y^+ - y^-,\bar{y}^+ - \bar{y}^-|x) \quad (3.15)$$

in rank-one fields $C_\pm(y\,\bar{y}|x)$ that solve the rank-one equations

$$D^L C_\pm(y\,\bar{y}|x) \pm e^{\alpha\beta'}\frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta'}}C_\pm(y\,\bar{y}|x) = 0, \quad (3.16)$$

which coincide with the Minkowski reduction of Eq. (2.23) and, up to a sign, with the flat limit of Eq. (2.7). The resulting currents reproduce the lower-spin and HS conserved currents built from massless fields, originally obtained in [15].

The change of minuses to pluses in the "half Fourier transform" (3.8) gives another set of operators

$$\chi_{+\alpha} = \frac{\partial}{\partial y^{+\alpha}}, \quad \bar{\chi}_{+\beta'} = \frac{\partial}{\partial \bar{y}^{+\beta'}}, \qquad \chi^{-\alpha} = y^{-\alpha} - x^{\alpha\beta'}\frac{\partial}{\partial \bar{y}^{+\beta'}}, \qquad \bar{\chi}^{-\alpha'} = \bar{y}^{-\alpha'} - x^{\beta\alpha'}\frac{\partial}{\partial y^{+\beta}} \quad (3.17)$$

that commute to $D^{tw}_{fl\,2}$ in (3.1) and hence also generate symmetry parameters and conserved currents. Generally, the following set of closed three-forms can be written with an arbitrary parameter $g(\xi,\bar{\xi},\chi,\bar{\chi})$:

$$\Omega_-(gJ) = \mathcal{H}^{\alpha\alpha'}\frac{\partial}{\partial y^{-\alpha}}\frac{\partial}{\partial \bar{y}^{-\alpha'}} \times g(\xi,\bar{\xi},\chi,\bar{\chi})J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0},$$

$$\Omega_+(gJ) = \mathcal{H}^{\alpha\alpha'}\frac{\partial}{\partial y^{+\alpha}}\frac{\partial}{\partial \bar{y}^{+\alpha'}} \times g(\xi,\bar{\xi},\chi,\bar{\chi})J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0},$$

$$\Omega_{\pm}(gJ) = \mathcal{H}^{\alpha\alpha'}\left(\frac{\partial}{\partial y^{-\alpha}}\frac{\partial}{\partial \bar{y}^{+\alpha'}} - \frac{\partial}{\partial y^{+\alpha}}\frac{\partial}{\partial \bar{y}^{-\alpha'}}\right) \times \times g(\xi,\bar{\xi},\chi,\bar{\chi})J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0}.$$

However, most of these forms turn out to be exact, giving rise to zero charges. As shown in the forthcoming publication [31], in both the Minkowski and AdS_4 cases, nontrivial charges (i.e., current cohomology) are fully represented by the closed three-forms

$$\begin{gathered}\mathcal{H}^{\alpha\alpha'}\frac{\partial}{\partial y^{-\alpha}}\frac{\partial}{\partial \bar{y}^{-\alpha'}}\eta(\xi,\bar{\xi},H_1-H_2) \times \\ \times J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0}, \\ \mathcal{H}^{\alpha\alpha'}\frac{\partial}{\partial y^{+\alpha}}\frac{\partial}{\partial \bar{y}^{+\alpha'}}\eta(\chi,\bar{\chi},H_1-H_2) \times \\ \times J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0},\end{gathered} \tag{3.18}$$

where

$$H_j = y^{j\alpha}\frac{\partial}{\partial y^{j\alpha}} - \bar{y}^{j\alpha'}\frac{\partial}{\partial \bar{y}^{j\alpha'}}.$$

We note that

$$(H_1-H_2)J = 4(h_+ - h_-)J$$

for the bilinear currents J in (3.15) with the fields $C_{\pm}$ of helicities $h_{\pm}$.

3.2. AdS_4

In the case of AdS_4, the rank-two unfolded equations, i.e., "current equations", are

$$D_2^{tw} J(y^{\pm},\bar{y}^{\pm}|x) = 0, \tag{3.19}$$

where

$$D_2^{tw} = D^L + \lambda e^{\alpha\beta'}\Big(y^+{}_\alpha \bar{y}^-{}_{\beta'} + y^-{}_\alpha \bar{y}^+{}_{\beta'} + + \frac{\partial^2}{\partial y^{+\alpha}\partial \bar{y}^{-\beta'}} + \frac{\partial^2}{\partial y^{-\alpha}\partial \bar{y}^{+\beta'}}\Big). \tag{3.20}$$

Again, current equations (3.19) imply that, being evaluated at

$$y^{\pm} = \bar{y}^{\pm} = 0,$$

three-forms (3.4)–(3.6) are closed.

3.2.1. The Howe-dual algebra

To classify different solutions of rank-two equation (3.19), we observe that the operators

$$\begin{gathered} f_+ = y^{+\nu}y^-_\nu - \frac{\partial^2}{\partial \bar{y}^{+\nu'}\bar{y}^-_{\nu'}}, \\ f_- = -\frac{\partial^2}{\partial y^{+\gamma}\partial y^-_\gamma} + \bar{y}^{+\gamma'}\bar{y}^-_{\gamma'}, \\ f_0 = y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha}\frac{\partial}{\partial y^{-\alpha}} - \bar{y}^{+\alpha'}\frac{\partial}{\partial \bar{y}^{+\alpha'}} - \\ -\bar{y}^{-\alpha'}\frac{\partial}{\partial \bar{y}^{-\alpha'}}, \end{gathered} \tag{3.21}$$

and

$$\begin{gathered} g_+ = y^{+\alpha}\frac{\partial}{\partial y^{-\alpha}} - \bar{y}^{+\alpha'}\frac{\partial}{\partial \bar{y}^{-\alpha'}}, \\ g_- = y^{-\alpha}\frac{\partial}{\partial y^{+\alpha}} - \bar{y}^{-\alpha'}\frac{\partial}{\partial \bar{y}^{+\alpha'}}, \\ g_0 = y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} + \bar{y}^{+\alpha'}\frac{\partial}{\partial \bar{y}^{+\alpha'}} - y^{-\alpha}\frac{\partial}{\partial y^{-\alpha}} - \\ -\bar{y}^{-\alpha'}\frac{\partial}{\partial \bar{y}^{-\alpha'}} \end{gathered} \tag{3.22}$$

commute to D_2^{tw}. These operators form two mutually commuting $\mathfrak{sl}_2$ algebras with the nonzero commutation relations

$$[f_+, f_-] = f_0, \quad [f_0, f_{\pm}] = \pm 2 f_{\pm};$$

$$[g_+, g_-] = g_0, \quad [g_0, g_{\pm}] = \pm 2 g_{\pm}.$$

The algebras of operators (3.21) and (3.22) are respectively referred to as vertical ${}^v\mathfrak{sl}_2$ and horizontal ${}^h\mathfrak{sl}_2$. The Cartan operator $f_0 \in {}^v\mathfrak{sl}_2$ in (3.21) is referred to as the rank-two helicity operator.

It is easy to see that

$$\begin{gathered} \mathcal{H}^{\alpha\alpha'}\frac{\partial^2}{\partial y^{-\alpha}\bar{y}^{-\alpha'}}\, f_-\, J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0} = \\ = \frac{1}{2\lambda}d\Big(H^{\alpha\beta}\frac{\partial^2}{\partial y^{-\alpha}\partial y^{-\beta}} \times \\ \times J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0}\Big), \\ \mathcal{H}^{\alpha\alpha'}\frac{\partial^2}{\partial y^{-\alpha}\partial\bar{y}^{-\alpha'}}f_+ \times \\ \times J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0} = \\ = -\frac{1}{2\lambda}d\Big(\overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial \bar{y}^{-\alpha'}\partial\bar{y}^{-\beta'}} \times \\ \times J(y^{\pm},\bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0}\Big), \end{gathered} \tag{3.23}$$

if J satisfies Eq. (3.19). We recall that the two-forms $H^{\alpha\beta}$ and $\overline{H}^{\alpha'\beta'}$ are defined in (2.8), while the three-form $\mathcal{H}^{\alpha\alpha'}$ is defined in (3.2).

The system of equations (3.19) decomposes into a set of subsystems associated with different elements of ${}^h\mathfrak{sl}_2 \oplus {}^v\,\mathfrak{sl}_2$-modules realized by rank-two fields. Let

$$Y = y^{+\alpha}y^{-}{}_{\alpha}, \quad \overline{Y} = \bar{y}^{+\alpha'}\bar{y}^{-}{}_{\alpha'}. \tag{3.24}$$

Any polynomial $P(y^{\pm})$ can be represented in the form

$$P(y^{\pm}) = \sum_{n,m,k=0}^{\infty} Y^n\, C^{n,m,k}_{\alpha(m+k)} y^{+\alpha(m)}y^{-\alpha(k)},$$

where multispinors $C^{n,m,k}_{\alpha(m+k)}$ are symmetric. It is easy to see that

$$\frac{\partial^2}{\partial y^-_\gamma \partial y^{+\gamma}}\Big(Y^n\, C^{n,m,k}_{\alpha(m+k)} y^{+\alpha(m)}y^{-\alpha(k)}\Big) = \\ = n(n+1+m+k)\,Y^{n-1}\,C^{n,m,k}_{\alpha(m+k)}y^{+\alpha(m)}y^{-\alpha(k)}.$$

It follows from this relation that lowest vectors F_m of the vertical ${}^v\mathfrak{sl}_2$ in (3.21), which satisfy the equation $f_-F_m = 0$, have the form

$$F_m(y,\bar{y},Y,\overline{Y}) = \\ = f^m(y,\bar{y},\overline{Y})\sum_{n=0}^{\infty} Y^n\overline{Y}^n \frac{1}{n!\,(1+m+n)!}, \tag{3.25}$$

where $f^m(y,\bar{y},\,\overline{Y})$ is an arbitrary function that satisfies the conditions

$$\frac{\partial^2}{\partial y^{+\gamma}\partial y^{-}{}_{\gamma}} f^m(y,\bar{y},\,\overline{Y}) = 0, \\ \Big(y^{+\gamma}\frac{\partial}{\partial y^{+\gamma}} + y^{-\gamma}\frac{\partial}{\partial y^{-}{}_{\gamma}}\Big) f^m(y,\bar{y},\,\overline{Y}) = \\ = m\, f^m(y,\bar{y},\,\overline{Y}). \tag{3.26}$$

We note that $F_m(y,\bar{y},Y,\overline{Y})$ (3.25) satisfies the eqution

$$\bigg(\Big(Y\frac{\partial}{\partial Y} + y^{j\alpha}\frac{\partial}{\partial y^{j\alpha}} + 1\Big)\frac{\partial}{\partial Y} - \overline{Y}\bigg)\, F_m(y,\bar{y},Y,\overline{Y}) = 0,$$

where the derivatives with respect to Y and y are treated as independent.

Since $\overline{f}_+ = f_-$, highest vectors are complex conjugate to the lowest ones. Therefore, the singlets $F_{m,m}$ of the vertical ${}^v\mathfrak{sl}_2$ in (3.21) have the form

$$F_{m,m}(y,\bar{y},Y,\overline{Y}) = \\ = s^m(y,\bar{y})\sum_{n\geq 0} Y^n\overline{Y}^n \frac{1}{(1+m+n)!n!}, \tag{3.27}$$

where polynomials $s^m(y,\bar{y})$ satisfy Eq. (3.26) along with the conjugate conditions

$$\frac{\partial^2}{\partial \bar{y}^{+\gamma'}\partial \bar{y}^-_{\gamma'}} s^m(y,\bar{y})(\bar{y}) = 0,$$

$$\Big(\bar{y}^{+\alpha'}\frac{\partial}{\partial \bar{y}^{+\alpha'}} + \bar{y}^{+\alpha'}\frac{\partial}{\partial \bar{y}^{+\alpha'}}\Big) s^m(y,\bar{y}) = m s^m(y,\bar{y}).$$

It is easy to see that the lowest vectors F_- and highest vectors F_+ of the horizontal ${}^h\mathfrak{sl}_2$ in (3.22) have the form

$$F_-\big(y^-,\bar{y}^-,\ (y^+{}_\alpha\,\bar{y}^-{}_{\beta'} + y^-{}_\alpha\,\bar{y}^+{}_{\beta'})\big),$$

$$F_+\big(y^+,\bar{y}^+,\ (y^+{}_\alpha\,\bar{y}^-{}_{\beta'} + y^-{}_\alpha\,\bar{y}^+{}_{\beta'})\big),$$

while the ${}^h\mathfrak{sl}_2$ singlets are

$$G\big(y^+{}_\alpha\,\bar{y}^-{}_{\beta'} + y^-{}_\alpha\,\bar{y}^+{}_{\beta'}\big),$$

where $F_\pm$ and G are arbitrary functions of their arguments.

We note that f_0 and the algebra ${}^h\mathfrak{sl}_2$ in (3.22) commute to $D^{tw}_{fl}{}_2$, while the flat limit of the operators $f_\pm$ gives the mutually commuting operators

$$f_{+fl} = -\frac{\partial^2}{\partial \bar{y}^{+\nu'}\bar{y}^-_{\nu'}}, \quad f_{-fl} = -\frac{\partial^2}{\partial y^{+\gamma}\partial y^-{}_\gamma}, \tag{3.28}$$

which commute to $D^{tw}_{fl}{}_2$.

3.2.2. Symmetry parameters of AdS_4 currents

Proceeding as in the Minkowski case in finding symmetry parameters of AdS_4 currents, we have to solve the equation

$$D^{ad}_2\,\eta(y^+,\bar{y}^+,u_-,\bar{u}_-|x) = 0, \tag{3.29}$$

$$D^{ad}_2 = D^L + \lambda e^{\alpha\beta'}\Big(-y_{+\alpha}\frac{\partial}{\partial \bar{u}^{-\beta'}} - \bar{y}_{+\beta'}\frac{\partial}{\partial u^{-\alpha}} + \\ + u_{-\alpha}\frac{\partial}{\partial \bar{y}^{+\beta'}} + \bar{u}_{-\beta'}\frac{\partial}{\partial y^{+\alpha}}\Big),$$

where D^{ad}_2 is again related to D^{tw}_2 via (3.8).

As in the Minkowski case, the space of solutions of the first-order system of partial differential equations (3.29) forms a commutative algebra that has two gradings

$$G^+ = \frac{1}{2}\left(y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} + \bar{u}_{-\alpha'}\frac{\partial}{\partial \bar{u}_{-\alpha'}}\right), \\ G_- = \frac{1}{2}\left(u_{-\alpha'}\frac{\partial}{\partial u_{-\alpha}} + \bar{y}^{+\alpha'}\frac{\partial}{\partial \bar{y}^{+\alpha'}}\right). \tag{3.30}$$

Since the compatibility of Eq. (3.29) is guaranteed by flatness condition (2.3), the space of solutions of (3.29) is isomorphic to the space of arbitrary functions of $y^+, \bar{y}^+, u_-, \bar{u}_-$, i.e., $\xi(y^+, \bar{y}^+, u_-, \bar{u}_-|x)$ is reconstructed via its values at any given point $x = x_0$. Since Eq. (3.29) is homogeneous in the variables $y^+, \bar{y}^+, u_-, \bar{u}_-$, its solutions can also be chosen to be homogeneous. Moreover, it suffices to find a complete set of solutions that have minimal grades with respect to both gradings in (3.30) and are therefore linear either in y^+ and $\bar{u}_-$ or in u_- and $\bar{y}^+$.

For this, we introduce Killing spinors $\mathrm{c}^\beta(x)$ and $\mathrm{s}^{\beta'}(x)$ that satisfy the equations

$$\begin{aligned} &D^L \mathrm{c}^\alpha(x) + \lambda e^{\alpha\beta'} \mathrm{s}_{\beta'}(x) = 0, \\ &D^L \mathrm{s}^{\beta'}(x) + \lambda e^{\alpha\beta'} \mathrm{c}_\alpha(x) = 0. \end{aligned} \tag{3.31}$$

Let a basis of this system be formed by four independent pairs of spinors $(\mathrm{c_a}^\beta(x), \mathrm{s_a}^{\beta'}(x))$ and $(\mathrm{c_{a'}}^\beta(x), \mathrm{s_{a'}}^{\beta'}(x))$ labeled by indices $\mathrm{a} = 1,2$ and $\mathrm{a}' = 1,2$. For example, basic solutions of (3.31) can be chosen to obey the following initial conditions at $x = 0$:

$$\mathrm{c_a}^\beta(0) = \delta_\mathrm{a}{}^\beta, \quad \mathrm{s_a}^{\beta'}(0) = 0,$$
$$\mathrm{c_{a'}}^\beta(0) = 0, \quad \mathrm{s_{a'}}^{\beta'}(0) = \delta_{\mathrm{a}'}{}^{\beta'}.$$

From these conditions, it follows that

$$\overline{\mathrm{c_a}^\beta(x)} = \mathrm{s_{a'}}^{\beta'}(x), \quad \overline{\mathrm{s_a}^{\beta'}(x)} = \mathrm{c_{a'}}^\beta(x).$$

A particular form of solutions $\mathrm{c_a}^\beta(x)$, $\mathrm{s_a}^{\beta'}(x)$, $\mathrm{c_{a'}}^\beta(x)$, $\mathrm{s_{a'}}^{\beta'}(x)$ depends on a chosen coordinate system.

