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POLARIZATION EFFECTS IN RADIATIVE DECAY OF A POLARIZED τ LEPTON

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The polarization effects in the one-meson radiative decay of a polarized τ lepton, $\tau^- \to \pi^- \gamma \nu_\tau$, are investigated. The inner bremsstrahlung and structural amplitudes are taken into account. The asymmetry of the differential decay width caused by the τ-lepton polarization and the Stokes parameters of the emitted photon itself are calculated depending on the polarization of the decaying τ lepton. These physical quantities are estimated numerically for an arbitrary direction of the τ lepton polarization 3-vector in the rest frame. The vector and axial-vector form factors describing the structure-dependent part of the decay amplitude are determined using the chiral effective theory with resonances (RχT).

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1. INTRODUCTION

As is known, the τ lepton is the only existing lepton that, due to its large mass, can decay into final states containing hadrons. The energy region of these decays corresponds to the hadron dynamics described by the nonperturbative QCD. Since the complete theory of nonperturbative QCD is absent at present, the phenomena in this energy region are described using various phenomenological approaches. To test different theoretical models, it is important to experimentally investigate the hadronization processes of weak currents. The semileptonic τ-lepton decays are very suitable for such investigations because the leptonic weak interaction is well understood in the Standard Model (SM). A review of the present status of τ physics can be found in Ref. [1].

In the last decade, experimental investigations of the τ lepton decays have been strongly extended due to the construction of the B-factories (BaBar, Belle) with a very high luminosity $L \approx 10^{34}$ cm$^{-2} \cdot$ s^{-1} [2]. At present, experiments at the B-factories led to the accumulation of the data sets of more than 10^9 τ-lepton pairs [3]. Interesting results obtained at the B-factories revived the plans for constructing new facilities such as SuperKEKB (Japan) and Super c–τ (Russia) [2, 4, 5]. These projects will use a new technique to collide the electron–positron beams, which permits increasing the existing luminosity by one or two orders of magnitude. The designed luminosity is $L \approx (1-2) \cdot 10^{35}$ cm$^{-2} \cdot$ s^{-1} for the Super c–τ and $L \approx 10^{36}$ cm$^{-2} \cdot$ s^{-1} for the SuperKEKB [2]. Besides, the Super c–τ and SuperKEKB factories can have a longitudinally polarized electron beam with the polarization degree of more than 80 %, which guarantees production of polarized τ leptons.

This very high luminosity of the planned τ factories will allow performing precise measurements of various decays of the τ lepton and hence searching for the manifestations of the new physics beyond the SM, such as the lepton flavor violation, CP/T violation in the lepton sector, and so on.

The simplest semileptonic τ-lepton decay is $\tau^- \to \pi^-(K^-)\nu_\tau$, but in this case, the hadronization of the weak currents is described by form factors at a fixed value of the momentum transfer squared t (the difference of the τ^- and ν_τ 4-momenta squared). The dependence of the form factors on this variable can be determined, in principle, in the transition $W \to \pi(K)\gamma$, where t is the squared invariant mass of the $\pi(K)$–γ system. This transition can be investigated in the τ-lepton radiative decay $\tau^- \to \pi^-(K^-)\nu_\tau\gamma$.

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The one-pseudoscalar-meson radiative τ-lepton decays have been investigated in a number of papers [4–9]. The authors of Ref. [6] obtained an expression for the double-differential decay rate for the $\tau^- \to \nu_\tau \pi^- \gamma$ decay in terms of the vector $v(t)$ and axial-vector $a(t)$ form factors. The numerical estimates were done for real parameterizations of these form factors using the vector-meson dominance approach. But the assumption that the form factors are real functions is not generally true because these form factors are complex functions in the time-like region of the momentum transfer squared, which is the case for the considered decay.

The author of Ref. [7] has studied the radiative decays $\tau^- \to \nu_\tau \pi^- \gamma$ and $\tau^- \to \nu_\tau \rho^- \gamma$, obtained analytic formulas for the differential decay rates, and evaluated them assuming that the form factors are constant. The authors of Ref. [8] have studied the decays $\tau^- \to \nu_\tau \pi^-(K^-)\gamma$. They obtained the photon energy spectrum, the meson–photon invariant mass distribution, and the integrated rates. The inner bremsstrahlung contribution to the decay rate contains infrared divergences and that is why the integrated decay rates must depend on the photon energy cut-off (or the meson–photon invariant mass cut-off). For the photon energy cut-off 100 MeV, the integrated decay rates $R = \Gamma(\tau \to \nu_\tau \pi \gamma)/\Gamma(\tau \to \nu_\tau \pi) = 1.4 \cdot 10^{-2}$ [6] and $R = 1.0 \cdot 10^{-2}$ [8] were obtained. We note that the leptonic radiative decay of the τ lepton $\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau \gamma$ was measured with the branching ratio $3.6 \cdot 10^{-3}$ [10]. Hence, we can expect that the one-pseudoscalar-meson radiative τ-lepton decay $\tau^- \to \nu_\tau \pi^- \gamma$ can also be measured experimentally, because theory predicts the value for its branching ratio of the same order as for the $\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau \gamma$ decay.

Some polarization observables in the decay $\tau^- \to \nu_\tau \pi^- \gamma$ were considered in Ref. [9]. The general expressions for the Stokes parameters of the produced photon have been calculated. The influence of the possible anomalous magnetic moment of the τ lepton and the existence of excited neutrinos on the matrix element of this decay are briefly discussed. The authors showed that a measurement of the dependence of the differential decay rate on the photon energy (at a fixed sum of the photon and pion energies) allows determining the moduli and phases of the form factors as functions of the variable t.

The τ-lepton radiative decays $\tau^- \to \nu_\tau \pi^-(K^-)\gamma$ were also studied in Refs. [11, 12] in the case of unpolarized particles. The light front quark model was used to evaluate the form factors $v(t)$ and $a(t)$ describing the structure-dependent contribution to these decays [11]. In the SM, the decay width was found to be

$$\Gamma = 1.62 \cdot 10^{-2} (3.86 \cdot 10^{-3}) \Gamma(\tau \to \nu\pi)$$

for the photon energy cut-off 50 (400) MeV. The same decays were studied in Ref. [12]. The photon energy spectrum, the pion–photon invariant mass distribution, and the integrated decay rate were calculated without free parameters and the authors obtained the decay width $1.46 \cdot 10^{-2} (2.7 \cdot 10^{-3}) \Gamma(\tau \to \nu\pi)$ for the same cut-off conditions.

The τ-lepton decay in the case of a virtual photon that converted into a lepton–antilepton pair was investigated in Refs. [13, 14]. This decay was not measured up to now, but the cross channel (namely, $\pi^+ \to e^+ \nu_e e^+ e^-$) has already been measured [15, 16]. This decay and the decay $\tau \to \pi l \bar{l} \nu_\tau$ probe the transition $W^* \to \pi\gamma^*$, where both bosons (W and the photon) are virtual. These decays complement the decay we consider in this paper, which can be a source of information about the transition $W^* \to \pi\gamma$. The vector and axial-vector form factors are functions of two variables (instead of one as in our case) due to the virtuality of the photon, and a third form factor appears in this case. The authors of [13, 14] calculated the branching ratios and di-lepton invariant mass spectra. They predicted that the process with $l = e^-$ should be measured soon at B-factories.