Evidently, the fundamental solutions

$$\begin{aligned} \varrho_\mathrm{a}(u_-, \bar{y}^+|x) &= \mathrm{c_a}^\nu(x) u_{-\nu} + \mathrm{s}_{\mathrm{a}\nu'}(x) \bar{y}^{+\nu'}, \\ \epsilon_\mathrm{a}(y^+, \bar{u}_-|x) &= \mathrm{c}_{\mathrm{a}\beta}(x) y^{+\beta} + \mathrm{s_a}^{\beta'}(x) \bar{u}_{-\beta'}, \\ \overline{\varrho}_{\mathrm{a}'}(\bar{u}_-, y^+|x) &= \mathrm{s_{a'}}^{\beta'}(x) \bar{u}_{-\beta'} + \mathrm{c}_{\mathrm{a}'\nu}(x) y^{+\nu}, \\ \overline{\epsilon}_{\mathrm{a}'}(\bar{y}^+, u_-|x) &= \mathrm{s}_{\mathrm{a}'\beta'}(x) \bar{y}^{+\beta'} + \mathrm{c_{a'}}^\beta(x) u_{-\beta} \end{aligned} \tag{3.32}$$

generate a commutative algebra $\mathcal{P}_{AdS}$ of solutions of (3.29) of the form

$$\eta'(y^+, \bar{y}^+, u_-, \bar{u}_-|x) = P(\varrho_\mathrm{a}, \epsilon_\mathrm{a}, \overline{\varrho}_{\mathrm{a}'}, \overline{\epsilon}_{\mathrm{a}'}). \tag{3.33}$$

As in the Minkowski case, substitution (3.11) maps $\mathcal{P}_{AdS}$ to the commutative algebra $\mathcal{R}_{AdS}$ of differential operators generated by[3)]

$$\varrho_\mathrm{a}(\partial_-, \bar{y}^+|x),\ \epsilon_\mathrm{a}(y^+, \bar{\partial}_-|x),$$
$$\overline{\varrho}_{\mathrm{a}'}(\bar{\partial}_-, y^+|x),\ \overline{\epsilon}_{\mathrm{a}'}(\bar{y}^+, \partial_-|x). \tag{3.34}$$

[3)] $\bar{\partial}_\pm$ and $\partial_\pm$ are a shorthand notation for $\partial/\partial\bar{y}^\pm$ and $\partial/\partial y^\pm$.

Again, it follows that

$$D_2^{tw}\big(\eta\, J(y^\pm, \bar{y}^\pm|x)\big) = 0$$

if $\eta \in \mathcal{R}_{AdS}$ and $J(y^\pm, \bar{y}^\pm|x)$ satisfies (3.19).

The commutative algebra $\mathcal{R}_{AdS}$ of the current parameters is a representation of the vertical ${}^v\mathfrak{sl}_2$ in (3.21). In particular,

$$[\varrho_\mathrm{a}(\partial_-, \bar{y}^+|x), f_+] = \epsilon_\mathrm{a}(y^+, \bar{\partial}_-|x),$$
$$[\epsilon_\mathrm{a}(y^+, \bar{\partial}_-|x), f_+] = 0,$$
$$[\overline{\epsilon}_{\mathrm{a}'}(\partial_-, \bar{y}^+|x), f_+] = \overline{\varrho}_{\mathrm{a}'}(\bar{\partial}_-, y^+|x),$$
$$[\overline{\varrho}_{\mathrm{a}'}(\bar{\partial}_-, y^+|x), f_+] = 0, \text{etc.}$$

On the other hand, parameters (3.34) are highest vectors of the horizontal ${}^h\mathfrak{sl}_2$ in (3.22):

$$\begin{aligned} &[\varrho_\mathrm{a}(\partial_-, \bar{y}^+|x), g_+] = [\epsilon_\mathrm{a}(y^+, \bar{\partial}_-|x), g_+] = \\ &\quad = [\overline{\epsilon}_{\mathrm{a}'}(\partial_-, \bar{y}^+|x), g_+] = [\overline{\varrho}_{\mathrm{a}'}(\bar{\partial}_-, y^+|x), g_+] = 0, \end{aligned}$$

while $g_- \in {}^h\mathfrak{sl}_2$ maps them to new parameters,

$$\begin{aligned} &[\varrho_\mathrm{a}(\partial_-, \bar{y}^+|x), g_-] = \varrho_\mathrm{a}(\partial_+, \bar{y}^-|x), \\ &[\epsilon_\mathrm{a}(y^+, \bar{\partial}_-|x), g_-] = -\epsilon_\mathrm{a}(y^-, \bar{\partial}_+|x), \text{ etc.} \end{aligned} \tag{3.35}$$

which follow from the original ones via exchange of pluses and minuses.

Since ${}^h\mathfrak{sl}_2$ commutes to D_2^{tw}, the new oscillators also commute to D_2^{tw}. The full list of covariantly constant spinors can be packed into the form

$$\varrho_\mathrm{a}^{n\hat{n}}, \quad \overline{\varrho}_{\mathrm{a}'}^{n\hat{n}}, \tag{3.36}$$

where $n = +,-$ and $\hat{n} = +,-$ are the respective indices of the doublet representations of ${}^v\mathfrak{sl}_2$ and ${}^h\mathfrak{sl}_2$. Namely,

$$\varrho_\mathrm{a}(\partial_-, \bar{y}^+|x) = -\varrho_\mathrm{a}^{+-}, \quad \epsilon_\mathrm{a}(y^+, \bar{\partial}_-|x) = \varrho_\mathrm{a}^{++},$$
$$\overline{\varrho}_{\mathrm{a}'}(\bar{\partial}_-, y^+|x) = \overline{\varrho}_\mathrm{a}^{++}, \quad \overline{\epsilon}_{\mathrm{a}'}(\bar{y}^+, \partial_-|x) = -\overline{\varrho}_{\mathrm{a}'}^{+-},$$
$$\overline{\varrho}_{\mathrm{a}'}(\bar{\partial}_+, y^-|x) = \overline{\varrho}_\mathrm{a}^{-+}, \quad \overline{\epsilon}_{\mathrm{a}'}(\bar{y}^-, \partial_+|x) = \overline{\varrho}_{\mathrm{a}'}^{--},$$
$$\varrho_\mathrm{a}(\partial_+, \bar{y}^-|x) = \varrho_\mathrm{a}^{--}, \quad \epsilon_\mathrm{a}(y^-, \bar{\partial}_+|x) = \varrho_\mathrm{a}^{-+}.$$

Since all oscillators (3.36) are covariantly constant, they have x-independent commutation relations

$$\begin{aligned} &[\varrho_\beta^{n\hat{k}},\, \varrho_\alpha^{m\hat{n}}] = \varepsilon^{nm} \varepsilon^{\hat{k}\hat{n}} \varepsilon_{\beta\alpha}, \\ &[\varrho_{\beta'}^{n\hat{k}},\, \varrho_{\alpha'}^{m\hat{n}}] = \varepsilon^{nm} \varepsilon^{\hat{k}\hat{n}} \varepsilon_{\beta'\alpha'}, \quad \varepsilon^{-+} = 1. \end{aligned} \tag{3.37}$$

In fact, as is explained in more detail in [31], covariantly constant spinors (3.36) are related to supergenerators of (conformal) SUSY.

The full set of parameters belongs to the space P of arbitrary functions of oscillators (3.36). This space is much bigger than the space of HS global symmetry parameters. As is shown in [31], most of the currents associated with elements of P are exact and hence generate no nontrivial charges, while the nontrivial currents are represented by the current cohomology in (3.18). (We note that the ambiguity in the dependence on $H_1 - H_2$ in (3.18) with ξ and χ replaced by ϱ and ε (3.36), respectively, is physically trivial, expressing the ambiguity in the normalization of the rank-one fields in formula (3.15).)

To introduce currents bilinear in rank-one fields, it is convenient to consider the operators $D^{tw}_{\pm}$ that differ from D^{tw} (2.10) by a sign in front of λ, such that the corresponding rank-one equations are

$$D^{tw}_{\pm} C_{\pm}(y,\bar{y}|x) = D^L C_{\pm}(y,\bar{y}|x) \pm \\ \pm \lambda e^{\alpha\beta'}\Big(y_\alpha \bar{y}_{\beta'} + \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta'}}\Big) C_{\pm}(y,\bar{y}|x). \quad (3.38)$$

Analogously to the Minkowski case, for any parameter $\eta \in \mathcal{R}_{AdS}$, Eq. (3.19) is solved by the bilinears

$$J(y^{\pm}\,\bar{y}^{\pm}|x) = \eta\, C_+(y^+ + y^-, \bar{y}^+ + \bar{y}^-|x) \times \\ \times C_-(y^+ - y^-, \bar{y}^+ - \bar{y}^-|x) \quad (3.39)$$

of rank-one fields $C_{\pm}\big(\sqrt{2}y, \sqrt{2}\bar{y}|x\big)$ that solve Eq. (3.38).

Now we are in a position to consider a deformation of the system (2.6), (2.7) combined with rank-two equations (3.19). We show in particular that upon bilinear substitution (3.15), the constructed deformed system leads to the Maxwell equations with a nonzero current and to the linearized Einstein equations with a nonzero stress tensor.

4. CURRENT DEFORMATION

To describe the current interactions of $4d$ massless fields, we look for a nontrivial deformation of the combination of rank-one and rank-two unfolded systems (2.6), (2.7), and (3.19). The form of the deformation is fixed by its formal consistency. The problem is solved in two steps. First, we consider the zero-form sector to find a gluing of the rank-two current module to the rank-one Weyl module. The result is presented in Sec. 4.1, while the details of the derivation are given in Appendix A. Second, the result for the gluing in the one-form sector is presented in Sec. 4.2, and the details are given in Appendices B, C, and D.

4.1. Current deformation in the zero-form sector

The deformation in the zero-form sector is independent of that in the one-form sector. On the other hand, because of the C-dependent part of Eq. (2.6), the form of the deformation in the zero-form sector affects the deformation in the one-form sector.

The most general consistent deformation of Eq. (2.7) by rank-two fields has the form

$$D^{tw} C(y,\bar{y}|x) + e^{\alpha\alpha'} F(\mathcal{N}_{\pm}, \overline{\mathcal{N}}_{\pm}) y^j{}_\alpha \bar{\partial}_{j\alpha'} \times \\ \times J(y^{\pm}, \bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0} + \\ + e^{\alpha\alpha'} \Phi(\mathcal{N}_{\pm}, \overline{\mathcal{N}}_{\pm}) \bar{y}^j{}_{\alpha'} \partial_{j\alpha} \times \\ \times I(y^{\pm}, \bar{y}^{\pm}|x)\Big|_{y^{\pm}=\bar{y}^{\pm}=0} = 0, \quad (4.1)$$

where D^{tw} is defined in (2.10), and $J(y^{\pm}, \bar{y}^{\pm})$ and $I(y^{\pm}, \bar{y}^{\pm})$ are rank-two fields satisfying unfolded field equations (3.19). The form of the gluing operators F and Φ is determined by the consistency of Eq. (4.1) analyzed in detail in Appendix A, which is the condition that the application of D^{tw} to (4.1) leads to the identity $0 = 0$ if the current fields $J(y^{\pm}, \bar{y}^{\pm}|x)$ and $I(y^{\pm}, \bar{y}^{\pm}|x)$ satisfy the current equation. Here, we use the notation

$$a^j b_j = a^+ b_+ - a^- b_-, \quad \mathcal{N}_{\pm} = y^\alpha \partial_{\pm\alpha}, \\ \overline{\mathcal{N}}_{\pm} = \bar{y}^{\alpha'} \bar{\partial}_{\pm\alpha'}. \quad (4.2)$$

The final result is

$$F(\mathcal{N}_{\pm}, \overline{\mathcal{N}}_{\pm}) = \sum_{m\geq 0} \sum_{n=0}^{m} a_{n,m} \mathfrak{F}^{n,m-n}(\mathcal{N}_{\pm}, \overline{\mathcal{N}}_{\pm}), \quad (4.3)$$

$$\Phi(\mathcal{N}_{\pm}, \overline{\mathcal{N}}_{\pm}) = \sum_{m\geq 0} \sum_{n=0}^{m} b_{n,m} \overline{\mathfrak{F}}^{n,m-n}(\mathcal{N}_{\pm}, \overline{\mathcal{N}}_{\pm}), \quad (4.4)$$

where $a_{n,m}$ and $b_{n,m}$ are arbitrary coefficients and

$$\mathfrak{F}^{n_+,n_-}(\mathcal{N}_{\pm}, \overline{\mathcal{N}}_{\pm}) = (\mathcal{N}_+)^{n_+} (\mathcal{N}_-)^{n_-} \times \\ \times \sum_{m\geq 0} \frac{(\overline{\mathcal{N}}_+ \mathcal{N}_- + \overline{\mathcal{N}}_- \mathcal{N}_+)^m}{m!(m+n_+ + n_- + 1)!}, \\ \overline{\mathfrak{F}}^{n_+,n_-}(\mathcal{N}_{\pm}, \overline{\mathcal{N}}_{\pm}) = (\overline{\mathcal{N}}_+)^{n_+} (\overline{\mathcal{N}}_-)^{n_-} \times \\ \times \sum_{m\geq 0} \frac{(\overline{\mathcal{N}}_+ \mathcal{N}_- + \overline{\mathcal{N}}_- \mathcal{N}_+)^m}{m!(m+n_+ + n_- + 1)!}. \quad (4.5)$$

As shown in Appendix D, the fields of the form $J = f_- J'$ and $I = f_+ I'$ give a D^{tw}-exact deformation (4.1), which can be removed by a local field redefinition.

We note that functions (4.5) can be expressed via the regular Bessel functions (see, e.g., [32])

$$I_{k+1}(2x^{\frac{1}{2}}) = x^{(k+1)/2} \sum_m \frac{x^m}{m!(m+k+1)!} \tag{4.6}$$

as follows:

$$\mathfrak{F}^{n,m} = \frac{(\mathcal{N}_+)^n(\mathcal{N}_-)^m}{\left(\overline{\mathcal{N}}_+\mathcal{N}_- + \overline{\mathcal{N}}_-\mathcal{N}_+\right)^{(n+m+1)/2}} \times \\ \times I_{n+m+1}\left(2\left(\overline{\mathcal{N}}_+\mathcal{N}_- + \overline{\mathcal{N}}_-\mathcal{N}_+\right)^{1/2}\right).$$

To see the origin of the ambiguity associated with the coefficients $a_{n,m}$ and $b_{n,m}$, we use the relations

$$\begin{gathered} [f_0, N_\pm] = -N_\pm, \quad [f_0, \overline{N}_\pm] = \overline{N}_\pm, \\ [g_0, \overline{N}_\pm] = \mp\overline{N}_\pm, \quad [g_0, N_\pm] = \pm N_\pm, \\ [g_\pm, N_\pm] = -N_\mp, \quad [g_\mp, N_\pm] = 0, \\ [g_\pm, \overline{N}_\pm] = \overline{N}_\mp, \quad [g_\mp, \overline{N}_\pm] = 0, \end{gathered} \tag{4.7}$$

whence it follows that

$$\left[g_\pm, \overline{\mathcal{N}}_+\mathcal{N}_- + \overline{\mathcal{N}}_-\mathcal{N}_+\right] = \left[g_0, \overline{\mathcal{N}}_+\mathcal{N}_- + \overline{\mathcal{N}}_-\mathcal{N}_+\right] = \\ = \left[f_0, \overline{\mathcal{N}}_+\mathcal{N}_- + \overline{\mathcal{N}}_-\mathcal{N}_+\right] = 0, \tag{4.8}$$

$$\begin{gathered} \left[g_-, \mathfrak{F}^{n_+,n_-}\right] = -n_-\mathfrak{F}^{n_++1,n_--1}, \\ \left[g_+, \mathfrak{F}^{n_+,n_-}\right] = -n_+\mathfrak{F}^{n_+-1,n_++1}, \\ \left[g_-, \overline{\mathfrak{F}}^{\bar{n}_+,\bar{n}_-}\right] = \bar{n}_-\overline{\mathfrak{F}}^{\bar{n}_++1,\bar{n}_--1}, \\ \left[g_+, \overline{\mathfrak{F}}^{\bar{n}_+,\bar{n}_-}\right] = \bar{n}_+\overline{\mathfrak{F}}^{\bar{n}_+-1,\bar{n}_-+1}. \end{gathered} \tag{4.9}$$

Here, f_a and g_b are respectively generators of ${}^v\mathfrak{sl}_2$ Eq. (3.21) and ${}^h\mathfrak{sl}_2$ Eq. (3.22).

On the other hand, the J and I-dependent terms in (4.1) are invariant under the action of f_0 and g_j on the variables $y^\pm$ and $\bar{y}^\pm$ simply because the result is zero at $y^\pm = \bar{y}^\pm = 0$. (However, this is not the case for the operators $f_\pm$, which contain second derivatives in $y^\pm$ and $\bar{y}^\pm$.) This means that the action of the rank-two helicity operator f_0 on the gluing functions is equivalent up to a sign to their action on J and I, respectively shifted to ∓ 2, because

$$[f_0, y^j{}_\alpha\bar{\partial}_{j\alpha'}] = 2y^j{}_\alpha\bar{\partial}_{j\alpha'}, \quad [f_0, \bar{y}^j{}_{\alpha'}\partial_{j\alpha}] = -2\bar{y}^j{}_{\alpha'}\partial_{j\alpha}.$$

For example,

$$0 = \left(f_0\mathfrak{F}^{k,n}y^j{}_\alpha\bar{\partial}_{j\alpha'}J(y^\pm,\bar{y}^\pm|x)\right)\Big|_{y^\pm=\bar{y}^\pm=0} = \\ = (2-k-n)\mathfrak{F}^{k,n}y^j{}_\alpha\bar{\partial}_{j\alpha'}J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} + \\ + \mathfrak{F}^{k,n}y^j{}_\alpha\bar{\partial}_{j\alpha'}f_0J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0}. \tag{4.10}$$

Analogously, the action of the horizontal operators g_j on the gluing functions is equivalent up to a sign to their action on J and I because the operators $y^j{}_\alpha\bar{\partial}_{j\alpha'}$ and their complex conjugate $\bar{y}^j{}_{\alpha'}\partial_{j\alpha}$ are invariant under ${}^h\mathfrak{sl}_2$, for example,

$$\begin{gathered} 0 = \Big(g_\pm\ \mathfrak{F}^{k,n}y^j{}_\alpha\bar{\partial}_{j\alpha'} \times \\ \times J(y^\pm,\bar{y}^\pm|x)\Big)\Big|_{y^\pm=\bar{y}^\pm=0} = \\ = [g_\pm, \mathfrak{F}^{k,n}]\,y^j{}_\alpha\bar{\partial}_{j\alpha'} \times \\ \times J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} + \\ + \mathfrak{F}^{k,n}y^j{}_\alpha\bar{\partial}_{j\alpha'}g_\pm J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0}, \\ 0 = \big(g_0\mathfrak{F}^{k,n}y^j{}_\alpha\bar{\partial}_{j\alpha'} \times \\ \times J(y^\pm,\bar{y}^\pm|x)\big)\Big|_{y^\pm=\bar{y}^\pm=0} = \\ = (-k+n)\mathfrak{F}^{k,n}y^j{}_\alpha\bar{\partial}_{j\alpha'} \times \\ \times J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} + \\ + \mathfrak{F}^{k,n}y^j{}_\alpha\bar{\partial}_{j\alpha'}g_0\,J(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0}. \end{gathered} \tag{4.11}$$

Since $\varphi(f,g)J$ and $\psi(f,g)I$ satisfy the rank-two equation and therefore provide new conserved currents for any functions $\varphi(f,g)J$ and $\psi(f,g)I$, the general deformation (4.1) realizes a representation of $\mathfrak{gl}_2$ formed by f_0 and ${}^h\mathfrak{sl}_2$. The application of f_0 and g_j to the deformation transforms the coefficients as finite-dimensional spin-$\frac{1}{2}(n+k)$ representations of $\mathfrak{gl}_2$. Indeed, deformation (4.1) for a spin s rank-one field with currents obeying

$$\begin{aligned} f_0J^{s-1} &= 2(s-1)J^{s-1}, \\ f_0I^{-s+1} &= -2(s-1)I^{-s+1}, \end{aligned} \tag{4.12}$$

is

$$D^{tw}C^s(y,\bar{y}|x) + e^{\alpha\alpha'}\sum_{m=0}^{2s} a_{m,2s}\mathfrak{F}^{m,2s-m}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm) \times \\ \times y^j{}_\alpha\bar{\partial}_{j\alpha'}J^{s-1}_{(2s-2m)}(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} = 0, \tag{4.13}$$

$$D^{tw}\overline{C}^{-s}(y,\bar{y}|x)+e^{\alpha\alpha'}\sum_{m=0}^{2s}\bar{a}_{m,2s}\overline{\mathfrak{F}}^{m,2s-m}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\times$$
$$\times\,\bar{y}^j{}_{\alpha'}\partial_{j\alpha}I^{-s+1}_{(2s-2m)}(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0}=0 \quad (4.14)$$

for $s>0$ and

$$D^{tw}C^0(y,\bar{y}|x)+e^{\alpha\alpha'}a_{0,0}\mathfrak{F}^{0,0}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)y^j{}_\alpha\bar{\partial}_{j\alpha'}\times$$
$$\times\,J^{-1}_{(0)}(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0}+$$
$$+\,e^{\alpha\alpha'}\bar{a}_{0,0}\overline{\mathfrak{F}}^{0,0}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\bar{y}^j{}_{\alpha'}\partial_{j\alpha}\times$$
$$\times\,I^1{}_{(0)}(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0}=0, \quad (4.15)$$

for $s=0$. Here, $J^p_{(k)}$ satisfies the relations

$$g_0J^p_{(k)}=k\,J^p_{(k)},$$
$$g_0\in{}^h\mathfrak{sl}_2,$$

and $a_{i,j}$ are arbitrary coefficients.

Since the deformation coefficients form finite-dimensional $\mathfrak{gl}_2$-modules, it suffices to consider the problem for any element of these modules. In Sec. 4.2 and in the examples in Sec. 5, we consider "${}^h\mathfrak{sl}_2$-highest deformations" with

$$a_{m,2s-m}=\delta^0_m a_{0,2s},\quad \bar{a}_{m,2s-m}=\delta^0_m\bar{a}_{0,2s}. \quad (4.16)$$

For the future convenience, we set

$$a_{0,2s}=\bar{a}_{0,2s}=2s+1.$$

To define the flat limit of the deformed equations (4.13), (4.14), and (4.15), it is necessary to introduce appropriate λ-dependent coefficients of the added deforming terms. It is evident that the terms

$$e^{\alpha\alpha'}\mathfrak{F}^{m,2s-m}y^j{}_\alpha\bar{\partial}_{j\alpha'}\times$$
$$\times\,(f^+)^nJ^{s-1}_{(2s-2m)}(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} \quad (4.17)$$

and

$$e^{\alpha\alpha'}\overline{\mathfrak{F}}^{m,2s-m}\bar{y}^j{}_{\alpha'}\partial_{j\alpha}\times$$
$$\times\,(f^-)^nI^{1-s}_{(2s-2m)}(y^\pm,\bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} \quad (4.18)$$

require some coefficient $a(\lambda^n)$ to yield the coefficient $a(1)$ after rescaling (2.20) in the flat limit $\lambda\to0$.

4.2. Current deformation in the one-form sector

Since zero-forms contribute to the right-hand sides of Eqs. (2.6), their formal consistency in the presence of deformation (4.1) requires an appropriate deformation in the one-form sector. Since the analysis of the deformation in the one-form sector is more complicated due to the gauge ambiguity, instead of considering the problem in full generality, we use an appropriate ansatz that not only guarantees the formal consistency but also gives rise to the correct current deformation of the dynamic equations.

The problem is considerably simplified by using the

$$\mathfrak{gl}_2=f_0\cup{}^h\mathfrak{sl}_2$$

symmetry acting on the gluing coefficients in (4.3) and (4.4) of deformation (4.1). Indeed, it allows us to first find the deformation in the one-form sector in the particular case of ${}^h\mathfrak{sl}_2$ highest-weight coefficients of the form (4.16) in (4.1) and then extend the result to arbitrary gluing coefficients by the action of ${}^h\mathfrak{sl}_2$ on the gluing functions.

Here, we present the final results of the "highest-weight" deformation. Details of their derivation are quite complicated and are presented in Appendices B and C.

First, for a given spin s, we introduce "seed current fields" $\mathcal{J}_{\mathrm{h},s}$ that solve Eq. (3.19) and obey the conditions

$$\begin{aligned}f_0\,\mathcal{J}_{\mathrm{h},s}(y^\pm,\bar{y}^\pm|x)&=2\mathrm{h}\,\mathcal{J}_{\mathrm{h},s}(y^\pm,\bar{y}^\pm|x),\\ g_0\mathcal{J}_{\mathrm{h},s}(y^\pm,\bar{y}^\pm|x)&=-2s\,\mathcal{J}_{\mathrm{h},s}(y^\pm,\bar{y}^\pm|x),\end{aligned} \quad (4.19)$$

where f_0 from (3.21) is the rank-two helicity operator, g_0 from (3.22) is the Cartan operator of ${}^h\mathfrak{sl}_2$, $\mathrm{h}=0$ for integer s and $\mathrm{h}=\pm1/2$ for half-integer s. The reality condition requires that $\mathcal{J}_{\mathrm{h},s}=\overline{\mathcal{J}}_{-\mathrm{h},s}$.

Given an integer spin $s\ge2$ and a seed current field $\mathcal{J}_{0,s}$, the deformed equation in the one-form sector is

$$D^{ad}\omega(y,\bar{y}|x)-\overline{H}^{\alpha'\beta'}\bar{\partial}_{\alpha'}\bar{\partial}_{\beta'}\overline{C}(0,\bar{y}|x)-$$
$$-\,H^{\alpha\beta}\partial_\alpha\partial_\beta C(y,0|x)=$$
$$=H^{\alpha\beta}\partial_{-\alpha}\partial_{-\beta}\sum_{k=0}^{s-2}\frac{(\mathcal{N}_-)^{s-k-2}(\overline{\mathcal{N}}_-)^{s+k}}{(s+k)!}\times$$
$$\times\,(f_-)^k\,\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0}+$$
$$+\,\overline{H}^{\alpha'\beta'}\bar{\partial}_{-\alpha'}\bar{\partial}_{-\beta'}\sum_{k=0}^{s-2}\frac{(\mathcal{N}_-)^{s+k}(\overline{\mathcal{N}}_-)^{s-k-2}}{(s+k)!}\times$$
$$\times\,(f_+)^k\,\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0}, \quad (4.20)$$

where $f_\pm\in{}^v\mathfrak{sl}_2$.

The associated deformation in the zero-form sector is

$$\begin{aligned} &D^{tw}C(y,\bar{y}|x)+\lambda(2s+1)e^{\mu\beta'}\mathfrak{F}^{0,2s}y^{j}{}_{\alpha}\bar{\partial}_{j\beta'}\times\\ &\quad\times\Big(f_{+}\Big)^{s-1}\mathcal{J}_{0,s}\big|_{y^{\pm}=\bar{y}^{\pm}=0}=0,\\ &D^{tw}\overline{C}(y,\bar{y}|x)+\lambda(2s+1)e^{\mu\beta'}\overline{\mathfrak{F}}^{0,2s}\partial_{j\mu}\bar{y}^{j}{}_{\beta'}\times\\ &\quad\times\Big(f_{-}\Big)^{s-1}\mathcal{J}_{0,s}\big|_{y^{\pm}=\bar{y}^{\pm}=0}=0, \end{aligned} \tag{4.21}$$

where $\mathfrak{F}^{0,2s}$ and $\overline{\mathfrak{F}}^{0,2s}$ are defined in (4.5).

Given a half-integer spin $s=l+1/2$ and seed current fields $\mathcal{J}_{\pm 1,s}$, the deformed equation in the one-form sector is

$$\begin{aligned} D^{ad}\omega(y,\bar{y}|x)&=\overline{H}^{\alpha'\beta'}\bar{\partial}_{\alpha'}\bar{\partial}_{\beta'}\overline{C}(0,\bar{y}|x)+\\ &+H^{\alpha\beta}\partial_{\alpha}\partial_{\beta}C(y,0|x)+\\ &+H^{\alpha\beta}\partial_{-\alpha}\partial_{-\beta}\Big\{\sum_{k=0}^{l-2}\frac{(\mathcal{N}_{-})^{l-k-2}(\overline{\mathcal{N}}_{-})^{l+1+k}}{(l+1+k)!}\times\\ &\times(f_{-})^{k}\mathcal{J}_{-1,s}+\sum_{k=0}^{l-1}\frac{(\mathcal{N}_{-})^{l-1-k}(\overline{\mathcal{N}}_{-})^{l+k}}{l(l+k)!}\times\\ &\times(f_{-})^{k}\mathcal{J}_{1,s}\Big\}\big|_{y^{\pm}=\bar{y}^{\pm}=0}+\\ &+\overline{H}^{\alpha'\beta'}\bar{\partial}_{-\alpha'}\bar{\partial}_{-\beta'}\Big\{\sum_{k=0}^{l-1}\frac{(\mathcal{N}_{-})^{l+k}(\overline{\mathcal{N}}_{-})^{l-k-1}}{l(l+k)!}\times\\ &\times(f_{+})^{k}\mathcal{J}_{-1,s}+\sum_{k=0}^{(l-2)}\frac{(\mathcal{N}_{-})^{l+1+k}(\overline{\mathcal{N}}_{-})^{l-k-2}}{(l+1+k)!}\times\\ &\times(f_{+})^{k}\mathcal{J}_{1,s}\Big\}\big|_{y^{\pm}=\bar{y}^{\pm}=0}. \end{aligned} \tag{4.22}$$

The associated deformation in the zero-form sector is

$$\begin{aligned} &D^{tw}C(y,\bar{y}|x)+\lambda(2s+1)e^{\mu\beta'}\mathfrak{F}^{0,2s}y^{j}{}_{\alpha}\times\\ &\quad\times\bar{\partial}_{j\beta'}\Big\{\Big(f_{+}\Big)^{l-1}\mathcal{J}_{1,s}+\\ &\quad+\frac{1}{l}\Big(f_{+}\Big)^{l}\mathcal{J}_{-1,s}\Big\}\big|_{y^{\pm}=\bar{y}^{\pm}=0}=0,\\ &D^{tw}\overline{C}(y,\bar{y}|x)+\lambda(2s+1)e^{\mu\beta'}\overline{\mathfrak{F}}^{0,2s}\times\\ &\quad\times\partial_{j\mu}\bar{y}^{j}{}_{\beta'}\Big\{\Big(f_{-}\Big)^{l-1}\mathcal{J}_{-1,s}+\\ &\quad+\frac{1}{l}\Big(f_{-}\Big)^{l}\mathcal{J}_{1,s}\Big\}\big|_{y^{\pm}=\bar{y}^{\pm}=0}=0. \end{aligned} \tag{4.23}$$

We note that these deformations are nontrivial if the seed current fields $\mathcal{J}_{\mathrm{h},s}$ in (4.19) are such that

$$\mathcal{J}_{\mathrm{h},s}(y^{\pm},\bar{y}^{\pm}|x)\big|_{y_{+}=\bar{y}_{+}=0}\neq 0.$$

5. CURRENT CONTRIBUTION TO DYNAMICAL EQUATIONS

We explain how the deformed unfolded equations affect the form of dynamical equations for massless fields. To obtain the usual current interactions, the rank-two fields should be realized as bilinears in massless fields,

$$\begin{aligned} \mathcal{J}_{0}=C_{+}\Big(y^{+}+y^{-},\bar{y}^{+}+\bar{y}^{-}\Big|x\Big)\times\\ \times C_{-}\Big(y^{+}-y^{-},\bar{y}^{+}-\bar{y}^{-}\Big|x\Big), \end{aligned} \tag{5.1}$$

where $C_{\pm}(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}\bar{y}|x)$ solve rank-one equations (3.38). For the future convenience, we use the decompositions

$$\begin{aligned} A(y^{\pm},\bar{y}^{\pm}|x)&=\sum_{m,\bar{m}}A^{m,\bar{m}}(y^{\pm},\bar{y}^{\pm}|x),\\ B(y,\bar{y}|x)&=\sum_{m,\bar{m}}B^{m,\bar{m}}(y,\bar{y}|x), \end{aligned} \tag{5.2}$$

where

$$\Big(y^{+\beta}\frac{\partial}{\partial y^{+\beta}}+y^{-\beta}\frac{\partial}{\partial y^{-\beta}}\Big)A^{m,\bar{m}}(y^{\pm},\bar{y}^{\pm}|x)=mA^{m,\bar{m}}(y^{\pm},\bar{y}^{\pm}|x),$$

$$\Big(\bar{y}^{+\beta'}\frac{\partial}{\partial\bar{y}^{+\beta'}}+\bar{y}^{-\beta'}\frac{\partial}{\partial\bar{y}^{-\beta'}}\Big)A^{m,\bar{m}}(y^{\pm},\bar{y}^{\pm}|x)=\bar{m}A^{m,\bar{m}}(y^{\pm},\bar{y}^{\pm}|x),$$

$$\Big(y^{\beta}\frac{\partial}{\partial y^{\beta}}\Big)B^{m,\bar{m}}(y,\bar{y}|x)=mB^{m,\bar{m}}(y,\bar{y}|x),$$

$$\Big(\bar{y}^{\beta'}\frac{\partial}{\partial\bar{y}^{\beta'}}\Big)B^{m,\bar{m}}(y,\bar{y}|x)=\bar{m}B^{m,\bar{m}}(y,\bar{y}|x).$$

5.1. Spin zero

Using (4.12), we consider J such that $f_0J=2J$. Equation (4.15) with

$$a_{0,0}=\bar{a}_{0,0}=1$$

gives

$$D^{L}{}_{\alpha\alpha'}C(0,0|x)+\lambda C_{\alpha\alpha'}(0,0|x)=0, \tag{5.3}$$

$$D^L{}_{\alpha\alpha'}C_{\beta\beta'}(0,0|x) + \lambda C_{\alpha\beta\alpha'\beta'}(0,0|x) + \lambda\varepsilon_{\alpha'\beta'}\varepsilon_{\alpha\beta}C(0,0|x) - \frac{\varepsilon_{\alpha'\beta'}}{2}\Big(\frac{\partial^2}{\partial y^{+\beta}\partial y^{-\alpha}} - \frac{\partial^2}{\partial y^{-\beta}\partial y^{+\alpha}}\Big)J(y^\pm,0|x)\big|_{y^\pm=\bar y^\pm=0} - \frac{\varepsilon_{\alpha\beta}}{2}\Big(\frac{\partial^2}{\partial \bar y^{+\beta'}\partial \bar y^{-\alpha'}} - \frac{\partial^2}{\partial \bar y^{-\beta'}\partial \bar y^{+\alpha'}}\Big)\overline{J}(0,\bar y^\pm|x)\big|_{y^\pm=\bar y^\pm=0} = 0.$$

Hence,

$$D^L_{\alpha\alpha'}D^{L\alpha\alpha'}C(0,0|x) = 4\lambda^2C(0,0|x) - 4\frac{\partial^2}{\partial y^+{}_\alpha\partial y^{-\alpha}}I(y^\pm,0|x) - 4\frac{\partial}{\partial^2\bar y^+{}_{\alpha'}\partial\bar y^{-\alpha'}}J(0,\bar y^\pm|x). \quad (5.4)$$

From (5.1), we obtain

$$D^L_{\alpha\alpha'}D^{L\alpha\alpha'}C(0,0|x) = 4\lambda^2C(0,0|x) + 4\overline{C}_{+\alpha'}(x)\overline{C}_-{}^{\alpha'}(x) + 4C_{+\alpha}(x)C_-{}^\alpha(x). \quad (5.5)$$

Remarkably, in the spin-zero sector, the proposed unfolded construction just reproduces Yukawa interaction since $C_{\pm\alpha}(x)$ are dynamical spin-1/2 fields. We note that a C^2 deformation, which one might naively expect in the spin-zero sector, does not appear in agreement with the fact that the construction in this paper is conformal, while the C^2 deformation is not conformal in four dimensions.