Because the SuperKEKB and Super c–τ factories plan to have a longitudinally polarized electron beam with the polarization degree about 80 %, it is worthwhile to investigate the effects caused by the polarization of the τ lepton. In this paper, we investigate the polarization effects in the one-meson radiative decay of the τ lepton, $\tau^- \to \pi^- \gamma \nu_\tau$. The decay polarization asymmetry and the Stokes parameters of the emitted photon itself are calculated for a polarized τ lepton. These observables are estimated numerically for an arbitrary polarization of the τ lepton.

The vector and axial-vector form factors (which are of theoretical and experimental interest), describing the structure-dependent part of the decay amplitude, are determined in the framework of the chiral effective theory with resonances (RχT) [17, 18]. The RχT is an extension of the chiral perturbation theory to the region of energies around 1 GeV, which explicitly includes the meson resonances. The corresponding Lagrangian contains a few free parameters, or coupling constants, and at the same time has a good predictive power. This approach has further theoretical developments, e. g., in Refs. [19, 20], and applications to various aspects of the meson phenomenology (see, e. g., review [21]). Here, we mention earlier studies of the e^+e^- annihilation to a pair of pseudoscalar mesons with final-state radiation [22], radiative decays with light scalar mesons [23], and two-photon form factors of the π^0, η, and η' mesons and three-pion of τ^- lepton [24].

This paper is organized as follows. In Sec. 2, the matrix element of the decay $\tau^- \to \nu_\tau \pi^- \gamma$ is considered, the phase-space factor of the final particles is introduced for an unpolarized and polarized τ lepton, and Stokes parameters and spin-correlation parameters of the photon are defined. This is done with the help of the current tensor $T^{\mu\nu}$ and two unit space-like orthogonal 4-vectors that describe polarization states of the photon and which we express via particle 4-momenta. In Sec. 3, the current tensor is calculated in terms of the particle 4-momenta and the τ lepton polarization 4-vector. In Sec. 4, we describe the chosen model for the vector and axial-vector form factors that enter the structural part of the decay amplitude. In Sec. 5, results of some analytic and numerical calculations are presented and illustrated. Section 6 contains a discussion and conclusion. In Appendix A, the RχT formalism is briefly reviewed. In Appendix B, we consider a polarization of the τ^- lepton in the annihilation process $e^+e^- \to \tau^+\tau^-$ near the threshold for a longitudinally polarized electron.

2. GENERAL FORMALISM

The main goal of our study is the investigation of polarization effects in the radiative semileptonic decay of a polarized τ lepton,

$$\tau^-(p) \to \nu_\tau(p') + \pi^-(q) + \gamma(k). \qquad (1)$$

2.1. Amplitude and decay width

The corresponding Feynman diagrams for the decay amplitude are shown in Fig. 1. Pole diagrams a and b describe the inner bremsstrahlung (IB) by charged particles in the point-like approximation; diagram c describes the so-called structural radiation.

Thus, we have [8, 25]

$$M_\gamma = M_{IB} + M_R.$$

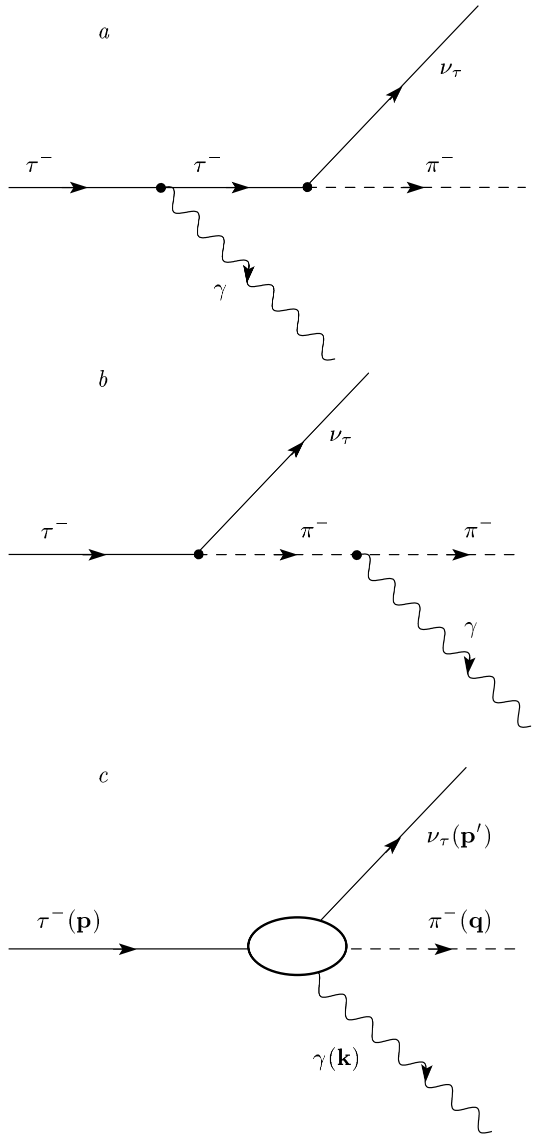

Fig. 1. Feynman diagrams for the radiative $\tau^- \to \pi^- + \nu_\tau + \gamma$ decay. Diagrams a and b correspond to the so-called structure-independent inner bremsstrahlung, for which it is assumed that the pion is a point-like particle. Diagram c represents the contribution of the structure-dependent part and is parameterized in terms of the vector and axial-vector form factors

The IB piece, in the case of a real photon ($k^2 = 0$), coincides with its so-called "contact limit" value and is given by

$$iM_{IB} = ZM\bar{u}(p')(1+\gamma_5) \times \times \left[\frac{\hat{k}\gamma^\mu}{2(kp)} + \frac{Ne_1^\mu}{(kp)(kq)}\right] u(p)\varepsilon_\mu^*(k), \qquad (2)$$

where the dimensional factor Z incorporates all constants: $Z = eG_F V_{ud} F_\pi$, M is the τ-lepton mass, and $\varepsilon_\mu(k)$ is the photon polarization 4-vector. Here, $e^2/4\pi = \alpha = 1/137$, $G_F = 1.166 \cdot 10^{-5}$ GeV^{-2} is the Fermi constant of the weak interactions, $V_{ud} = 0.9742$ is the corresponding element of the CKM-matrix [26], and $F_\pi = 92.42$ MeV is the constant that determines the decay $\pi^- \to \mu^- \bar{\nu}_\mu$. The remaining notation is

$$e_1^\mu = \frac{1}{N}\big[(pk)q^\mu - (qk)p^\mu\big], \quad (e_1k) = 0, \quad e_1^2 = -1,$$
$$N^2 = 2(qp)(pk)(qk) - M^2(qk)^2 - m^2(pk)^2,$$

where m is the pion mass.

The structure radiation in decay (1) arises due to the possibility of the virtual radiative transition

$$W^- \to \pi^- + \gamma.$$

We write the corresponding amplitude in a standard form in terms of two complex form factors, a vector $v(t)$ and an axial vector $a(t)$ [12, 25],

$$iM_R = \frac{Z}{M^2}\bar{u}(p')(1+\gamma_5)\Big\{i\gamma_\alpha(\alpha\mu kq)v(t) - \\ - \big[\gamma^\mu(qk) - q^\mu\hat{k}\big]a(t)\Big\}u(p)\varepsilon_\mu^*(k), \quad (3)$$

where $t = (k+q)^2$ and

$$(\alpha\mu kq) = \epsilon^{\alpha\mu\nu\rho}k_\nu q_\rho, \quad \epsilon^{0123} = +1,$$
$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \mathrm{Tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda = -4i\epsilon^{\mu\nu\rho\lambda}.$$

We can see that both matrix elements, M_{IB} and M_R, satisfy the gauge invariance condition, and it is valid for M_R for any choice of form factors.