5.2. Spin 1/2

Let $f_0J = J$. Equations (4.13) and (4.14) with

$$a_{m,2s-m} = \bar a_{m,2s-m} = 2\delta^0_m$$

give

$$\begin{gathered} D^L{}_{\alpha\alpha'}C_\mu(0,0|x) + \lambda C_{\mu\alpha\alpha'}(0,0|x) + \varepsilon_{\mu\alpha}\frac{\partial}{\partial\bar y^{-\alpha'}}\overline{J}(0,\bar y^-|x)|_{\bar y^-=0} = 0\,, \\ D^L{}_{\alpha\alpha'}\overline{C}_{\mu'}(0,0|x) + \lambda\overline{C}_{\alpha\mu'\alpha'}(0,0|x) + \varepsilon_{\mu'\alpha'}\frac{\partial}{\partial y^{-\alpha}}J(y^-,0|x)|_{y^-=0} = 0. \end{gathered} \quad (5.6)$$

It follows from (5.6) that

$$\begin{gathered} D^L{}_{\alpha\alpha'}C^\alpha(0,0|x) - 2\frac{\partial}{\partial\bar y^{-\alpha'}}\overline{J}(0,\bar y^-|x)|_{\bar y^-=0} = 0, \\ D^L{}_{\alpha\alpha'}\overline{C}^{\alpha'}(0,0|x) - 2\frac{\partial}{\partial y^{-\alpha}}J(y^-,0|x)|_{y^-=0} = 0. \end{gathered} \quad (5.7)$$

Substituting the bilinear $\overline{J}$ and J from (5.1) built from fermions and bosons gives

$$\begin{gathered} D^L{}_{\alpha\alpha'}C^\alpha(x) - \sqrt2\overline{C}'_{+\alpha}(x)\overline{C}_-(x) + \sqrt2\overline{C}_+(x)\overline{C}_{-\alpha'}(x) = 0, \\ D^L{}_{\alpha\alpha'}\overline{C}^{\alpha'}(x) - \sqrt2C_{+\alpha}(x)C_-(x) + \sqrt2C_+(x)C_{-\alpha}(x) = 0, \end{gathered} \quad (5.8)$$

which is the Yukawa interaction in the spin-1/2 sector.

5.3. Maxwell equations

Let $f_0J = 0$. Then the reality condition requires that $\overline{J} = J$. Equation (2.6) still has the form

$$D^{ad}\omega(x) = \overline{H}^{\alpha'\beta'}\overline{C}_{\alpha'\beta'}(x) + H^{\alpha\beta}C_{\alpha\beta}(x). \quad (5.9)$$

This identifies for $C_{\alpha\beta}(x)$ and $\overline{C}_{\alpha'\beta'}(x)$ involve selfdual and anti-selfdual parts of the Maxwell field strength. The consistency conditions of (5.9) imply the Bianchi identities

$$D^{ad}\big(H^{\alpha\beta}C_{\alpha\beta}(x) + \overline{H}^{\alpha'\beta'}\overline{C}_{\alpha'\beta'}(x)\big) = 0. \quad (5.10)$$

Deformed equation (4.21) for $s = 1$ at $y = \bar y = 0$ gives

$$D^L{}_{\alpha\alpha'}C_{\mu\nu}(0,0|x) + \lambda C_{\mu\nu\alpha\alpha'}(0,0|x) + \Big(\varepsilon_{\mu\alpha}\frac{\partial^2}{\partial\bar y^{-\alpha'}\partial y^{-\nu}}J(y^\pm,\bar y^\pm|x) + \varepsilon_{\nu\alpha}\frac{\partial^2}{\partial\bar y^{-\alpha'}\partial y^{-\mu}}J(y^\pm,\bar y^\pm|x)\Big)\Big|_{y^\pm=\bar y^\pm=0} = 0. \quad (5.11)$$

It follows from (5.11) that in accordance with decompositions (5.2),

$$D^{L\mu}{}_{\alpha'}C_{\mu\nu}(0,0|x) + 3\lambda\frac{\partial^2}{\partial\bar y^{-\alpha'}\partial y^{-\nu}} \times J^{1,1}(y^\pm,\bar y^\pm|x) = 0. \quad (5.12)$$

By virtue of (5.12) along with the identities

$$\begin{gathered} H^{\alpha\beta}\wedge e^{\mu\mu'} = \epsilon^{\alpha\mu}\mathcal{H}^{\beta\mu'} + \epsilon^{\beta\mu}\mathcal{H}^{\alpha\mu'}, \\ \overline{H}^{\alpha'\beta'}\wedge e^{\mu\mu'} = -\epsilon^{\alpha'\mu'}\mathcal{H}^{\mu\beta'} - \epsilon^{\beta'\mu'}\mathcal{H}^{\mu\alpha'}, \end{gathered} \quad (5.13)$$

we have

$$H^{\alpha\beta}e^{\nu\nu'}D^L{}_{\nu\nu'}C_{\alpha\beta}(x) = 2\mathcal{H}^{\beta\nu'}D^{L\alpha}{}_{\nu'}C_{\alpha\beta} = -6\lambda\mathcal{H}^{\beta\nu'}\frac{\partial^2}{\partial y^{-\beta}\partial y^{-\nu'}}J^{1,1}(y^\pm,\bar y^\pm|x).$$

Analogously,

$$\overline{H}^{\alpha'\beta'}D^L\overline{C}_{\alpha'\beta'}(x) = 6\lambda\mathcal{H}^{\beta\nu'}\frac{\partial^2}{\partial y^{-\beta}\partial y^{-\nu'}}J^{1,1}(y^\pm,\bar y^\pm|x).$$

Hence, it follows that, as anticipated, Bianchi identities (5.10) are respected and

$$D^L\left(H^{\alpha\beta}C_{\alpha\beta}(x)-\overline{H}^{\alpha'\beta'}\overline{C}_{\alpha'\beta'}(x)\right)= = -12\lambda\mathcal{H}^{\beta\nu'}\frac{\partial^2}{\partial y^{-\beta}\partial y^{-\nu'}}J^{1,1}(y^{\pm},\bar{y}^{\pm}|x). \quad (5.14)$$

This just reproduces the Maxwell equations with a nonzero current.

For J in (5.1) built from scalars and spinors, we respectively have

$$\mathcal{H}^{\beta\nu'}\frac{\partial^2}{\partial y^{-\beta}\partial y^{-\nu'}}J^{1,1}(y^{\pm},\bar{y}^{\pm}|x)=\frac{1}{3\lambda}\times \times\mathcal{H}^{\beta\nu'}\left(-C_-(x)\frac{\partial}{\partial x^{\beta\nu'}}C_+(x)+C_+(x)\frac{\partial}{\partial x^{\beta\nu'}}C_-(x)\right),$$

$$\mathcal{H}^{\beta\nu'}\frac{\partial^2}{\partial y^{-\beta}\partial y^{-\nu'}}J^{1,1}(y^{\pm},\bar{y}^{\pm}|x)= =\frac{-1}{3\lambda}\mathcal{H}^{\beta\nu'}C_{+\beta}(x)\overline{C}_{-\nu'}(x),$$

which are the standard expressions for spin-one currents.

5.4. Spin 3/2

Using decomposition (5.2), from Eq. (4.22), we have

$$D^L\omega^{0,1}(0,\bar{y})-\lambda e^{\beta\beta'}\bar{y}_{\beta'}\frac{\partial}{\partial y^{\beta}}\omega^{1,0}(y,0|x)= =\overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial\bar{y}^{\alpha'}\partial\bar{y}^{\beta'}}\overline{C}(0,\bar{y}|x)+2H^{\alpha\beta}\bar{y}^{\beta'}\frac{\partial}{\partial\bar{y}^{-\beta'}}\times \times\frac{\partial^2}{\partial y^{-\alpha}\partial y^{-\beta}}\mathcal{J}_+^{2,1}(y^{\pm},\bar{y}^{\pm}|x), \quad (5.15)$$

$$D^L\omega^{1,0}(y,0|x)-\lambda e^{\beta\beta'}y_{\beta}\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{0,1}(0,\bar{y}|x)= =H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}C(y,0|x)+2\overline{H}^{\alpha'\beta'}y^{\beta}\frac{\partial}{\partial y^{-\beta}}\times \times\frac{\partial^2}{\partial\bar{y}^{-\alpha'}\partial\bar{y}^{-\beta'}}\mathcal{J}_-^{1,2}(y^{\pm},\bar{y}^{\pm}|x). \quad (5.16)$$

Substituting

$$\omega^{j,k}=e^{\alpha\beta'}\omega^{j,k}{}_{\alpha\beta'}$$

in (5.15) and (5.16), we obtain spin-3/2 massless equations in AdS_4 in the form

$$\begin{aligned} &D^L{}_{\beta\beta'}\omega^{0,1}{}_{\alpha}{}^{\beta'}(0,\bar{y})- \\ &-\lambda\bar{y}_{\beta'}\frac{\partial}{\partial y^{\beta}}\omega^{1,0}{}_{\alpha}{}^{\beta'}(y,0|x)= \\ &=2\bar{y}^{\beta'}\frac{\partial}{\partial\bar{y}^{-\beta'}}\frac{\partial^2}{\partial y^{-\alpha}\partial y^{-\beta}}\mathcal{J}_+^{2,1}(y^{\pm},\bar{y}^{\pm}|x), \\ &D^L{}_{\beta\beta'}\omega^{1,0\beta}{}_{\alpha'}(y,0|x)- \\ &-\lambda y_{\beta}\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{0,1\beta}{}_{\alpha'}(0,\bar{y}|x)= \\ &=2y^{\beta}\frac{\partial}{\partial y^{-\beta}}\frac{\partial^2}{\partial\bar{y}^{-\alpha'}\partial\bar{y}^{-\beta'}}\mathcal{J}_-^{1,2}(y^{\pm},\bar{y}^{\pm}|x). \end{aligned} \quad (5.17)$$

Substituting the bilinear current $\mathcal{J}_+=\overline{\mathcal{J}}_-$ from (5.1) gives

$$\begin{aligned} &\frac{\partial}{\partial\bar{y}^{-\nu'}}D^L{}_{\alpha\beta'}\omega^{0,1}{}_{\alpha}{}^{\beta'}(0,\bar{y})+ \\ &+\lambda\frac{\partial}{\partial y^{\alpha}}\omega^{1,0}{}_{\alpha\nu'}(y,0|x)= \\ &=\sqrt{2}\Big(-C^{2,0}_{+\alpha\alpha}(0,0|x)\overline{C}^{0,1}_{-\nu'}(0,0|x)- \\ &-C^{0,0}_{+\alpha\nu'}(0,0|x)C^{1,0}_{-\alpha}(0,0|x)\Big)+(+\leftrightarrow-), \\ &\frac{\partial}{\partial y^{-\nu}}D^L{}_{\beta\beta'}\omega^{1,0\beta}{}_{\alpha'}(y,0|x)+ \\ &+\lambda\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{0,1}{}_{\nu\alpha'}(0,\bar{y}|x)= \\ &=\sqrt{2}\Big(-\overline{C}^{0,2}_{+\alpha'\alpha'}(0,0|x)C^{1,0}_{-\nu}(0,0|x)- \\ &-\overline{C}^{0,0}_{+\nu\alpha'}(0,0|x)\overline{C}^{0,1}_{-\alpha'}(0,0|x)\Big)+(+\leftrightarrow-). \end{aligned} \quad (5.18)$$

This is the Rarita–Schwinger equation with the supercurrent built from a scalar and a spinor.

5.5. Spin two

In the case $s=2$, it follows from conditions (4.19) and (4.12) that $f_0\,\mathcal{J}_0=0$ and

$$\left(y^{-\alpha}\partial_{-\alpha}+\bar{y}^{-\alpha'}\bar{\partial}_{-\alpha'}-4\right)\mathcal{J}_0(y^{\pm},\bar{y}^{\pm}|x)\big|_{y_+=\bar{y}_+=0}=0.$$

From Eq. (4.20), we hence obtain

$$D^{ad}\omega(y,\bar y|x) = \overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial\bar y^{\alpha'}\partial\bar y^{\beta'}}\overline{C}(0,\bar y|x) + H^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha\partial y^\beta}C(y,0|x) + \frac{1}{2}\overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial\bar y^{-\alpha'}\partial\bar y^{-\beta'}}(\mathcal{N}_-)^2 \times \mathcal{J}_0(y^\pm,\bar y^\pm|x)\big|_{y^\pm=\bar y^\pm=0} + \frac{1}{2}H^{\alpha\beta}\frac{\partial^2}{\partial y^{-\alpha}\partial y^{-\beta}}(\overline{\mathcal{N}}_-)^2 \times \mathcal{J}_0(y^\pm,\bar y^\pm|x)\big|_{y^\pm=\bar y^\pm=0}. \quad (5.19)$$

In accordance with decompositions (5.2), this gives

$$D^L\omega^{1,1}(y,\bar y|x) = \lambda e^{\alpha\beta'}\bar y_{\beta'}\frac{\partial}{\partial y^\alpha}\omega^{2,0}(y,0|x) + \lambda e^{\alpha\beta'}y_\alpha\frac{\partial}{\partial\bar y^{\beta'}}\omega^{0,2}(0,\bar y|x), \quad (5.20)$$

$$D^L\omega^{0,2}(0,\bar y) = \lambda e^{\alpha\beta'}\bar y_{\beta'}\frac{\partial}{\partial y^\alpha}\omega^{1,1}(y,\bar y|x) + \overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial\bar y^{\alpha'}\partial\bar y^{\beta'}}\overline{C}(0,\bar y|x) + H^{\alpha\beta}\bar y^{\alpha'}\bar y^{\beta'}\frac{\partial^2}{\partial\bar y^{-\alpha'}\partial\bar y^{-\beta'}}\frac{\partial^2}{\partial y^{-\alpha}\partial y^{-\beta}} \times \mathcal{J}_0^{2,2}(y^\pm,\bar y^\pm|x), \quad (5.21)$$

$$D^L\omega^{2,0}(y,0|x) = \lambda e^{\alpha\beta'}y_\alpha\frac{\partial}{\partial\bar y^{\beta'}}\omega^{1,1}(y,\bar y|x) + H^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha\partial y^\beta}C(y,0|x) + \overline{H}^{\alpha'\beta'}y^\alpha y^\beta\frac{\partial^2}{\partial y^{-\alpha}\partial y^{-\beta}}\frac{\partial^2}{\partial\bar y^{-\alpha'}\partial\bar y^{-\beta'}} \times \mathcal{J}_0^{2,2}(y^\pm,\bar y^\pm|x). \quad (5.22)$$