The form factors play an important role in the low-energy hadron phenomenology. However, the experimental values of $v(0)$ and $a(0)$ have uncertainties in both absolute values and signs [10]. Of course, this complicates a search for the signals of the new physics beyond the SM in future experiments with high statistics at τ-factories [27, 28].

To determine the vector and axial-vector form factors, we use the model based on the resonance chiral theory [17]. A brief physical description of the theoretical approach to this problem is given in Appendix A and Sec. 4. In accordance with the results of the theoretical model used, we can write the form factors in the form

$$a(t) = -f_A(t)\frac{M^2}{\sqrt{2}mF_\pi}, \quad v(t) = -f_V(t)\frac{M^2}{\sqrt{2}mF_\pi},$$

where $f_A(t)$ and $f_V(t)$ are defined in Sec. 4.

We choose the normalization such that the differential width of decay (1), in terms of the matrix element M_γ, has the following form in the τ-lepton rest frame:

$$d\Gamma = \frac{1}{4M(2\pi)^5}|M_\gamma|^2\frac{d^3k}{2\omega}\frac{d^3q}{2\epsilon}\delta(p'^2), \quad (4)$$

where ω and ϵ are the energies of the photon and π meson. When writing $|M_\gamma|^2$, we have to use the respective relations

$$u(p)\bar{u}(p) = \hat{p} + M, \quad u(p)\bar{u}(p) = (\hat{p}+M)(1+\gamma_5\hat{S})$$

for unpolarized and polarized τ-lepton decays. Here, S is the τ-lepton polarization 4-vector.

2.2. Phase space factor

It is convenient to analyze events of τ decay in its rest frame. In this system, in the unpolarized case, $|M_\gamma|^2$ depends only on the photon and pion energies ω and ϵ. Then the phase space factor for the unpolarized τ can be written as [9]

$$d\Phi = \frac{d^3k}{2\omega}\frac{d^3q}{2\epsilon}\delta(p'^2) = \pi^2 d\omega\, d\epsilon, \quad (5)$$

and the range of the energies is defined by the inequalities

$$\begin{gathered} \frac{M^2+m^2-2M\epsilon}{2(M-\epsilon+|\mathbf{q}|)} \le \omega \le \frac{M^2+m^2-2M\epsilon}{2(M-\epsilon-|\mathbf{q}|)}, \\ m \le \epsilon \le \frac{M^2+m^2}{2M}, \\ \frac{M^2+m^2+4\omega(\omega-M)}{2(M-2\omega)} \le \epsilon \le \frac{M^2+m^2}{2M}, \\ 0 \le \omega \le \frac{M^2-m^2}{2M}. \end{gathered} \quad (6)$$

Because the τ radiative decay amplitude depends on the invariant variable $t = (k+q)^2 = M(2\epsilon + 2\omega - M)$ via the vector and axial-vector form factors in amplitude (3), we can integrate the differential width with respect to ϵ (or ω) at fixed values of t to investigate these form factors. This can be done noting that

$$d\epsilon d\omega = \frac{1}{2M}d\epsilon\, dt = \frac{1}{2M}d\omega\, dt$$

and

$$\begin{gathered} \frac{t^2+m^2M^2}{2Mt} \le \epsilon \le \frac{M^2+m^2}{2M}, \\ \frac{t-m^2}{2M} \le \omega \le \frac{M(t-m^2)}{2t}, \quad m^2 \le t \le M^2. \end{gathered} \quad (7)$$

The integration regions for the variables (ϵ, ω), (ϵ, t), and (ω, t) are shown in Fig. 2.

In the polarized case, we have an additional independent 4-vector S. In the τ-lepton rest frame, $S = (0, \mathbf{n})$. We define a coordinate system with the z axis directed along the vector $\mathbf{n}$ and the pion 3-momentum lying in the xz plane, as shown in Fig. 3. If we use the δ-function $\delta(p'^2)$ to eliminate integration over the azimuthal angle φ, phase space (5) can be rewritten in the form

$$d\Phi = \frac{\pi}{2}\frac{dc_1dc_2d\omega d\epsilon}{K}, \\ K = s_1s_2|\sin\varphi|\sqrt{1-c_1^2-c_2^2-c_{12}^2+2c_1c_2c_{12}}, \quad (8)$$

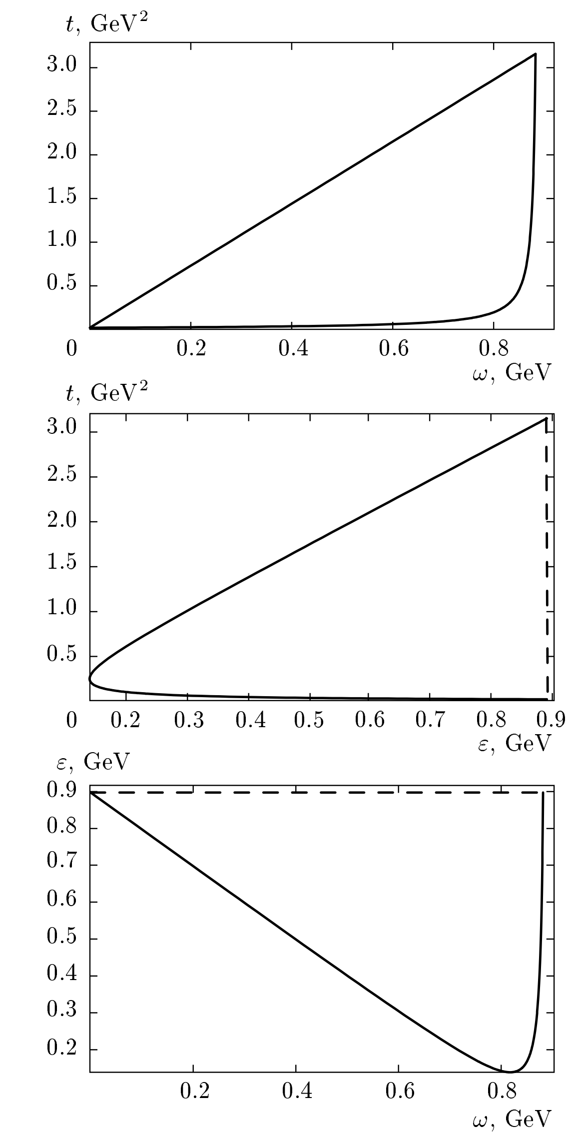

Fig. 2. The ranges for the radiative τ^- decay for different sets of kinematical variables. ω and ε — the energies of the photon and the pion, t — the invariant variable. The line equations are defined by inequalities (6) and (7)

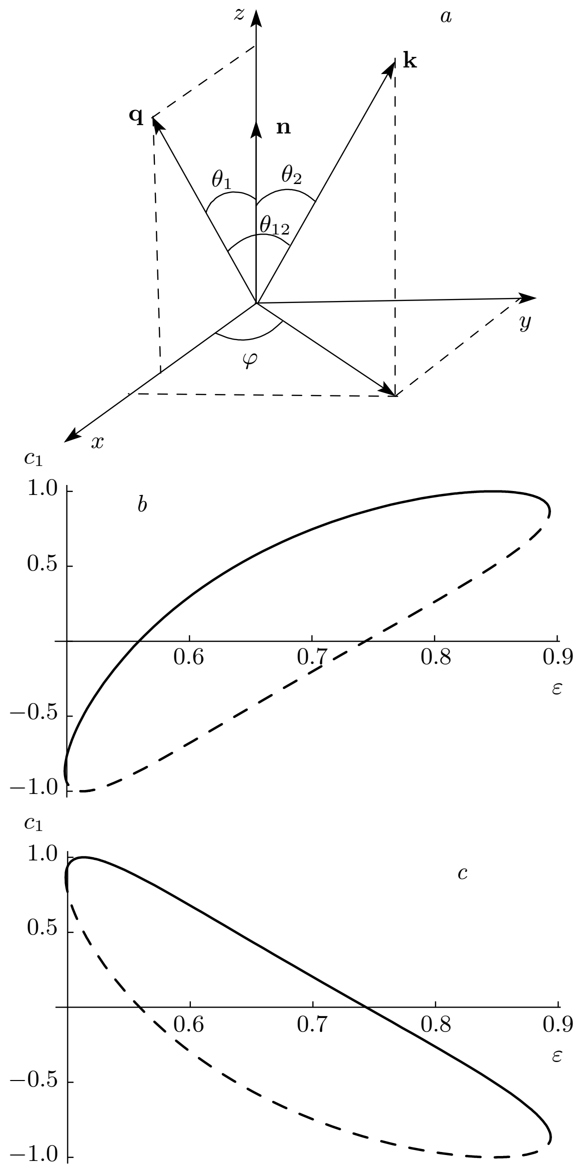

Fig. 3. Definition of the angles for a polarized radiative τ decay at rest (a); the limits of variation of c_1 at $\omega = 0.4$ GeV and $c_2 = 150^\circ$ (b), 30° (c): the solid (dashed) line corresponds to c_{1+} (c_{1-})

where $c_1(s_1) = \cos\theta_1(\sin\theta_1)$, $c_2(s_2) = \cos\theta_2(\sin\theta_2)$, and $c_{12} = \cos\theta_{12}$, and θ_1, θ_2, and θ_{12} are the respective angles between $\mathbf{n}$ and $\mathbf{q}$, $\mathbf{n}$ and $\mathbf{k}$, and $\mathbf{q}$ and $\mathbf{k}$. In this case, we can study the spin-dependent effects caused by only those terms in the matrix element squared that are independent of the azimuthal angle, namely, (Sk) and (Sq). The contribution of the term containing $(Spqk)$ vanishes when we integrate over φ in the whole region $(0, 2\pi)$.

The ranges of c_1 and c_2 are defined by the condition of positivity of the expression in the radicand in Eq. (8); they are shown in Fig. 3. A simple calculation gives

$$c_{1-} \le c_1 \le c_{1+}, \quad -1 \le c_2 \le 1,$$

$$c_{1\pm} = c_2 c_{12} \pm s_2 s_{12},$$

where $s_{12} = \sin\theta_{12}$. Besides, the integration in the above limits gives

$$\int \frac{dc_1}{K} = \int \frac{dc_2}{K} = \pi,$$
$$\int \frac{c_1 dc_1}{K} = \pi c_2 c_{12}, \quad \int \frac{c_2 dc_2}{K} = \pi c_1 c_{12}. \tag{9}$$

If $|M_\gamma|^2$ has no angular dependence, it corresponds to the unpolarized case.

2.3. Decay asymmetry and Stokes parameters

To determine moduli and phases of the form factors, it is necessary to measure some polarization observables. The simplest of such observables are the so-called single-spin quantities: the asymmetries caused by the τ-lepton polarization and the photon Stokes parameters. We also consider the double-spin quantities: the dependence of the Stokes parameters on the τ-lepton polarization.

Generally, the polarization properties of a photon are described by its Stokes parameters. At this point, we have to clarify the terminology used. The measurable Stokes parameters $\bar{\xi}_i$, $i = 1, 2, 3$, define the covariant spin-density matrix of the photon in terms of the $\bar{\xi}_i$ and two independent polarization 4-vectors $e_{1\mu}$ and $e_{2\mu}$ [29]:

$$\rho_{\mu\nu} = \frac{1}{2}\big[e_{1\mu}e_{1\nu} + e_{2\mu}e_{2\nu} + \bar{\xi}_1(e_{1\mu}e_{2\nu} + e_{1\nu}e_{2\mu}) - - i\bar{\xi}_2(e_{1\mu}e_{2\nu} - e_{1\nu}e_{2\mu}) + \bar{\xi}_3(e_{1\mu}e_{1\nu} - e_{2\mu}e_{2\nu})\big], \tag{10}$$
$$e_1^2 = e_2^2 = -1, \quad (ke_1) = (ke_2) = 0.$$

If the parameters $\bar{\xi}_i$ are measured, the matrix element squared can be written as a contraction of the current tensor $T^{\mu\nu}$ and the $\rho_{\mu\nu}$ matrix

$$|M_\gamma|^2 = T^{\mu\nu}\rho_{\mu\nu}, \tag{11}$$

where the current tensor $T^{\mu\nu}$ obeys the evident conditions due to the electromagnetic current conservation:

$$k_\mu T^{\mu\nu} = T^{\mu\nu}k_\nu = 0.$$

The polarization states of a real photon are described by two independent purely spatial polarization vectors $\mathbf{l}_1$ and $\mathbf{l}_2$, which are both perpendicular to the photon 3-momentum $\mathbf{k}$. In our case, in the rest frame of the decaying τ^- lepton, it is convenient to take $\mathbf{l}_1$ in the decay plane and $\mathbf{l}_2$ to be perpendicular to this plane. We can determine two covariant polarization 4-vectors that in the rest frame coincide with $\mathbf{l}_1$ and $\mathbf{l}_2$. These vectors are

$$\varepsilon_1^\mu = e_1^\mu - \frac{1}{N}\left[(qp) - M^2\frac{(qk)}{(pk)}\right]k^\mu,$$
$$\varepsilon_2^\mu \equiv e_2^\mu = \frac{(\mu pqk)}{N}, \tag{12}$$

where e_1^μ and N are defined after Eq. (2).

It is easy to verify that in the rest frame,

$$\varepsilon_{1\mu} = (0, \mathbf{l}_1), \quad \varepsilon_{2\mu} = (0, \mathbf{l}_2), \quad \mathbf{l}_1^2 = \mathbf{l}_2^2 = 1,$$
$$\mathbf{l}_1 = \frac{\mathbf{q} - \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{q})}{\sqrt{\mathbf{q}^2 - (\hat{\mathbf{k}}\cdot\mathbf{q})^2}}, \quad \mathbf{l}_2 = \hat{\mathbf{k}} \times \mathbf{l}_1, \quad \hat{\mathbf{k}} = \frac{\mathbf{k}}{\omega},$$
$$\mathbf{l}_1 \cdot \hat{\mathbf{k}} = \mathbf{l}_2 \cdot \hat{\mathbf{k}} = 0.$$