Introducing

$$\omega^{j,k} = e^{\alpha\beta'}\omega^{j,k}{}_{\alpha\beta'},$$

from Eq. (5.20) we obtain

$$D^L{}_{\beta\beta'}\omega^{1,1}{}_\beta{}^{\beta'}(y,\bar y|x) = \lambda\bar y_{\beta'}\frac{\partial}{\partial y^\beta}\omega^{2,0}{}_\beta{}^{\beta'}(y,0|x) + \lambda y_\beta\frac{\partial}{\partial\bar y^{\beta'}}\omega^{0,2}{}_\beta{}^{\beta'}(0,\bar y|x), \quad (5.23)$$

$$D^L{}_{\beta\beta'}\omega^{1,1\beta}{}_{\beta'}(y,\bar y|x) = \lambda\bar y_{\beta'}\frac{\partial}{\partial y^\beta}\omega^{2,0\beta}{}_{\beta'}(y,0|x) + \lambda y_\beta\frac{\partial}{\partial\bar y^{\beta'}}\omega^{0,2\beta}{}_{\beta'}(0,\bar y|x). \quad (5.24)$$

Equation (5.21) gives (omitting the arguments)

$$D^L{}_{\beta\beta'}\omega^{0,2}{}_\beta{}^{\beta'} = \bar y^{\alpha'}\bar y^{\beta'}\frac{\partial^2}{\partial\bar y^{-\alpha'}\partial\bar y^{-\beta'}} \times \frac{\partial^2}{\partial y^{-\beta}\partial y^{-\beta}}\mathcal{J}_0^{2,2} + \lambda\bar y_{\beta'}\frac{\partial}{\partial y^\beta}\omega^{1,1}{}_\beta{}^{\beta'}, \quad (5.25)$$

$$D^L{}_{\beta\beta'}\omega^{2,0\beta}{}_{\beta'} = y^\alpha y^\beta\frac{\partial^2}{\partial y^{-\alpha}\partial y^{-\beta}}\frac{\partial^2}{\partial\bar y^{-\beta'}\partial\bar y^{-\beta'}}\mathcal{J}_0^{2,2} + \lambda y_\beta\frac{\partial}{\partial\bar y^{\beta'}}\omega^{1,1\beta}{}_{\beta'}. \quad (5.26)$$

Equations (5.23) and (5.24) express the Lorentz connection $\omega^{2,0}$ and $\omega^{0,2}$ via derivatives of the vierbein $\omega^{1,1}$, while Eqs. (5.25) and (5.26) contain the Bianchi identities for Eq. (5.20),

$$\frac{\partial^2}{\partial\bar y^{\nu'}\partial\bar y^{\nu'}}D^L{}_{\beta\beta'}\omega^{0,2}{}_\beta{}^{\beta'}(0,\bar y|x) = \frac{\partial^2}{\partial y^\beta\partial y^\beta}D^L{}_{\nu\nu'}\omega^{2,0\nu}{}_{\nu'}(y,0|x), \quad (5.27)$$

and the linearized Einstein equations

$$\frac{\partial^2}{\partial\bar y^{\nu'}\partial\bar y^{\nu'}}D^L{}_{\beta\beta'}\omega^{0,2}{}_\beta{}^{\beta'}(0,\bar y|x) - 2\lambda\frac{\partial^2}{\partial\bar y^{\nu'}\partial y^\beta}\omega^{1,1}{}_{\beta\nu'}(y,\bar y|x) = 2\frac{\partial^2}{\partial\bar y^{-\nu'}\partial\bar y^{-\nu'}}\frac{\partial^2}{\partial y^{-\beta}\partial y^{-\beta}}\mathcal{J}_0^{2,2}(y^\pm,\bar y^\pm|x), \quad (5.28)$$

which contain the contribution of the stress tensor.

Substituting the bilinear $\mathcal{J}_0$ from (5.1) gives the linearized Einstein equations

$$\frac{\partial^2}{\partial\bar y^{\nu'}\partial\bar y^{\nu'}}D^L{}_{\beta\beta'}\omega^{0,2}{}_\beta{}^{\beta'}(0,\bar y) - 2\lambda\frac{\partial}{\partial\bar y^{\nu'}}\frac{\partial}{\partial y^\beta}\omega^{1,1}{}_{\beta\nu'}(y,\bar y|x) = 2\Big(C^{2,0}_{+\alpha\alpha}(0,0|x)\overline{C}^{0,2}_{-\alpha'\alpha'}(0,0|x) + C^{1,0}_{+\alpha\alpha\alpha'}(0,0|x)\overline{C}^{0,1}_{-\alpha'}(0,0|x) + C^{0,0}_{+\alpha\alpha'}(0,0|x)\overline{C}^{0,0}_{-\alpha\alpha'}(0,0|x) + (+\leftrightarrow-)\Big)$$

with the stress tensor of massless fields of spins 0, 1/2 and 1 (we recall that $C^{2,0}_{+\alpha\alpha}(0,0|x)$ and $\overline{C}^{0,2}_{-\alpha'\alpha'}(0,0|x)$ describe the selfdual and anti-selfdual combinations of the spin-one field strength).

5.6. Higher spins

5.6.1. Integer spins

For any integer $s \geq 2$ and a real seed current field $\mathcal{J}_0 = \overline{\mathcal{J}}_0$, we should obtain equations for the components $\omega_{\alpha\alpha'}{}^{m,n}$ of

$$\omega^{m,n} = e^{\alpha\alpha'} \omega_{\alpha\alpha'}{}^{m,n}.$$

In particular, for

$$m = s-1-k, \quad n = s-1+k, \quad k = -1, 0, 1$$

and with decomposition (5.2) for ω, it follows from (4.20) that

$$D^L \omega^{s-1,s-1}(y,\bar{y}|x) = \lambda e^{\alpha\beta'} \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s,s-2}(y,\bar{y}|x) + \lambda e^{\alpha\beta'} y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-2,s}(y,\bar{y}|x), \quad (5.29)$$

$$D^L \omega^{s,s-2}(y,\bar{y}|x) = \lambda e^{\alpha\beta'} y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-1,s-1}(y,\bar{y}|x) + \lambda e^{\alpha\beta'} \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s+1,s-3}(y,\bar{y}|x) + \overline{H}^{\alpha'\beta'} \frac{\partial^2}{\partial \bar{y}^{-\alpha'} \partial \bar{y}^{-\beta'}} \times \frac{1}{s!} (\mathcal{N}_-)^s (\overline{\mathcal{N}}_-)^{s-2} \mathcal{J}_0^{s,s}(y^\pm, \bar{y}^\pm|x), \quad (5.30)$$

$$D^L \omega^{s-2,s}(y,\bar{y}|x) = \lambda e^{\alpha\beta'} \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s-1,s-1}(y,\bar{y}|x) + \lambda e^{\alpha\beta'} y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-3,s+1}(y,\bar{y}|x) + H^{\alpha\beta} \frac{\partial^2}{\partial y^{-\alpha} \partial y^{-\beta}} \times \frac{1}{s!} (\mathcal{N}_-)^{s-2} (\overline{\mathcal{N}}_-)^s \mathcal{J}_0^{s,s}(y^\pm, \bar{y}^\pm|x). \quad (5.31)$$

Hence it follows that (omitting the arguments)

$$e^{\mu\mu'} e^{\nu\nu'} D^L_{\mu\mu'} \omega^{s-1,s-1}{}_{\nu\nu'} = \lambda e^{\alpha\beta'} e^{\nu\nu'} \bar{y}_{\beta'} \times \frac{\partial}{\partial y^\alpha} \omega^{s,s-2}{}_{\nu\nu'} + \lambda e^{\alpha\beta'} e^{\nu\nu'} y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-2,s}{}_{\nu\nu'}, \quad (5.32)$$

$$e^{\mu\mu'} e^{\nu\nu'} D^L_{\mu\mu'} \omega^{s,s-2}{}_{\nu\nu'} = \lambda e^{\alpha\beta'} e^{\nu\nu'} y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-1,s-1}{}_{\nu\nu'} + \lambda e^{\alpha\beta'} e^{\nu\nu'} \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s+1,s-3}{}_{\nu\nu'} + \overline{H}^{\alpha'\beta'} \frac{\partial^2}{\partial \bar{y}^{-\alpha'} \partial \bar{y}^{-\beta'}} \frac{1}{s!} (\mathcal{N}_-)^s (\overline{\mathcal{N}}_-)^{s-2} \mathcal{J}_0^{s,s}, \quad (5.33)$$

$$e^{\mu\mu'} e^{\nu\nu'} D^L_{\mu\mu'} \omega^{s-2,s}{}_{\nu\nu'} = \lambda e^{\alpha\beta'} \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s-1,s-1} + \lambda e^{\alpha\beta'} e^{\nu\nu'} y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-3,s+1}{}_{\nu\nu'} + H^{\alpha\beta} \frac{\partial^2}{\partial y^{-\alpha} \partial y^{-\beta}} \times \frac{1}{s!} (\mathcal{N}_-)^{s-2} (\overline{\mathcal{N}}_-)^s \mathcal{J}_0^{s,s}. \quad (5.34)$$

Therefore,

$$\begin{aligned} D^L_{\alpha\mu'} \omega^{s-1,s-1}{}_\alpha{}^{\mu'} &= \lambda \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s,s-2}{}_\alpha{}^{\beta'} + \lambda y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-2,s}{}_\alpha{}^{\beta'}, \\ D^L_{\mu\beta'} \omega^{s-1,s-1\mu}{}_{\beta'} &= \lambda \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s,s-2\alpha}{}_{\beta'} + \lambda y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-2,s\alpha}{}_{\beta'}, \end{aligned} \quad (5.35)$$

$$\begin{aligned} D^L_{\alpha\mu'} \omega^{s,s-2}{}_\alpha{}^{\mu'} &= \lambda y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-1,s-1}{}_\alpha{}^{\beta'} + \lambda \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s+1,s-3}{}_\alpha{}^{\beta'}, \\ D^L_{\mu\beta'} \omega^{s-2,s\mu}{}_{\beta'} &= \lambda \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s-1,s-1\alpha}{}_{\beta'} + \lambda y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-3,s+1\alpha}{}_{\beta'}, \end{aligned} \quad (5.36)$$

$$D^L_{\alpha\mu'} \omega^{s-2,s}{}_\alpha{}^{\mu'} = \lambda \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s-1,s-1}{}_\alpha{}^{\beta'} + \lambda y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-3,s+1}{}_\alpha{}^{\beta'} + \frac{\partial^2}{\partial y^{-\alpha} \partial y^{-\alpha}} \frac{1}{s!} (\mathcal{N}_-)^{s-2} (\overline{\mathcal{N}}_-)^s \mathcal{J}_0^{s,s}, \quad (5.37)$$

$$D^L_{\mu\beta'} \omega^{s,s-2\mu}{}_{\beta'} = \lambda y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{s-1,s-1\alpha}{}_{\beta'} + \lambda \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{s+1,s-3\alpha}{}_{\beta'} + \frac{\partial^2}{\partial \bar{y}^{-\beta'} \partial \bar{y}^{-\beta'}} \frac{1}{s!} (\mathcal{N}_-)^s (\overline{\mathcal{N}}_-)^{s-2} \mathcal{J}_0^{s,s}. \quad (5.38)$$

Substituting the bilinear $\mathcal{J}_0$ from (5.1) gives

$$D^L_{\alpha\mu'}\omega^{s-2,s}{}_{\alpha}{}^{\mu'} = \lambda\bar{y}_{\beta'}\frac{\partial}{\partial y^{\alpha}}\omega^{s-1,s-1}{}_{\alpha}{}^{\beta'} + \lambda y_{\alpha}\frac{\partial}{\partial \bar{y}^{\beta'}}\omega^{s-3,s+1}{}_{\alpha}{}^{\beta'} + \frac{\partial^2}{\partial y^{-\alpha}\partial y^{-\alpha}}\frac{\left(\mathcal{N}_-\right)^{s-2}\left(\overline{\mathcal{N}}_-\right)^{s}}{s!}\times \sum_{p,\, n+m=s-p}\Big(C_+^{p+n,n}\left(y_-,\bar{y}_-|x\right)\times C_-^{m,p+m}\left(-y_-,-\bar{y}_-|x\right)+cc\Big)\Big|_{y_-=\bar{y}_-=0}, \quad (5.39)$$

$$D^L_{\mu\beta'}\omega^{s,s-2\;\mu}{}_{\beta'} = \lambda y_{\alpha}\frac{\partial}{\partial \bar{y}^{\beta'}}\omega^{s-1,s-1\,\alpha}{}_{\beta'} + \lambda\bar{y}_{\beta'}\frac{\partial}{\partial y^{\alpha}}\omega^{s+1,s-3\,\alpha}{}_{\beta'} + \frac{\partial^2}{\partial \bar{y}^{-\beta'}\partial \bar{y}^{-\beta'}}\frac{\left(\mathcal{N}_-\right)^{s}\left(\overline{\mathcal{N}}_-\right)^{s-2}}{s!}\times \sum_{p,\, n+m=s-p}\Big(C_+^{p+n,n}\left(y_-,\bar{y}_-|x\right)\times C_-^{m,p+m}\left(-y_-,-\bar{y}_-|x\right)+cc\Big)\Big|_{y_-=\bar{y}_-=0}. \quad (5.40)$$

To obtain the dynamical spin-s equations with the current corrections, it remains to project out the terms that contain $\omega^{s-3,s+1}$ and $\omega^{s+1,s-3}$. This is achieved by contracting the free indices in (5.39) with $y^{\alpha}y^{\alpha}$ and in (5.40) with $\bar{y}^{\beta'}\bar{y}^{\beta'}$. The resulting equations describe the contribution of HS currents in [15] to the right-hand sides of Fronsdal's equations in AdS_4.

That the currents do not contribute to Eqs. (5.36) is a manifestation of conformal invariance of the currents, which, being traceless, cannot contribute to the trace part of the Fronsdal equations contained in Eq. (5.36).

5.6.2. Half-integer spins

Using decomposition (5.2), for a half–integer s, we obtain from (4.22) that

$$D^L\omega^{[s]-1,[s]}(y,\bar{y}|x) = \lambda e^{\alpha\beta'}y_{\alpha}\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{[s]-2,[s]+1}(y,\bar{y}|x) + \lambda e^{\alpha\beta'}\bar{y}_{\beta'}\frac{\partial}{\partial y^{\alpha}}\omega^{[s],[s]-1}(y,\bar{y}|x) + H^{\alpha\beta}\frac{\partial^2}{\partial y_-{}^{\alpha}\partial y_-{}^{\beta}}\frac{\left(\mathcal{N}_-\right)^{[s]-1}\left(\overline{\mathcal{N}}_-\right)^{[s]}}{[s]\,[s]!}\times \mathcal{J}_+^{[s]+1,[s]}(y^{\pm},\bar{y}^{\pm}|x), \quad (5.41)$$

$$D^L\omega^{[s],[s]-1}(y,\bar{y}|x) = \lambda e^{\alpha\beta'}\bar{y}_{\beta'}\frac{\partial}{\partial y^{\alpha}}\omega^{[s]+1,[s]-2}(y,\bar{y}|x) + \lambda e^{\alpha\beta'}y_{\alpha}\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{[s]-1,[s]}(y,\bar{y}|x) + \overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial \bar{y}_-\partial \bar{y}_-{}^{\beta'}}\frac{\left(\mathcal{N}_-\right)^{[s]}\left(\overline{\mathcal{N}}_-\right)^{[s]-1}}{[s]\,[s]!}\times \mathcal{J}_-^{[s],[s]+1}(y^{\pm},\bar{y}^{\pm}|x), \quad (5.42)$$

where

$$\overline{\mathcal{J}}_+ = \mathcal{J}_-.$$

Hence (omitting the arguments),

$$D^L_{\alpha\mu'}\omega^{[s]-1,[s]}{}_{\alpha}{}^{\mu'} = \lambda y_{\alpha}\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{[s]-2,[s]+1}{}_{\alpha}{}^{\beta'} + \lambda\bar{y}_{\beta'}\frac{\partial}{\partial y^{\alpha}}\omega^{[s],[s]-1}{}_{\alpha}{}^{\beta'} + \frac{\partial^2}{\partial y_-{}^{\alpha}\partial y_-{}^{\alpha}}\frac{\left(\mathcal{N}_-\right)^{[s]-1}\left(\overline{\mathcal{N}}_-\right)^{[s]}}{[s][s]!}\mathcal{J}_+^{[s]+1,[s]}, \quad (5.43)$$

$$D^L_{\mu\beta'}\omega^{[s]-1,[s]\mu}{}_{\beta'} = \lambda y_{\alpha}\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{[s]-2,[s]+1\,\alpha}{}_{\beta'} + \lambda\bar{y}_{\beta'}\frac{\partial}{\partial y^{\alpha}}\omega^{[s],[s]-1\,\alpha}{}_{\beta'}, \quad (5.44)$$

$$D^L_{\mu\beta'}\omega^{[s],[s]-1\mu}{}_{\beta'} = \lambda\bar{y}_{\beta'}\frac{\partial}{\partial y^{\alpha}}\omega^{[s]+1,[s]-2\,\alpha}{}_{\beta'} + \lambda y_{\alpha}\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{[s]-1,[s]\alpha}{}_{\beta'} + \frac{\partial^2}{\partial \bar{y}_-{}^{\beta'}\partial \bar{y}_-{}^{\beta'}}\frac{\left(\mathcal{N}_-\right)^{[s]}\left(\overline{\mathcal{N}}_-\right)^{[s]-1}}{[s][s]!}\mathcal{J}_-^{[s],[s]+1}, \quad (5.45)$$

$$D^L_{\alpha\mu'}\omega^{[s],[s]-1}{}_{\alpha}{}^{\mu'} = \lambda\bar{y}_{\beta'}\frac{\partial}{\partial y^{\alpha}}\omega^{[s]+1,[s]-2}{}_{\alpha}{}^{\beta'} + \lambda y_{\alpha}\frac{\partial}{\partial\bar{y}^{\beta'}}\omega^{[s]-1,[s]}{}_{\alpha}{}^{\beta'}. \quad (5.46)$$