Therefore, the set of the unit 3-vectors $\mathbf{l}_1$, $\mathbf{l}_2$, and $\hat{\mathbf{k}}$ forms a right-hand system of rectangular coordinates. Due to the electromagnetic current conservation, we can replace $\varepsilon_{1\mu}$ by $e_{1\mu}$ in calculations of any observable. Thus, in Eq. (10), we use polarization 4-vectors in form (12). Then the matrix element squared becomes

$$|M_\gamma|^2 = \frac{1}{2}\big[\Sigma + \Sigma_i\,\bar{\xi}_i\big], \tag{13}$$

where

$$\Sigma = T^{\mu\nu}(e_{1\mu}e_{1\nu} + e_{2\mu}e_{2\nu}), \quad \Sigma_1 = T^{\mu\nu}(e_{1\mu}e_{2\nu} + e_{1\nu}e_{2\mu}),$$
$$\Sigma_2 = -iT^{\mu\nu}(e_{1\mu}e_{2\nu} - e_{1\nu}e_{2\mu}),$$
$$\Sigma_3 = T^{\mu\nu}(e_{1\mu}e_{1\nu} - e_{2\mu}e_{2\nu}).$$

It is obvious that the parameters $\bar{\xi}_i$ depend on the properties of detectors that analyze the polarization states of the photon and are independent of the production mechanism. But the quantities Σ and Σ_i are defined only by the decay amplitude and thus determine polarization properties of the photon itself in decay (1) [29]. To study predictions of different theoretical models for these quantities, we can write

$$|M_\gamma|^2 = \Sigma + \Sigma_i$$

instead of expression (13).

For a polarized τ lepton, the current tensor is given by

$$T_{\mu\nu} = T^0_{\mu\nu} + T^S_{\mu\nu},$$

where $T^S_{\mu\nu}$ depends on the polarization 4-vector of the τ lepton and $T^0_{\mu\nu}$ is independent of it. In this case, we can write

$$\Sigma = \Sigma^0 + \Sigma^S, \quad \Sigma_i = \Sigma_i^0 + \Sigma_i^S,$$

and define the physical quantities

$$A^S = \frac{\Sigma^S d\Phi}{\Sigma^0 d\Phi}, \quad \xi_i = \frac{\Sigma_i^0 d\Phi}{\Sigma^0 d\Phi}, \quad \xi_i^S = \frac{\Sigma_i^S d\Phi}{\Sigma^0 d\Phi}, \tag{14}$$

which completely describe the polarization effects in the considered decay.

The quantity A^S represents the polarization asymmetry of the differential decay width caused by the τ-lepton polarization. The ξ_i are the Stokes parameters of the photon itself if the τ lepton is unpolarized, and the ξ_i^S are correlation parameters describing the influence of τ-lepton polarization on the photon Stokes parameters.

Thus, to analyze the polarization phenomena in process (1), we have to study both the spin-independent and spin-dependent parts of the differential width. In accordance with Eq. (4), these are

$$\frac{d\Gamma_0}{d\Phi} = g\Sigma^0, \quad \frac{d\Gamma_0^S}{d\Phi} = g\Sigma^S, \quad \frac{d\Gamma_i}{d\Phi} = g\Sigma_i^0,$$

$$\frac{d\Gamma_i^S}{d\Phi} = g\Sigma_i^S, \quad g = \frac{1}{4\,M(2\,\pi)^5}.$$

We note that by partial integration in the numerators and denominator in relations (14), we can also define and study the corresponding reduced polarization parameters.

If we record the photon and pion energies, the parameters $\xi_1(\omega,\epsilon)$ and $\xi_3(\omega,\epsilon)$ describe linear polarizations of the photon and the parameter $\xi_2(\omega,\epsilon)$ describes the circular one. The last parameter does not depend on the choice of two polarization vectors defined by relations (12) in this paper. On the contrary, each of the parameters $\xi_1(\omega,\epsilon)$ and $\xi_3(\omega,\epsilon)$ depends on the axes relative to which it is defined, and only the quantity $\sqrt{\xi_1(\omega,\epsilon)^2+\xi_3(\omega,\epsilon)^2}$ remains invariant. In principle, we can choose the polarization vectors e_1' and e_2' such that, for example, $\xi_1'(\omega,\epsilon)$ vanishes; then

$$\xi_3'(\omega,\epsilon) = \sqrt{\xi_1(\omega,\epsilon)^2+\xi_3(\omega,\epsilon)^2}$$

(and vice versa).

This statement can be verified by a simple rotation of the 4-vectors e_1 and e_2 in the plane perpendicular to the direction $\mathbf{k}$ [29],

$$\begin{aligned} e'_{1\mu} &= e_{1\mu}\cos\beta - e_{2\mu}\sin\beta, \\ e'_{2\mu} &= e_{1\mu}\sin\beta + e_{2\mu}\cos\beta \end{aligned} \tag{15}$$

and hence in terms of e_1' and e_2', using definitions (13) and (14), we have

$$\xi_1'(\omega,\epsilon) = \xi_1(\omega,\epsilon)\cos 2\beta + \xi_3(\omega,\epsilon)\sin 2\beta,$$

$$\xi_3'(\omega,\epsilon) = -\xi_1(\omega,\epsilon)\sin 2\beta + \xi_3(\omega,\epsilon)\cos 2\beta,$$

$$\xi_2'(\omega,\epsilon) = \xi_2(\omega,\epsilon).$$

For example, taking,

$$\cos 2\beta = \frac{\xi_1(\omega,\epsilon)}{\sqrt{\xi_1(\omega,\epsilon)^2+\xi_3(\omega,\epsilon)^2}},$$

$$\sin 2\beta = \frac{\xi_3(\omega,\epsilon)}{\sqrt{\xi_1(\omega,\epsilon)^2+\xi_3(\omega,\epsilon)^2}},$$

we easy obtain

$$\xi_1'(\omega,\epsilon) = \sqrt{\xi_1(\omega,\epsilon)^2+\xi_3(\omega,\epsilon)^2}, \quad \xi_3'(\omega,\epsilon) = 0.$$

If the experimental setup allows fixing the decay plane, the Stokes parameter ξ_3 defines the probability of the photon linear polarization along two orthogonal directions: $\mathbf{l}_1$ and $\mathbf{l}_2$. If $\xi_3 = 1$ ($\xi_3 = -1$), the photon is fully polarized along the direction $\mathbf{l}_1$ ($\mathbf{l}_2$), and its polarization vector lies in the decay plane (is perpendicular to it). In general, the probability of the linear polarization in the decay plane is $(1+\xi_3)/2$ and in the plane perpendicular to it, $(1-\xi_3)/2$.

The parameter ξ_1 determines the probability of linear polarization in the planes rotated through the angle $\phi = \pm 45°$ around the $\mathbf{k}$-direction relative to the decay one. The full linear polarization at $\phi = +45°$ $(-45°)$ occurs at $\xi_1 = 1$ (-1). The corresponding probabilities, in the general case, are $(1+\xi_1)/2$ and $(1-\xi_1)/2$. Thus, we can say that the circular polarization degree of a photon equals $\xi_2(\omega,\epsilon)$ ($\xi_2(\omega,\epsilon) = 1$ or -1 corresponds to full right or left circular polarization), and the linear one equals $\sqrt{\xi_1(\omega,\epsilon)^2+\xi_3(\omega,\epsilon)^2}$. Nevertheless, both parameters $\xi_1(\omega,\epsilon)$ and $\xi_3(\omega,\epsilon)$ are measurable and carry different information about the decay mechanism. But to define them separately, we have to determine the plane $(\mathbf{q},\mathbf{k})$ in every event. The same also applies to the reduced photon polarization parameters, for example, $\xi_i(\omega)$, and so on.

2.4. Polarization of τ lepton

Before proceeding, we briefly discuss the possible polarization states of a τ lepton created at τ-factories in the electron–positron annihilation process with a longitudinally polarized electron beam:

$$e^- + e^+ \to \tau^- + \tau^+.$$

Simple calculations in the lowest approximation of QED show that polarization of τ arises if at least one of the colliding beams is polarized. For example, in the case of a longitudinally polarized electron beam, the τ^- lepton has longitudinal and transverse polarizations in the reaction plane relative to the τ lepton momentum direction (see Appendix B for the details)

$$\begin{gathered} P^L = \frac{2\lambda\cos\theta}{Q}, \quad P^T = \frac{4\lambda M\sin\theta}{\sqrt{s}Q}, \\ Q = 1+\cos^2\theta + \frac{4M^2}{s}\sin^2\theta, \end{gathered} \tag{16}$$

where λ is the beam polarization degree, θ is the c.m.s. angle between 3-momenta of the electron and τ^- lepton, and s is the total c.m.s. energy squared. If we select and analyze the events with a longitudinally (transversally) polarized τ^-, then the 3-vector $\mathbf{n}$ in the rest frame (see Fig. 3) lies in the annihilation reaction plane and is directed along (perpendicular to) the c.m.s. 3-momentum of τ^-.

3. CALCULATION OF THE CURRENT TENSOR $T^{\mu\nu}$

The current tensor $T^{\mu\nu}$ contains three contributions: the IB, the resonance ones, and the interference between the IB and resonance amplitudes. We divide every contribution into symmetric and antisymmetric parts with respect to the Lorentz indices. The symmetric part contributes to Σ, Σ_1, and Σ_3, whereas the antisymmetric one, to Σ_2 only.

Below, in the formulas for the different parts of the current tensor, we omit the terms proportional to the 4-vectors k_μ and k_ν because they do not contribute to the observables.

3.1. Inner bremsstrahlung contribution

For the IB contribution, we have

$$T^{\mu\nu}_{IB} = 4Z^2M^2\big[S^{\mu\nu}_{IB} + iA^{\mu\nu}_{IB}\big],$$

where the symmetric part is

$$S^{\mu\nu}_{IB} = \frac{(qk)-(pk)}{(pk)^2}\big[(pk)+M(kS)\big]g^{\mu\nu} + \\ + \frac{N^2}{(pk)^2(qk)^2}\big[M^2-m^2+2M(p'S)\big]e_1^\mu e_1^\nu + \\ + \frac{NM}{(pk)^2(qk)}\big[(e_1l_p)^{\mu\nu}-(e_1l_q)^{\mu\nu}\big], \quad (17)$$

and antisymmetric one is

$$A^{\mu\nu}_{IB} = \frac{(pk)-(qk)}{(pk)^2}[M(\mu\nu kS)-(\mu\nu pk)] + \\ + \frac{N^2}{(pk)^2(qk)}[e_1e_2]^{\mu\nu} - \\ - \frac{MN}{(pk)^2(qk)}[e_1^\mu(\nu p'kS)-e_1^\nu(\mu p'kS)]. \quad (18)$$

Here, we use the notation

$$l^\mu_p = (pk)S^\mu-(kS)p^\mu, \quad l^\mu_q = (qk)S^\mu-(kS)q^\mu,$$

$$(ab)^{\mu\nu} = a^\mu b^\nu + a^\nu b^\mu, \quad [ab]^{\mu\nu} = a^\mu b^\nu - a^\nu b^\mu,$$

$$(kl_p) = (kl_q) = 0.$$

We note that the antisymmetric part can be written in different equivalent forms. Indeed, we can derive another form using the well-known relation

$$g^{\alpha\beta}(\mu\nu\lambda\rho) = g^{\alpha\mu}(\beta\nu\lambda\rho) + g^{\alpha\nu}(\mu\beta\lambda\rho) + \\ + g^{\alpha\lambda}(\mu\nu\beta\rho) + g^{\alpha\rho}(\mu\nu\lambda\beta).$$

3.2. Resonance contribution

As concern the resonance contribution to the current tensor $T^{\mu\nu}$, we write it in the form

$$T^{\mu\nu}_R = \frac{8Z^2}{M^4}\big[S^{\mu\nu}_R + iA^{\mu\nu}_R\big],$$

where both the symmetric and antisymmetric parts include four independent pieces. They are proportional to $|a(t)|^2$, $|v(t)|^2$, $\mathrm{Re}(a(t)v^*(t))$, and $\mathrm{Im}(a(t)v^*(t))$. Denoting the respective symmetric pieces as S_{Ra}, S_{Rv}, S_{Rr}, and S_{Ri}, and the antisymmetrical ones as A_{Ra}, A_{Rv}, A_{Rr}, and A_{Ri}, we have

$$S^{\mu\nu}_{Ra} = |a(t)|^2\big\{(qk)^2[M(p'S)-(pp')]g^{\mu\nu} + \\ + 2N^2e_1^\mu e_1^\nu + NM(e_1l_q)^{\mu\nu}\big\}, \quad (19)$$

$$A^{\mu\nu}_{Ra} = |a(t)|^2(qk)\big\{(qp')(\mu\nu pk)-(qp)(\mu\nu p'k) - \\ - M[(qp')(\mu\nu Sk)-(qS)(\mu\nu p'k)]\big\}, \quad (20)$$

$$S^{\mu\nu}_{Rv} = |v(t)|^2\big\{\big[M(p'S)-(pp')\big](qk)^2g^{\mu\nu}+2N^2e_2^\mu e_2^\nu - \\ - NM\big[e_2^\mu(\nu qkS)+e_2^\nu(\mu qkS)\big]\big\}, \quad (21)$$

$$A^{\mu\nu}_{Rv} = |v(t)|^2(\mu\nu qk)\big\{(qk)[(pq)-(pk)]-m^2(pk) - \\ - M[(qp')(kS)-(p'k)(qS)]\big\}, \quad (22)$$

$$S^{\mu\nu}_{Rr} = 2\,\mathrm{Re}\{a^*(t)v(t)\}g^{\mu\nu}(qk)\big\{(qk)\big[(pk)-(pq)\big] + \\ + m^2(pk) - M\big[(kp')(Sq)-(Sk)(p'q)\big]\big\}, \quad (23)$$

$$A^{\mu\nu}_{Rr} = 2\,\mathrm{Re}\{a^*(t)v(t)\}\bigg\{(qk)[(pp')-M(p'S)]\times \\ \times(\mu\nu qk) - N^2[e_1e_2]^{\mu\nu} + \frac{1}{2}NM\big(e_1^\mu(\nu qkS) - \\ - e_1^\nu(\mu qkS)+[e_2l_q]^{\mu\nu}\big)\bigg\}, \quad (24)$$

$$S^{\mu\nu}_{Ri} = 2\,\mathrm{Im}\{a^*(t)v(t)\}\bigg\{\frac{1}{2}NM\big[-(e_2l_q)^{\mu\nu} + \\ + e_1^\mu(\nu qkS)+e_1^\nu(\mu qkS)\big] - N^2(e_1e_2)^{\mu\nu}\bigg\}, \quad (25)$$

$$A^{\mu\nu}_{Ri} = 0. \tag{26}$$

The interference between the vector and axial-vector contributions of the resonance amplitudes is sensitive to the relative sign of the axial-vector and vector couplings and its separation can be used to fix the sign of the ratio $f_V(0)/f_A(0)$.