Substituting the bilinear

$$\mathcal{J}_+ = C_+(y_- + y_+, \bar{y}_- + \bar{y}_+ |x) \overline{C}_-(y_+ - y_-, \bar{y}_+ - \bar{y}_- |x)$$

and

$$\mathcal{J}_- = \overline{\mathcal{J}_+}$$

from (5.1) gives

$$D^L_{\alpha\mu'} \omega^{[s]-1,[s]}{}_\alpha{}^{\mu'} = \lambda y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{[s]-2,[s]+1}{}_\alpha{}^{\beta'} + \lambda \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{[s],[s]-1}{}_\alpha{}^{\beta'} + \frac{\partial^2}{\partial y^{-\alpha} \partial y^{-\alpha}} \times \frac{(\mathcal{N}_-)^{[s]-1} (\overline{\mathcal{N}}_-)^{[s]}}{[s][s]!} \times \sum_{p,\, n+m=[s]-p} \Big(C_+^{p+n+1,n}(y_-, \bar{y}_- |x) \times C_-^{m,p+m}(-y_-, -\bar{y}_- |x) + C_+^{m,p+m}(y_-, \bar{y}_- |x) \times C_-^{p+n+1,n}(-y_-, -\bar{y}_- |x) \Big) \Big|_{y_- = \bar{y}_- = 0}, \quad (5.47)$$

$$D^L_{\mu\beta'} \omega^{[s],[s]-1\mu}{}_{\beta'} = \lambda \bar{y}_{\beta'} \frac{\partial}{\partial y^\alpha} \omega^{[s]+1,[s]-2\alpha}{}_{\beta'} + \lambda y_\alpha \frac{\partial}{\partial \bar{y}^{\beta'}} \omega^{[s]-1,[s]\alpha}{}_{\beta'} + \frac{\partial^2}{\partial \bar{y}^{-\beta'} \partial \bar{y}^{-\beta'}} \times \frac{(\mathcal{N}_-)^{[s]} (\overline{\mathcal{N}}_-)^{[s]-1}}{[s][s]!} \times \sum_{p,\, n+m=[s]-p} \Big(\overline{C}_+^{n,p+n+1}(y_-, \bar{y}_- |x) \times \overline{C}_-^{p+m,m}(-y_-, -\bar{y}_- |x) + \overline{C}_+^{p+m,m}(y_-, \bar{y}_- |x) \times \overline{C}_-^{n,p+n+1}(-y_-, -\bar{y}_- |x) \Big) \Big|_{y_- = \bar{y}_- = 0}. \quad (5.48)$$

Projecting out the terms that contain the extra fields $\omega^{[s]-2,[s]+1}$ and $\omega^{[s]+1,[s]-21}$ by respectively contracting the free indices with $y^\alpha y^\alpha$ and $\bar{y}^{\beta'} \bar{y}^{\beta'}$, we obtain the Fang–Fronsdal field equations [28] in AdS_4 with the conformal currents in the right-hand sides.

6. CONCLUSION

In this paper, the unfolded equations for free massless fields of all spins are extended to current interactions. The resulting equations have linear form where the currents are realized as the rank-two linear fields discussed in [7]. More precisely, the construction in [7] deals with conformal currents built from $4d$ massless fields. Correspondingly, in this paper, we describe interactions of massless fields with conformal currents. We have checked in detail how usual current interactions for lower spins as well as their generalization to the HS sector are reproduced. Remarkably, the same system reproduces Yukawa interactions in the sector of spins zero and half.

More precisely, the set of currents that results from the construction in [13] is infinitely degenerate, with most of the currents being exact, describing no charge conservation. However, the infinite set of currents of a given spin contains one member that involves a minimal number of derivatives of the constituent fields and is not exact. In this respect, the set of currents resulting from our construction is analogous to that considered recently in the case of any dimension in [33], which is also infinitely degenerate (however, our construction contains HS currents built from fields of different integer and half-integer spins, while only the HS currents built from a scalar field were considered in [33]). We stress that exact currents may also play a nontrivial role in the interacting theory: the difference is that nontrivial currents (elements of the current cohomology) describe minimal HS interactions, while the exact currents (also known as improvements) describe nonminimal HS interactions of the anomalous magnetic moment type, which may also be important in the full interacting HS theory.

The analysis in this paper is performed in the AdS_4 background. The unfolded machinery makes is technically as simple as that in the Minkowski case. This should be compared with other approaches to the analysis of HS conserved currents in the AdS background [34–37]. (We note that the case of AdS_3 was considered in [38, 39].)

An interesting problem for the future is to see how the results in this paper are reproduced by the full nonlinear system of equations of motion that is known for HS fields both in AdS_4 [4] and in AdS_d [5] (see also reviews [6, 25]). This may help to reach better understanding of the full nonlinear problem and allow interpreting interactions as a linear problem that involves fields that can be interpreted either as free fields in higher dimensions or as currents in AdS_4. It should be noted, however, that to proceed along this direction, it is necessary to extend our results to the case of non-gauge-invariant HS currents built from HS gauge connection one-forms rather than from the gauge-invariant generalized Weyl zero-forms like the generalized Bell–Robinson tensors in [15]. The complication is that currents of this type, like, e.g., the stress tensor built from HS gauge fields, are not gauge invariant, as was pointed out in [40]. In fact, it is this property that leads to peculiarities of the HS interactions [41], which require additional interactions with higher derivatives and a nonzero cosmological constant to restore the gauge invariance [29]. It would be interesting to see how this works within the approach presented in this paper.

One of the conclusions of this paper is that within the unfolded dynamics approach, at least some of the interactions can be interpreted in terms of free fields in higher dimensions. The remarkable feature of the unfolded approach is that it makes it easy to put field theories in different dimensions on the same footing. The only source of nonlinearity comes from the realization of higher-dimensional fields as bilinears in the lower-dimensional ones, as in Eq. (3.15). We note that from this perspective, the results in this paper are somewhat reminiscent of the correspondence between pairs of massless fields in two dimensions and sources of massless fields in four dimensions observed in [42]. It would be interesting to reconsider the analysis in [42] in the framework of the unfolded machinery. Also, it is interesting to extend our analysis to dynamical systems in different dimensions. In particular, in accordance with the results in Ref. [11], $3d$ conformal currents should be identified with $4d$ massless fields and $6d$ conformal currents should be identified with $10d$ conformal fields.

More generally, it is tempting to further elaborate the interpretation of the obtained results in the context of the AdS/CFT correspondence. Moreover, we believe that the further analysis of HS gauge theories within the unfolded approach may help to understand the origin of the remarkable interplay between space–times of different dimensions suggested by the AdS/CFT correspondence [1–3] but going beyond the standard AdS/CFT interpretations of HS theories [43–51]. The results in this paper indicate that HS theories, which involve infinite towers of massless fields associated with infinite-dimensional HS symmetries, suggest that the usual space–time picture we are used to work with results from localization of an infinite dimensional space by virtue of chosen dynamical systems as discussed in Ref. [11]. We also interpret the results in this paper as further evidence in favor of the idea of an infinite chain of dualities that relate the spaces $\mathcal{M}_M$ with different M, as suggested in Ref. [10].

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APPENDIX A

Weyl sector gluing operators

In Sec. 4, we introduced the gluing operators, polynomial in the operators $\mathcal{N}_\pm$ and $\overline{\mathcal{N}}_\pm$ in (4.2). Here, we present the details of the derivation.

The following simple properties of an arbitrary function $\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)$ are used below:

$$\begin{gathered}
\Big[\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm), y^{j\mu}\Big] = y^\mu \frac{\partial}{\partial \mathcal{N}_j}\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm),\\
\Big[\frac{\partial}{\partial y^\mu}, g(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\Big] = \frac{\partial}{\partial \mathcal{N}_j}\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\frac{\partial}{\partial y^{j\mu}},\\
\Big[\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm), \bar{y}^{j\mu'}\Big] = \bar{y}^{\mu'} \frac{\partial}{\partial \overline{\mathcal{N}}_j}\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm),\\
\Big[\frac{\partial}{\partial \bar{y}^{\mu'}}, \mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\Big] = \frac{\partial}{\partial \overline{\mathcal{N}}_j}\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\frac{\partial}{\partial \bar{y}^{j\mu'}},
\end{gathered} \tag{A.1}$$

$$\begin{gathered}
\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\, y^k{}_\alpha F\Big|_{y^\pm=\bar{y}^\pm=0} = \\
= y_\alpha \frac{\partial}{\partial \mathcal{N}_k}\,\mathcal{G}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\, F\Big|_{y^\pm=\bar{y}^\pm=0} \quad \forall F(y^\pm).
\end{gathered} \tag{A.2}$$

For the future convenience, we introduce a set of functions

$$\begin{gathered}
\mathfrak{F}_K{}^{n_+,n_-}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm) = (\mathcal{N}_+)^{n_+}(\mathcal{N}_-)^{n_-} \times \\
\times \sum_{m\geq 0}\frac{(\overline{\mathcal{N}}_+\mathcal{N}_- + \overline{\mathcal{N}}_-\mathcal{N}_+)^m}{m!(m+n_++n_-+K)!},
\end{gathered} \tag{A.3}$$

which have useful properties

$$\begin{gathered}
\frac{\partial}{\partial \overline{\mathcal{N}}_\pm}\mathfrak{F}_K{}^{n_+,n_-} = \mathcal{N}_\mp \mathfrak{F}_{K+1}{}^{n_+,n_-},\\
\Big\{K + \mathcal{N}_A \frac{\partial}{\partial \mathcal{N}_A}\Big\}\mathfrak{F}_K{}^{n_+,n_-} = \mathfrak{F}_{K-1}{}^{n_+,n_-},\\
\Big(\frac{\partial^2}{\partial\mathcal{N}_+\partial\overline{\mathcal{N}}_-} + \frac{\partial^2}{\partial\mathcal{N}_-\partial\overline{\mathcal{N}}_+}\Big)\mathfrak{F}_K{}^{n_+,n_-} = \\
= (K-1)\mathfrak{F}_{K+1}{}^{n_+,n_-} + \mathfrak{F}_K{}^{n_+,n_-}.
\end{gathered} \tag{A.4}$$

We note that the function $\mathfrak{F}^{n_+,n_-}$ used through out the paper coincides with $\mathfrak{F}_1{}^{n_+,n_-}$. Functions (A.3) are related to the regular Bessel functions $I_k(x)$ (see, e. g., [32]) as

$$\frac{\mathfrak{F}_K{}^{n_+,n_-}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)}{(\mathcal{N}_+)^{n_+}(\mathcal{N}_-)^{n_-}} = f_{n_++n_-+K}(\overline{\mathcal{N}}_+\mathcal{N}_- + \overline{\mathcal{N}}_-\mathcal{N}_+),$$

$$f_k(r) = r^{-k/2} I_k(2r^{1/2}).$$

The deformed conformal equations are of the form

$$D^{tw}C + e^{\mu\nu'} G^k_j B^j_{k\mu\nu'} J \Big|_{y^\pm=\bar{y}^\pm=0} = 0, \qquad (A.5)$$

where $B^k_{j\mu\nu'}$ are bilinear in $\partial_{k\alpha}$, $y^{j\alpha}$, $\bar{\partial}_{k\alpha'}$, $\bar{y}^{j\alpha'}$ with $j,k = \{+,-\}$, namely,

$$\begin{aligned} B^k_{j\ \alpha\alpha'} &= y^k{}_\alpha \bar{\partial}_{j\alpha'}, \quad \overline{B}^k_{j\ \alpha\alpha'} = \bar{y}^k{}_{\alpha'} \partial_{j\alpha}, \\ B^{kj}{}_{\alpha\alpha'} &= y^k{}_\alpha \bar{y}^j{}_{\alpha'}, \quad B_{kj\ \alpha\alpha'} = \partial_{k\alpha} \bar{\partial}_{j\alpha'}, \end{aligned} \qquad (A.6)$$

$G_a(\mathcal{N}_\pm, \overline{\mathcal{N}}_\pm)$ are some gluing operators, D^{tw} is the rank-one twisted covariant derivative (2.10), and the rank-two current field $J(y^\pm, \bar{y}^\pm)$ satisfies the current equation (3.19). The system of equations (3.19) decomposes into a set of subsystems associated with different elements of ${}^v\mathfrak{sl}_2$-modules realized by bilinear operators B_a in (A.6).

The consistency condition for Eq. (A.5)

$$\begin{aligned} \Big(H^{\mu\alpha}\varepsilon^{\nu'\beta'} + H^{\nu'\beta'}\varepsilon^{\mu\alpha}\Big)\Big\{ & \big(y_\alpha \bar{y}_{\beta'} + \partial_\alpha \bar{\partial}_{\beta'}\big) G^k_j B^j_{k\mu\nu'} - \\ - G^k_j B^j_{k\mu\nu'} \big(y^+{}_\alpha \bar{y}^-{}_{\beta'} & + y^-{}_\alpha \bar{y}^+{}_{\beta'} + \partial_{-\alpha}\bar{\partial}_{+\beta'} + \\ & + \partial_{+\alpha}\bar{\partial}_{-\beta'}\big)\Big\} J(y^\pm, \bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} = 0 \qquad (A.7) \end{aligned}$$

imposes restrictions on the gluing operators G_a analyzed below. Evidently, Eq. (A.7) decomposes into a set of subsystems characterized by different eigenvalues of the rank-two helicity operator f_0 in (3.21). We begin with the simpler Minkowski case, and then show that the obtained solution also works in AdS_4.

A.1. Minkowski case

We first consider ${}^v\mathfrak{sl}_2$ highest element $B^k_{j\alpha\alpha'}$ in (A.6), which satisfies the relation

$$[f_0, B^k_{j\alpha\alpha'}] = 2B^k_{j\alpha\alpha'}.$$

In this case, the flat limit of Eq. (A.7) gives along with (A.2),

$$\begin{aligned} e^{\mu\alpha'} e^{\alpha\beta'} \Big(\partial_\alpha \bar{\partial}_{\beta'} y_\mu F^j \bar{\partial}_{j\nu'} - y_\mu F^j \bar{\partial}_{j\nu'} \times \\ \times \big(\partial_{-\alpha}\bar{\partial}_{+\beta'} + \partial_{+\alpha}\bar{\partial}_{-\beta'}\big)\Big) \times \\ \times J(y^\pm, \bar{y}^\pm|x)\Big|_{y^\pm=\bar{y}^\pm=0} = 0, \qquad (A.8) \end{aligned}$$

where

$$F^j = \frac{\partial}{\partial \mathcal{N}_i} G^j_i. \qquad (A.9)$$

Hence, by virtue of (A.1), we have

$$\begin{aligned} H^{\mu'\beta'}\Big(\bar{\partial}_{\mu'}\Big\{2 + \mathcal{N}_K \frac{\partial}{\partial \mathcal{N}_K}\Big\} F^j - \\ - \mathcal{N}_+ F^j \bar{\partial}_{-\mu'} - \mathcal{N}_- F^j \bar{\partial}_{+\mu'}\Big)\bar{\partial}_{j\beta'} = 0, \qquad (A.10) \end{aligned}$$

$$\begin{aligned} H^{\mu\alpha} y_\alpha \bar{\partial}_j{}^{\mu'}\Big(\bar{\partial}_{\mu'}\partial_\mu F^j - F^j \times \\ \times \big(\partial_{-\mu}\bar{\partial}_{+\mu'} + \partial_{+\mu}\bar{\partial}_{-\mu'}\big)\Big) = 0. \qquad (A.11) \end{aligned}$$

This gives the following conditions for $F^\pm$ in (A.9):

$$\begin{aligned} &\Big\{2 + \mathcal{N}_K \frac{\partial}{\partial \mathcal{N}_K}\Big\} \frac{\partial F^+}{\partial \overline{\mathcal{N}}_+} - \mathcal{N}_- F^+ = 0, \\ &\Big\{2 + \mathcal{N}_K \frac{\partial}{\partial \mathcal{N}_K}\Big\} \frac{\partial F^-}{\partial \overline{\mathcal{N}}_-} - \mathcal{N}_+ F^- = 0, \\ &\Big\{2 + \mathcal{N}_K \frac{\partial}{\partial \mathcal{N}_K}\Big\}\Big(\frac{\partial F^+}{\partial \overline{\mathcal{N}}_-} + \frac{\partial F^-}{\partial \overline{\mathcal{N}}_+}\Big) - \\ &\qquad -\mathcal{N}_- F^- - \mathcal{N}_+ F^+ = 0, \\ &\qquad \frac{\partial}{\partial \mathcal{N}_-} F^+ + \frac{\partial}{\partial \mathcal{N}_+} F^- = 0, \\ &\Big(\frac{\partial^2}{\partial \mathcal{N}_+ \partial \overline{\mathcal{N}}_-} + \frac{\partial^2}{\partial \mathcal{N}_- \partial \overline{\mathcal{N}}_+} - 1\Big) F^+ = 0, \\ &\Big(\frac{\partial^2}{\partial \mathcal{N}_- \partial \overline{\mathcal{N}}_+} + \frac{\partial^2}{\partial \mathcal{N}_+ \partial \overline{\mathcal{N}}_-} - 1\Big) F^- = 0. \end{aligned} \qquad (A.12)$$

Elementary straightforward analysis shows that $F^\pm$ have form (A.9), i. e.,

$$F^\pm = \frac{\partial}{\partial \mathcal{N}_i} G^\pm_i$$

with

$$\begin{aligned} G^+_+ = -G^-_- &= \sum_{n_+,\, n_- \ge 0} a_{n_+,n_-} \mathfrak{F}_1{}^{n_+,n_-}, \\ G^-_+ &= -G^+_- = 0. \end{aligned} \qquad (A.13)$$

The corresponding deformation, i. e., the second term in the left-hand side of Eq. (A.5), is

$$\begin{aligned} e^{\mu\beta'\beta} \sum_{n_+,\, n_- \ge 0} a_{n_+,n_-} \mathfrak{F}_1{}^{n_+,n_-} \times \\ \times \big(y^+{}_\mu \bar{\partial}_{+\beta'} - y^-{}_\mu \bar{\partial}_{-\beta'}\big) J\Big|_{y^\pm=\bar{y}^\pm=0}, \qquad (A.14) \end{aligned}$$

where a_{n_+,n_-} are arbitrary coefficients. We note that the ambiguity in the coefficients a_{n_+,n_-} is in accordance with the ambiguity of contributions of different spin fields to the currents.