3.3. IB–resonance interference

The interference between the IB and resonance amplitudes is more sensitive to all the resonance parameters because $a(t)$ and $v(t)$ enter it linearly. It is very important to find such a polarization observable where the interference contribution would be enhanced relative to the background created by the pure IB and resonance contributions. For the current tensor caused by the IB–resonance interference, we set

$$T^{\mu\nu}_{IR} = \frac{8Z^2}{M(pk)}\left[S^{\mu\nu}_{IR} + iA^{\mu\nu}_{IR}\right].$$

Again, we have four symmetric and antisymmetric terms, which are given by

$$S^{\mu\nu}_{IRra} = \mathrm{Re}(a(t))\Big\{(qk)[(pk)(Sp') - (kS)(pp') - M(p'k)]g^{\mu\nu} + \frac{2N^2}{(qk)}[M + (Sp')]e_1^\mu e_1^\nu + \frac{N(pp')}{(qk)}(e_1 l_q)^{\mu\nu} + N(e_1 l_p)^{\mu\nu}\Big\}, \tag{27}$$

$$A^{\mu\nu}_{IRra} = \mathrm{Re}(a(t))\Big\{\big[(qS)\big(2(pk) - (qk) + (pq)\big) - M(qk)\big](\mu\nu pk) + \big[(pq)\big(2(pk) - (qk) + (pq)\big) - M^2(m^2 + (qk))\big](\mu\nu kS) + M\big[(qk) - M(qS)\big](\mu\nu qk)\Big\}, \tag{28}$$

$$S^{\mu\nu}_{IRia} = -\mathrm{Im}(a(t)) \times N\left[(qk)(e_2 S)g^{\mu\nu} + \frac{(e_1 Q)^{\mu\nu}}{(qk)}\right], \tag{29}$$

$$A^{\mu\nu}_{IRia} = \mathrm{Im}(a(t)) \times N\left[[e_1 l_p]^{\mu\nu} - \left(1 + \frac{(pp')}{(qk)}\right)[e_1 l_q]^{\mu\nu}\right], \tag{30}$$

where

$$Q^\mu = N(e_2 S)q^\mu - (qk)(Spq\mu),$$

$$S^{\mu\nu}_{IRrv} = \mathrm{Re}(v(t))\Big\{(p'k)\big[M(qk) + (pq)(kS) - (pk)(qS)\big]g^{\mu\nu} + 2\frac{N^2}{(qk)}(qS)e_1^\mu e_1^\nu + N\left[\frac{(pq)}{(qk)}(e_1 l_q)^{\mu\nu} - \left(\frac{1}{2} + \frac{m^2}{(qk)}\right)(e_1 l_p)^{\mu\nu} + \frac{1}{2}\big(e_2^\mu(\nu pkS) + e_2^\nu(\mu pkS)\big)\right]\Big\}, \tag{31}$$

$$A^{\mu\nu}_{IRrv} = \mathrm{Re}(v(t))\Big\{\big[M(p'k) + (pp')(kS) - (pk)(p'S)\big](\mu\nu qk) - \frac{N^2}{(qk)}[M + (p'S)][e_1 e_2]^{\mu\nu} + N\left[\frac{(pp')}{(qk)}\big[e_2 l_q\big]^{\mu\nu} + \big[e_2 l_p\big]^{\mu\nu}\right]\Big\}, \tag{32}$$

$$S^{\mu\nu}_{IRiv} = \mathrm{Im}(v(t))N\Big\{-\frac{N}{(qk)}[M + (p'S)](e_1 e_2)^{\mu\nu} + \left(\frac{1}{2} + \frac{(pp')}{(qk)}\right)\big(e_1^\mu(\nu qkS) + e_1^\nu(\mu qkS)\big) - \frac{1}{2}\big(e_2(2l_p - l_q)\big)^{\mu\nu}\Big\}, \tag{33}$$

$$A^{\mu\nu}_{IRiv} = \mathrm{Im}(v(t))N\Big\{\left(\frac{1}{2} + \frac{m^2}{(qk)}\right)\big[e_1 l_p\big]^{\mu\nu} - \frac{(pq)}{(qk)}\big[e_1 l_q\big]^{\mu\nu} + \frac{1}{2}\big[e_2^\mu(\nu pkS) - e_2^\nu(\mu pkS)\big] - \big[e_2^\mu(\nu qkS) - e_2^\nu(\mu qkS)\big]\Big\}. \tag{34}$$

In the above formulas, we omit terms proportional to the 4-vectors k_μ and k_ν because they do not contribute to any observables.

The expression for the current tensor $T^{\mu\nu}$ allows us to derive all the polarization observables in a Lorentz-invariant form by contracting this tensor with an appropriate combination of the 4-vectors e_1 and e_2. The set of needed formulas is

$$e_1^2 = e_2^2 = -1, \quad (e_1 e_2) = 0,$$

$$(e_1 l_p) = -(e_2 pkS) = \\ = \frac{1}{N}\Big\{(Sq)(pk)^2 + (Sk)\big[M^2(qk) - (pq)(pk)\big]\Big\}, \\ (e_2 l_p) = (e_1 pkS) = \frac{(pk)}{N}(Spqk), \\ (e_1 l_q) = -(e_2 qkS) = \frac{1}{N}\Big\{(Sq)(pk)(qk) + \\ + (Sk)\big[(pq)(qk) - m^2(pk)\big]\Big\}, \\ (e_2 l_q) = (e_1 qkS) = \frac{(qk)}{N}(Spqk), \\ (e_1 e_2 qk) = -(qk), \quad (e_1 e_2 pk) = -(pk), \\ (e_1 e_2 Sk) = -(Sk), \\ (e_1 Q) = \frac{m^2(pk) - (pq)(qk)}{N}(Spqk), \\ (e_2 Q) = \frac{(qk)}{N}\Big\{(kS)\big[(pq)^2 - M^2 m^2\big] + \\ + (qS)\big[M^2(qk) - (pq)(pk)\big]\Big\}. \tag{35}$$

In the presence of a polarized τ lepton, the structure of the differential width and the Stokes parameters of the photon are much richer. As we saw above, in a polarized electron–positron annihilation, the created τ leptons have essential longitudinal or transverse polarizations. In both cases, in the rest frame of τ, its polarization 4-vector is $S^\mu = (0, \mathbf{n})$, and choosing a coordinate system as shown in Fig. 3, we have

$$(qS) = -|\mathbf{q}|c_1, \quad (kS) = -\omega c_2, \tag{36}$$

$$(Spqk) = M\big(\mathbf{n}[\mathbf{qk}]\big) = \\ = \operatorname{sign}(\sin\varphi) M |\mathbf{q}| \omega \sqrt{1 - c_1^2 - c_2^2 - c_{12}^2 + 2c_1 c_2 c_{12}},$$

where $\sin\varphi$ defines the y-component of the 3-vector $\mathbf{k}$, namely $k_y = \omega \sin\theta_2 \sin\varphi$.

If we sum events with all possible values of the azimuthal angle φ, the spin correlation $(Spqk)$, which is perpendicular to the plane $(\mathbf{q}, \mathbf{k})$, does not contribute. On the other hand, spin correlations in the plane $(\mathbf{q}, \mathbf{k})$, caused by the (Sq) and (Sk) terms, being integrated over c_1 (or over c_2), are always proportional to c_2 (or c_1), as follows from relations (9) and (36).