For the complex conjugate $\overline{B^k}_{p\alpha\alpha'}$ satisfying

$$[f_0, \overline{B^k}_{p\alpha\alpha'}] = -2\overline{B^k}_{p\alpha\alpha'},$$

the gluing operators $\overline{G}_a$ are

$$\overline{G}^+_+ = -\overline{G}^-_- = \sum_{\bar n_+,\,\bar n_-\geq 0} \bar a_{\bar n_+,\bar n_-}\overline{\mathfrak{F}}_1{}^{\bar n_+,\bar n_-}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm),$$

$$\overline{G}^-_+ = -\overline{G}^+_- = 0,$$

where $\bar a_{\bar n_+,\bar n_-}$ are arbitrary coefficients and

$$\overline{\mathfrak{F}}_K{}^{\bar n_+,\bar n_-}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm) = (\overline{\mathcal{N}}_+)^{\bar n_+}(\overline{\mathcal{N}}_-)^{\bar n_-}\times \times\sum_{m\geq 0}\frac{(\overline{\mathcal{N}}_+\mathcal{N}_- + \overline{\mathcal{N}}_-\mathcal{N}_+)^m}{m!(m+\bar n_+ + \bar n_- + K)!} \quad \text{(A.15)}$$

is complex conjugate to $\mathfrak{F}_K{}^{n_+,n_-}$ in (A.3). The corresponding deformation is

$$e^{\mu\nu'}\sum_{\bar n_+,\,\bar n_-\geq 0}\bar a_{\bar n_+,\bar n_-}\overline{\mathfrak{F}}_1{}^{\bar n_+,\bar n_-}\times \times\,\bar y^j_{\nu'}\partial_{j\mu}J\Big|_{y^\pm=\bar y^\pm=0}, \quad \text{(A.16)}$$

where we use the notation

$$a^j b_j = a^+b_+ - a^-b_-.$$

We note that the operators $y^j_\mu\bar\partial_{j\beta'}$ and $\bar y^j_{\nu'}\partial_{j\mu}$ in deformations (A.14) and (A.16) are invariant under ${}^h\mathfrak{sl}_2$.

It is also not difficult to see that deformation (A.5) with the remaining B^a in (A.6) satisfying

$$[f_0, B^a] = 0$$

is trivial, i. e., can be removed by a local field redefinition (in other words, it is D^{tw}_{fl}-exact on solutions of the current equation).

A.2. AdS

In the AdS_4 case, the gluing coefficients remain the same as in the Minkowski case. For example, we consider $B^{a\mu}_{\beta'}$ of the form $y^{+\mu}\bar\partial_{+\beta'} - y^{-\mu}\bar\partial_{-\beta'}$ found above. Equation (A.7) then gives

$$\Big(H^{\mu\alpha}\varepsilon^{\mu'\beta'} + H^{\mu'\beta'}\varepsilon^{\mu\alpha}\Big) + +\Big\{\partial_\mu\bar\partial_{\mu'}y_\alpha F^j\bar\partial_{j\beta'} - y_\alpha F^j\bar\partial_{j\beta'}\left(\partial_{-\mu}\bar\partial_{+\mu'} + \partial_{+\mu}\bar\partial_{-\mu'}\right) + + y_\mu\bar y_{\mu'}y_\alpha F^j\bar\partial_{j\beta'} - y_\alpha F^j\bar\partial_{j\beta'}\left(y^+_\mu\,\bar y^-{}_{\mu'} + y^-_{\mu'}\,\bar y^+_{\mu'}\right)\Big\}\times \times J(y^\pm,\bar y^\pm|x)\Big|_{y^\pm=\bar y^\pm=0} = 0. \quad \text{(A.17)}$$

We can see that (A.17) holds if $F^\pm(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)$ satisfy conditions (A.12) and the relation

$$y_\alpha y_\mu\bar y_{\mu'}\Big\{1 - \Big(\frac{\partial}{\partial\overline{\mathcal{N}}_-}\frac{\partial}{\partial\mathcal{N}_+} + \frac{\partial}{\partial\overline{\mathcal{N}}_+}\frac{\partial}{\partial\mathcal{N}_-}\Big)\Big\}F^j\bar\partial_{j\beta'}\times \times J(y^\pm,\bar y^\pm|x)\Big|_{y^\pm=\bar y^\pm=0} - - y_\alpha y_\mu\varepsilon_{\beta'\mu'}\Big\{\frac{\partial}{\partial\mathcal{N}_-}F^+ + \frac{\partial}{\partial\mathcal{N}_+}F^-\Big\}\times \times J(y^\pm,\bar y^\pm|x)\Big|_{y^\pm=\bar y^\pm=0} = 0, \quad \text{(A.18)}$$

which holds by virtue of (A.12). Hence, the deformation of the form (A.14) remains consistent in the AdS_4 case as well. The complex conjugate case is analogous.

Analogously to the Minkowski case, it is not difficult to see that the consistent deformed equations (A.5) with B^a obeying

$$[f_0, B^a] = 0$$

are trivial (D^{tw}-exact) for any current field J.

APPENDIX B

Spin-$s\geq 2$ one-form sector

Since zero-forms contribute to the right-hand sides of Eqs. (2.6), their formal consistency in presence of deformation (4.1) requires an appropriate deformation in the one-form sector,

$$D^{ad}\omega(y,\bar y|x) = \overline{H}^{\alpha'\beta'}\bar\partial_{\alpha'}\bar\partial_{\beta'}\overline{C}(0,\bar y|x) + + H^{\alpha\beta}\partial_\alpha\partial_\beta C(y,0|x) + + \overline{H}^{\alpha'\beta'}\overline{G}_{\alpha'\beta'}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\,\mathcal{I}(y^\pm,\bar y^\pm|x)\big|_{y^\pm=\bar y^\pm=0} + + H^{\alpha\beta}G_{\alpha\beta}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)\,\mathcal{J}(y^\pm,\bar y^\pm|x)\big|_{y^\pm=\bar y^\pm=0} \quad \text{(B.1)}$$

for some gluing operators $G_{\alpha\beta}$ and $\overline{G}_{\alpha'\beta'}$ and current fields $\mathcal{I}$ and $\mathcal{J}$ with $\mathcal{N}_\pm$ and $\overline{\mathcal{N}}_\pm$ defined in (4.2).

Let $s\geq 2$. (The case $s = 3/2$ is special and is considered in Appendix C.)

Since the ${}^h\mathfrak{sl}_2$ in (3.22) acts on current fields $\mathcal{J}$ and $\mathcal{I}$ and hence on the gluing functions, it is convenient to require that $G_{\alpha\beta}$ and $\overline{G}_{\alpha'\beta'}$ be highest vectors with respect to ${}^h\mathfrak{sl}_2$, by setting

$$G_{\alpha\beta} = \partial_{-\alpha}\partial_{-\beta}G^{s-1}(\mathcal{N}_-,\overline{\mathcal{N}}_-), \qquad \overline{G}_{\alpha'\beta'} = \bar\partial_{-\alpha'}\bar\partial_{-\beta'}\overline{G}^{s-1}(\mathcal{N}_-,\overline{\mathcal{N}}_-), \quad \text{(B.2)}$$

where G^{s-1} and $\overline{G}^{s-1}$ are some degree-$2(s-1)$ homogeneous polynomials in $\mathcal{N}_-$ and $\overline{\mathcal{N}}_-$, to match the fact that the one-forms ω are degree-$2(s-1)$ homogeneous polynomials in y and $\bar y$.

Taking the form of the ${}^{h}\mathfrak{sl}_2$ highest-weight deformation in the zero-form sector into account, namely, Eqs. (4.13) and (4.14) with

$$\bar{a}_{m,2s-m} = a_{m,2s-m} = \delta^0_m a_{0,2s},$$

and setting

$$a_{0,2s} = 2s+1$$

for definiteness, we can see that the consistency condition for Eq. (B.1) imposes the following conditions on the current fields J, $\overline{J}$, $\mathcal{J}$, and $\mathcal{I}$:

$$D^{ad}\Big(\overline{H}^{\alpha'\beta'}\overline{G}_{\alpha'\beta'}(\mathcal{N}_-,\overline{\mathcal{N}}_-)\,\mathcal{I}(y^\pm,\bar{y}^\pm|x)\big|_{y^\pm=\bar{y}^\pm=0} + H^{\alpha\beta}G_{\alpha\beta}(\mathcal{N}_-,\overline{\mathcal{N}}_-)\,\mathcal{J}(y^\pm,\bar{y}^\pm|x)\big|_{y^\pm=\bar{y}^\pm=0}\Big) = \frac{2}{(2s-2)!}\mathcal{H}^{\alpha\beta'}\partial_{-\alpha}\overline{\partial}_{-\beta'}\times \times\Big\{(\mathcal{N}_-)^{2s-2}J - (\overline{\mathcal{N}}_-)^{2s-2}\overline{J}\Big\}\Big|_{y^\pm=\bar{y}^\pm=0}. \tag{B.3}$$

Substituting (B.2) in (B.3), and using (3.19) and (5.13) along with the evident identities

$$G(\mathcal{N}_-,\overline{\mathcal{N}}_-)\Big(\partial_{-\gamma}\partial_+{}^{\gamma} - f_-\Big)\mathcal{J}\big|_{y^\pm=\bar{y}^\pm=0} \equiv 0,$$

$$\overline{G}(\mathcal{N}_-,\overline{\mathcal{N}}_-)\Big(\bar{\partial}_{-\gamma'}\bar{\partial}_+{}^{\gamma'} - f_+\Big)\mathcal{I}\big|_{y^\pm=\bar{y}^\pm=0} \equiv 0,$$

where f_- and f_+ are generators of ${}^{h}\mathfrak{sl}_2$, we obtain

$$\lambda\partial_{-\alpha}\bar{\partial}_{-\beta'}\Big(-\overline{\mathcal{N}}_-\frac{\partial}{\partial\mathcal{N}_-}\overline{G}^{s-1}\mathcal{I} + \overline{G}^{s-1}f_+\mathcal{I} + \mathcal{N}_-\frac{\partial}{\partial\overline{\mathcal{N}}_-}G^{s-1}\mathcal{J} - G^{s-1}f_-\mathcal{J}\Big)\big|_{y^\pm=\bar{y}^\pm=0} = \frac{1}{(2s-2)!}\partial_{-\alpha}\overline{\partial}_{-\beta'}\times \times\Big\{(\mathcal{N}_-)^{2s-2}J - (\overline{\mathcal{N}}_-)^{2s-2}\overline{J}\Big\}\Big|_{y^\pm=\bar{y}^\pm=0}. \tag{B.4}$$

This equation can be easily solved by the ansatz

$$\overline{G}^{s-1} = (\overline{\mathcal{N}}_-)^{2s-2}, \quad G^{s-1} = (\mathcal{N}_-)^{2s-2}.$$

As shown in Appendix D, currents of the form

$$J = -\lambda(2s-2)!f_-\mathcal{J}, \quad \overline{J} = -\lambda(2s-2)!f_+\mathcal{I}, \tag{B.5}$$

which solve (B.4), lead to a trivial deformation in the zero-form sector and hence to a trivial deformation in the one-form sector.

The proper strategy is to start with some "seed current field" $\tilde{\mathcal{J}}_{(l)}$ under the conditions

$$f_0\tilde{\mathcal{J}}_{(l)} = 2(l-s)\tilde{\mathcal{J}}_{(l)} \tag{B.6}$$

with some integer l in the interval $2 \le l \le 2s-2$. Setting

$$G^{s-1}\mathcal{J} = G^{s-1}_{(l)}\mathcal{J}_{(l)} = \frac{1}{(l-1)!}\times \times\sum_{k=0}^{l-2}\frac{(\mathcal{N}_-)^{l-k-2}(\overline{\mathcal{N}}_-)^{2s-l+k}}{(2s-l+k)!}(f_-)^k\,\tilde{\mathcal{J}}_{(l)}, \tag{B.7}$$

$$\overline{G}^{s-1}\mathcal{I} = \overline{G}^{s-1}_{(l)}\mathcal{I}_{(l)} = \frac{1}{(2s-l-1)!}\times \times\sum_{k=0}^{(2s-l-2)}\frac{(\mathcal{N}_-)^{l+k}(\overline{\mathcal{N}}_-)^{2s-l-k-2}}{(l+k)!}(f_+)^k\,\tilde{\mathcal{J}}_{(l)} \tag{B.8}$$

in Eq. (B.4), we obtain

$$\lambda\mathcal{H}^{\alpha\beta'}\partial_{-\alpha}\bar{\partial}_{-\beta'}\frac{(\mathcal{N}_-)^{2s-2}}{(2s-2)!(2s-l-1)!}(f_+)^{2s-l-1}\,\tilde{\mathcal{J}}_{(l)} - \frac{(\overline{\mathcal{N}}_-)^{2s-2}}{(2s-2)!(l-1)!}(f_-)^{l-1}\,\tilde{\mathcal{J}}_{(l)}\Big)\big|_{y^\pm=\bar{y}^\pm=0} = \frac{1}{(2s-2)!}\mathcal{H}^{\alpha\beta'}\partial_{-\alpha}\overline{\partial}_{-\beta'}\times \times\Big\{(\mathcal{N}_-)^{2s-2}J - (\overline{\mathcal{N}}_-)^{2s-2}\overline{J}\Big\}\Big|_{y^\pm=\bar{y}^\pm=0}. \tag{B.9}$$

Then

$$J = \lambda\frac{1}{(2s-l-1)!}(f_+)^{2s-l-1}\,\tilde{\mathcal{J}}_{(l)},$$

$$\overline{J} = \lambda\frac{1}{(l-1)!}(f_-)^{l-1}\,\tilde{\mathcal{J}}_{(l)}$$

solve (B.9). The resulting deformed equations are

$$D^{ad}\omega(y,\bar{y}|x) - \overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial\bar{y}^{\alpha'}\partial\bar{y}^{\beta'}}\overline{C}(0,\bar{y}|x) - H^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha\partial y^\beta}C(y,0|x) = H^{\alpha\beta}\frac{\partial}{\partial y_-{}^{\alpha}}\frac{\partial}{\partial y_-{}^{\beta}}\times \times\sum_{k=0}^{l-2}\frac{(\mathcal{N}_-)^{l-k-2}(\overline{\mathcal{N}}_-)^{2s-l+k}}{(2s-l+k)!(l-1)!}\times \times(f_-)^k\,\tilde{\mathcal{J}}_{(l)}\big|_{y^\pm=\bar{y}^\pm=0} + \overline{H}^{\alpha'\beta'}\frac{\partial}{\partial\bar{y}_-{}^{\alpha'}}\frac{\partial}{\partial\bar{y}_-{}^{\beta'}}\times \times\sum_{k=0}^{(2s-l-2)}\frac{(\mathcal{N}_-)^{l+k}(\overline{\mathcal{N}}_-)^{2s-l-k-2}}{(l+k)!(2s-l-1)!}\times \times(f_+)^k\,\tilde{\mathcal{J}}_{(l)}\big|_{y^\pm=\bar{y}^\pm=0} \tag{B.10}$$

and

$$\begin{gathered}D^{tw}C(y,\bar{y}|x) + \\ + e^{\mu\beta'}\mathfrak{F}_1{}^{0,2s}y^j_\mu\bar{\partial}_{j\beta'}\frac{\lambda(2s+1)}{(2s-l-1)!}\times \\ \times (f_+)^{2s-l-1}\,\tilde{\mathcal{J}}_{(l)}\big|_{y^\pm=\bar{y}^\pm=0} = 0, \\ D^{tw}\overline{C}(y,\bar{y}|x) + e^{\mu\beta'}\overline{\mathfrak{F}}_1{}^{0,2s}\partial_{j\mu}\bar{y}^j{}_{\beta'}\times \\ \times\frac{\lambda(2s+1)}{(l-1)!}(f_-)^{l-1}\,\tilde{\mathcal{J}}_{(l)}\big|_{y^\pm=\bar{y}^\pm=0} = 0,\end{gathered} \qquad (B.11)$$

where $\mathfrak{F}_1{}^{0,2s}$ is given by (A.3) with $n_+ = 0$, $n_- = 2s$.

As shown in Appendix D, the final result is independent of the choice of $\tilde{\mathcal{J}}_{(l)}$. Namely, up to D^{ad}-exact one-forms and D^{tw}-exact zero-forms, the final result remains the same upon the identification

$$\tilde{\mathcal{J}}_{(l+1)} = \frac{1}{(2s-l-1)}f_+\tilde{\mathcal{J}}_{(l)}.$$

On the other hand, in the flat limit, this procedure works properly only for $|l-s| \le \frac{1}{2}$. For this reason, formulas (4.20) and (4.22) were presented for $|l-s| \le 1/2$ with the following identifications of the current fields $\mathcal{J}_{\mathrm{h},s}$ in (4.19):

$$\begin{aligned}\mathcal{J}_{0,s} &= \frac{\tilde{\mathcal{J}}_{(s)}}{(s-1)!} \quad \text{for integer } s, \\ \mathcal{J}_{\pm1,s} &= \frac{\tilde{\mathcal{J}}_{(s\pm\frac{1}{2})}}{(s-\frac{1}{2})!} \quad \text{for half-integer } s.\end{aligned} \qquad (B.12)$$

We note that for any $G(\mathcal{N}_-,\overline{\mathcal{N}}_-)$ with $\mathcal{N}_-$ and $\overline{\mathcal{N}}_-$ defined in (4.2) and an arbitrary integer $m \ge 0$

$$\begin{gathered}\Big(\mathrm{ad}^m_{g_-}\Big(\partial_{-\alpha}\partial_{-\beta}G(\mathcal{N}_-,\overline{\mathcal{N}}_-)\,(f_-)^k\Big) - \\ - \partial_{-\alpha}\partial_{-\beta}G(\mathcal{N}_-,\overline{\mathcal{N}}_-)\,(f_-)^k\big(-g_-\big)^m\Big)\times \\ \times\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0} = 0 \qquad (B.13)\end{gathered}$$

because g_- (3.22) is zero at $y^\pm = \bar{y}^\pm = 0$ and $[f_a, g_b] = 0$ by virtue of (3.21), (3.22) (recall that $\mathrm{ad}_x(y) = [x,y]$). The complex conjugate formula is analogous.

Since $G_{\alpha\beta}$ and $\overline{G}_{\alpha'\beta'}$ in (B.2) are highest ${}^h\mathfrak{sl}_2$-vectors, $\mathrm{ad}^m_{g_-}(G_{\alpha\beta})$ and $\mathrm{ad}^m_{g_-}(\overline{G}_{\alpha'\beta'})$ in the zero-form sector reproduce the current deformations of the dynamical equations, associated with arbitrary gluing coefficients in (4.3) and (4.4).

As an application of this mechanism, we observe that Eq. (B.13) implies that the deformation

$$\begin{gathered}D^{ad}\omega(y,\bar{y}|x) - \overline{H}^{\alpha'\beta'}\bar{\partial}\alpha'\bar{\partial}\beta'\overline{C}(0,\bar{y}|x) - \\ - H^{\alpha\beta}\partial_\alpha\partial_\beta C(y,0|x) = (-1)^m H^{\alpha\beta}\times \\ \times\mathrm{ad}^m_{g_-}\left(\partial_{-\alpha}\partial_{-\beta}\sum_{k=0}^{s-2}\frac{(\mathcal{N}_-)^{s-k-2}\,(\overline{\mathcal{N}}_-)^{s+k}}{(s+k)!}\right)\times \\ \times(f_-)^k\,\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0} + (-1)^m\overline{H}^{\alpha'\beta'}\times \\ \times\mathrm{ad}^m_{g_-}\left(\bar{\partial}_{-\alpha'}\bar{\partial}_{-\beta'}\sum_{k=0}^{s-2}\frac{(\mathcal{N}_-)^{s+k}\,(\overline{\mathcal{N}}_-)^{s-k-2}}{(s+k)!}\right)\times \\ \times(f_+)^k\,\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0}\end{gathered} \qquad (B.14)$$

is consistent for any $m \ge 0$. By virtue of (4.21), the associated deformations in the zero-form sector are

$$\begin{gathered}D^{tw}C(y,\bar{y}|x) + \lambda(2s+1)e^{\alpha\beta'}\mathfrak{F}^{0,2s}y^j{}_\alpha\bar{\partial}_{j\beta'}\times \\ \times\Big(f_+\Big)^{s-1}(g_-)^m\,\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0} = 0,\end{gathered}$$

$$\begin{gathered}D^{tw}\overline{C}(y,\bar{y}|x) + \lambda(2s+1)e^{\alpha\beta'}\overline{\mathfrak{F}}^{0,2s}\partial_{j\alpha}\bar{y}^j{}_{\beta'}\times \\ \times\Big(f_-\Big)^{s-1}(g_-)^m\,\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0} = 0.\end{gathered}$$

Since

$$\overline{g_\pm} = -g_\pm,$$

the reality conditions require considering the horizontal algebra $\mathfrak{sl}_2$ spanned by

$$g_+ =: ig_+, \quad g_- =: -ig_-, \quad g_0.$$

Therefore, according to (4.9) and (4.11), the deformed equations in the zero-form sector can be rewritten as

$$\begin{gathered}D^{tw}C(y,\bar{y}|x) + e^{\alpha\beta'}\frac{\lambda(-i)^m\,(2s+1)!}{m!}\times \\ \times\mathfrak{F}^{m,2s-m}y^j{}_\alpha\bar{\partial}_{j\beta'}\Big(f_+\Big)^{s-1}\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0} = 0, \\ D^{tw}\overline{C}(y,\bar{y}|x) + e^{\alpha\beta'}\frac{\lambda(i)^m\,(2s+1)!}{m!}\times \\ \times\overline{\mathfrak{F}}^{m,2s-m}\partial_{j\alpha}\bar{y}^j{}_{\beta'}\Big(f_-\Big)^{s-1}\mathcal{J}_{0,s}\big|_{y^\pm=\bar{y}^\pm=0} = 0,\end{gathered} \qquad (B.15)$$

which gives the general result that all zero-form gluing operators (4.3) and (4.4) are relevant, which allows us to conclude that formulas (B.14) contain all possible nontrivial current deformations of integer-spin fields in the one-form sector.

The case of half-integer spins is analogous.

APPENDIX C

Spin-3/2 one-form sector

The case $s = 3/2$ is special. We seek for solution of (B.3) in the form

$$\mathcal{J} = \tilde{\mathcal{J}}_{(1)}, \quad \mathcal{I} = \tilde{\mathcal{J}}_{(-1)}, \quad \tilde{\mathcal{J}}_{(1)} = \overline{\tilde{\mathcal{J}}_{(-1)}}, \qquad \text{(C.1)}$$

where

$$f_0 \tilde{\mathcal{J}}_{(\pm 1)} = \pm \tilde{\mathcal{J}}_{(\pm 1)}.$$

Setting

$$G^{1/2}_{\alpha\beta} = \partial_{-\alpha}\partial_{-\beta}\overline{\mathcal{N}}_-, \quad \overline{G}^{1/2}_{\alpha'\beta'} = \bar{\partial}_{-\alpha'}\bar{\partial}_{-\beta'}\mathcal{N}_-, \qquad \text{(C.2)}$$

and substituting (C.1) in (B.4), we obtain

$$\begin{aligned} \lambda\partial_{-\alpha}\bar{\partial}_{-\beta'}\Big(-\overline{\mathcal{N}}_-\tilde{\mathcal{J}}_{(-1)} + \mathcal{N}_- f_+ \tilde{\mathcal{J}}_{(-1)} + \\ + \mathcal{N}_-\tilde{\mathcal{J}}_{(1)} - \overline{\mathcal{N}}_- f_- \tilde{\mathcal{J}}_{(1)}\Big)\Big|_{y^\pm=\bar{y}^\pm=0} = \\ = \partial_{-\alpha}\overline{\partial}_{-\beta'}\left\{\mathcal{N}_- J - \overline{\mathcal{N}}_-\overline{J}\right\}\Big|_{y^\pm=\bar{y}^\pm=0}. \end{aligned} \qquad \text{(C.3)}$$

As a result, the expressions

$$J = \lambda\tilde{\mathcal{J}}_{(1)} + \lambda f_+ \tilde{\mathcal{J}}_{(-1)}, \quad \overline{J} = \lambda\tilde{\mathcal{J}}_{(-1)} + \lambda f_- \tilde{\mathcal{J}}_{(1)} \qquad \text{(C.4)}$$

solve Eq. (C.3) and the deformed equation is

$$\begin{aligned} D^{ad}\omega(y,\bar{y}|x) = \overline{H}^{\alpha'\beta'}\bar{\partial}_{\alpha'}\bar{\partial}_{\beta'}\overline{C}(0,\bar{y}|x) + \\ + H^{\alpha\beta}\partial_\alpha\partial_\beta C(y,0|x) + \\ + H^{\alpha\beta}\partial_{-\alpha}\partial_{-\beta}\overline{\mathcal{N}}_- \times \\ \times \tilde{\mathcal{J}}_{(1)}\big|_{y^\pm=\bar{y}^\pm=0} + \\ + \overline{H}^{\alpha'\beta'}\bar{\partial}_{-\alpha'}\bar{\partial}_{-\beta'}\mathcal{N}_-\tilde{\mathcal{J}}_{(-1)}\big|_{y^\pm=\bar{y}^\pm=0}. \end{aligned} \qquad \text{(C.5)}$$

This result coincides with (4.22) at $s = 3/2$ under the convention that all terms containing $\sum_{k=0}^{-1}(\ldots)$ or $\sum_{k=2}^{1}(\ldots)$ are zero.

APPENDIX D

D.1. Trivial gluings

Here, we identify a class of currents that upon substitution in Eqs. (4.1) do not lead to a nontrivial deformation of the massless field equations, being removable by a local field redefinition. Also, deformation (4.20) in the one-form sector is shown to be insensitive to a particular choice of the seed current field $\tilde{\mathcal{J}}_{(l)}$ in (B.6).

D.2. Trivial gluings in the zero-form sector

Using the relations

$$\begin{aligned} \mathcal{N}_+\partial_{-\alpha} - \mathcal{N}_-\partial_{+\alpha} = y^\gamma\partial_{+\gamma}\partial_{-\alpha} - y^\gamma\partial_{-\gamma}\partial_{+\alpha} = \\ = y^\gamma\varepsilon_{\gamma\alpha}\partial_{+\beta}\partial_-{}^\beta = y_\alpha\,\partial_{+\beta}\partial_-{}^\beta \end{aligned}$$

and taking properties (A.4) of $\mathfrak{F}_K{}^{n_+,n_-}(\mathcal{N}_\pm,\overline{\mathcal{N}}_\pm)$ into account, for any Minkowski current field J_{fl}, we obtain

$$\begin{aligned} D^{tw}_{fl}\mathfrak{F}_0{}^{n_+,n_-}\,J_{fl}\Big|_{y^\pm=\bar{y}^\pm=0} = \\ = -e^{\mu\beta'}\mathfrak{F}_1{}^{n_+,n_-}\left(y^+{}_\mu\,\bar{\partial}_{+\beta'} - y^-{}_\mu\,\bar{\partial}_{-\beta'}\right)f_{-fl}J_{fl}\Big|_{y^\pm=\bar{y}^\pm=0}, \end{aligned}$$

where

$$f_{-fl} = -\partial_{+\gamma}\partial_-{}^\gamma$$

(see (3.28)). Analogously, for any AdS current field J,

$$\begin{aligned} \lambda^{-1}D^{tw}\mathfrak{F}_0{}^{n_+,n_-}\,J\Big|_{y^\pm=\bar{y}^\pm=0} = -e^{\mu\beta'}\mathfrak{F}_1{}^{n_+,n_-}\times \\ \times\left(y^+{}_\mu\,\bar{\partial}_{+\beta'} - y^-{}_\mu\,\bar{\partial}_{-\beta'}\right)f_- J\Big|_{y^\pm=\bar{y}^\pm=0}, \end{aligned} \qquad \text{(D.1)}$$

where

$$f_- = -\partial_{+\gamma}\partial_-{}^\gamma + \bar{y}^{+\gamma'}\bar{y}^-_{\gamma'}$$

(see (3.21)).

Therefore, the equation

$$\begin{aligned} D^{tw}C(y,\bar{y}|x) + e^{\mu\beta'}\mathfrak{F}_1{}^{n_+,n_-}\times \\ \times\left(y^+{}_\mu\bar{\partial}_{+\beta'} - y^-{}_\mu\bar{\partial}_{-\beta'}\right)f_- J\Big|_{y^\pm=\bar{y}^\pm=0} = 0 \end{aligned} \qquad \text{(D.2)}$$

follows from a local field redefinition of the twisted equation

$$\begin{aligned} D^{tw}(C(y,\bar{y}|x) - \lambda^{-1}\mathfrak{F}_0{}^{n_+,n_-}\times \\ \times J(y^\pm,\bar{y}^\pm|x))\Big|_{y^\pm=\bar{y}^\pm=0} = 0. \end{aligned} \qquad \text{(D.3)}$$

The same is true in the flat limit. Complex conjugate formulas are analogous.

D.3. Trivial gluings in the one-form sector

We let $\Delta_{s,l}(J_{(l)})$ denote the deformation term in the right-hand side of (B.10) and show that the deformation

$$\begin{aligned} D^{ad}\omega(y,\bar{y}|x) - \overline{H}^{\alpha'\beta'}\frac{\partial^2}{\partial\bar{y}^{\alpha'}\partial\bar{y}^{\beta'}}\overline{C}(0,\bar{y}|x) - \\ - H^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha\partial y^\beta}C(y,0|x) = \\ = \Delta_{s,l+1}\left(f_+\tilde{\mathcal{J}}_{(l)}\right) - (2s-l-1)\Delta_{s,l}\left(\tilde{\mathcal{J}}_{(l)}\right), \end{aligned} \qquad \text{(D.4)}$$

where $\tilde{\mathcal{J}}_{(l)}$ with $l \geq 2$ satisfies Eq. (B.6), is trivial. We consider

$$\Omega = \lambda^{-1} e^{\alpha\beta'} \partial_{-\alpha} \bar{\partial}_{-\beta'} \times \times \sum_{k=-1}^{l-2} \frac{(\mathcal{N}_-)^{l-k-2} (\overline{\mathcal{N}}_-)^{2s-l+k}}{(2s-l+k)!(l)!} (f_-)^{k+1} \tilde{\mathcal{J}}_{(l)}.$$

Straightforwardly, we can show that

$$D^{ad}\Omega - \Delta_{s,l+1} \left(f_+ \tilde{\mathcal{J}}_{(l)} \right) = -(2s-l-1)\Delta_{s,l} \left(\tilde{\mathcal{J}}_{(l)} \right) + + \overline{H}^{\alpha'\beta'} \bar{\partial}_{-\alpha'} \bar{\partial}_{-\beta'} \frac{(\overline{\mathcal{N}}_-)^{2s-2}}{(2s-2)!(l)!} (f_-)^l \times \times \tilde{\mathcal{J}}_{(l)} \big|_{y^\pm = \bar{y}^\pm = 0}. \quad \text{(D.5)}$$

Proceeding as in Appendix B, we can see that the corresponding deformation in the zero-form sector is proportional to

$$\overline{\mathfrak{F}}_0{}^{0,2s} f_+ (f_-)^l \mathcal{J}_{(l)} \big|_{y^\pm = \bar{y}^\pm = 0}. \quad \text{(D.6)}$$

Since $l > 1$, Eq. (D.6) can be rewritten as

$$\overline{\mathfrak{F}}_0{}^{0,2s} f_- \widetilde{\mathcal{J}_{(l)}} \big|_{y^\pm = \bar{y}^\pm = 0} \quad \text{(D.7)}$$

for some current field $\widetilde{\mathcal{J}_{(l)}}$. By virtue of Eq. (D.1), this implies that zero-form deformation (D.6) is trivial, resulting from a local field redefinition. It remains to observe that the one-form deformation is indeed trivial by virtue of (D.5).

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