4. AXIAL-VECTOR AND VECTOR FORM FACTORS IN THE RχT

From the Lagrangians in Appendix A, we can obtain the photon coupling to the pseudoscalar mesons P in the form

$$\mathcal{L}_{\gamma PP} = ieB^\mu \mathrm{Tr}\,(Q, [\Phi, \partial_\mu \Phi]) = \\ = ieB^\mu(\pi^+ \overleftrightarrow{\partial_\mu} \pi^- + K^+ \overleftrightarrow{\partial_\mu} K^-), \\ \overleftrightarrow{\partial_\mu} \equiv \overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu}. \tag{37}$$

The axial transition $W^\pm \to \pi^\pm(K^\pm)$ is described by

$$\mathcal{L}_{WP} = -\frac{gF}{2}\mathrm{Tr}\,(W_\mu \partial^\mu \Phi) = \\ = \frac{gF}{2} W_\mu^+ (V_{ud}\partial^\mu \pi^- + V_{us}\partial^\mu K^-) + \mathrm{H.c.} \tag{38}$$

The vertex with an additional photon, $W^\pm \to \gamma\pi^\pm(K^\pm)$, is generated from

$$\mathcal{L}_{WP\gamma} = -\frac{i}{2}egFB^\mu \mathrm{Tr}\,([Q, \Phi]W_\mu) = \\ = \frac{i}{2}egFB^\mu W_\mu^+ (V_{ud}\pi^- + V_{us}K^-) + \mathrm{H.c.} \tag{39}$$

To evaluate the resonance contribution to the axial-vector form factor, we need the vertex of the $W^\pm \to a_1^\pm(K_1^\pm)$ transition

$$\mathcal{L}_{WA} = -\frac{1}{4}gF_A \mathrm{Tr}\,(W_{\mu\nu}A^{\mu\nu}) = \\ = -\frac{gF_A}{2}\partial_\mu W_\nu^+ (V_{ud} a_1^{-\,\mu\nu} + V_{us} K_1^{-\,\mu\nu}) + \mathrm{H.c.} \tag{40}$$

The transition $a_1^\pm(K_1^\pm) \to \gamma\pi^\pm(K^\pm)$ is described by

$$\mathcal{L}_{AP\gamma} = -i\frac{eF_A}{2F} F^{\mu\nu}(\pi^- a^+_{1\mu\nu} + K^- K^+_{1\mu\nu}) + \mathrm{H.c.} \tag{41}$$

The transition $W^\pm \to \pi^\pm \rho^0$ is generated from the Lagrangian

$$\mathcal{L}_{WPV} = -i\frac{gG_V}{\sqrt{2}F}\mathrm{Tr}\,(V^{\mu\nu}[W_\mu, \partial_\nu \Phi]) + \\ + i\frac{gF_V}{4\sqrt{2}F}\mathrm{Tr}\,(V^{\mu\nu}[W_{\mu\nu}, \Phi]) \tag{42}$$

with the notation

$$W_{\mu\nu} = W^+_{\mu\nu}T_+ + W^-_{\mu\nu}T_-, \\ W^\pm_{\mu\nu} \equiv \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu. \tag{43}$$

Keeping the contribution from the neutral vector mesons in Eq. (42), we obtain

$$\mathcal{L}_{WPV} = i\frac{gG_V}{\sqrt{2}F} W_\mu^+ \Bigg[-\sqrt{2}V_{ud}\partial_\nu \pi^- \rho^{0\,\mu\nu} + \\ + V_{us}\partial_\nu K^- \left(\phi - \frac{1}{\sqrt{2}}\rho^0 - \frac{1}{\sqrt{2}}\omega\right)^{\mu\nu}\Bigg] - \\ - i\frac{gF_V}{4\sqrt{2}F} W^+_{\mu\nu}\Bigg[-\sqrt{2}V_{ud}\pi^- \rho^{0\,\mu\nu} + V_{us}K^- \times \\ \times \left(\phi - \frac{1}{\sqrt{2}}\rho^0 - \frac{1}{\sqrt{2}}\omega\right)^{\mu\nu}\Bigg] + \mathrm{H.c.} \tag{44}$$

Finally, we need the term describing a transition of the neutral vector mesons to a photon,

$$\mathcal{L}_{V\gamma} = -i\frac{eF_V}{\sqrt{2}F} F^{\mu\nu}\mathrm{Tr}\,(V_{\mu\nu}\,Q) = = eF_V F^{\mu\nu}\left(\frac{1}{2}\rho^0_{\mu\nu} + \frac{1}{6}\omega_{\mu\nu} - \frac{1}{3\sqrt{2}}\phi_{\mu\nu}\right) \quad (45)$$

with $F^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$.

Collecting the vertices for the transition $W^- \to \to a_1^- \to \pi^-\gamma$ from (40) and (41), and the vertices for the transition $W^- \to \pi^-\rho^0 \to \pi^-\gamma$ from (44) and (45), we obtain the axial-vector form factor

$$f_A(t) = \frac{\sqrt{2}m_{\pi\pm}}{F_\pi} \times \times \left[\frac{F_A^2}{m_a^2 - t - im_a\Gamma_a(t)} + \frac{F_V(2G_V - F_V)}{m_\rho^2}\right], \quad (46)$$

where $\Gamma_a(t)$ is the decay width of the a_1-meson.

We note that the axial-vector form factor in Eq. (46) is normalized at $t = 0$ as

$$f_A(0) = \frac{\sqrt{2}m_{\pi\pm}}{F_\pi}\left[\frac{F_A^2}{m_a^2} + \frac{F_V(2G_V - F_V)}{m_\rho^2}\right], \quad (47)$$

which is consistent with the chiral expansion in the order $\mathcal{O}(p^4)$ (see also Refs. [30, 31]) in terms of the low-energy constants,

$$f_A(0) = \frac{4\sqrt{2}m_{\pi\pm}}{F_\pi}(L_9 + L_{10}), \quad L_9 = \frac{F_V G_V}{2m_\rho^2}, \quad L_{10} = \frac{F_A^2}{4m_a^2} - \frac{F_V^2}{4m_\rho^2}. \quad (48)$$

The expressions for L_9 and L_{10} follow from the resonance saturation of the low-energy constants [17, 18].

The masses of the $\rho(770)$ and $a_1(1260)$ mesons in Eq. (46) are $m_\rho = 0.7755$ GeV and $m_a = 1.230$ GeV. The width of the $a_1(1260)$ meson in Eq. (46) is taken from Ref. [32]:

$$\Gamma_a(t) = \Gamma_0 g(t)/g(m_a^2), \quad g(t) = \left(1.623t + 10.38 - \frac{9.23}{t} + \frac{0.65}{t^2}\right) \times \times\,\theta(t - (m_\rho + m_\pi)^2) + + 4.1(t - 9m_\pi^2)^3\big[1.0 - 3.3(t - 9m_\pi^2) + + 5.8(t - 9m_\pi^2)^2\big]\theta(t - 9m_\pi^2)\theta((m_\rho + m_\pi)^2 - t). \quad (49)$$

It is implied in this equation that t is in GeV^2, the masses are in GeV ($m_\pi = m_{\pi\pm}$), and all numbers are in appropriate powers of GeV.

The values of the coupling constants F_A, F_V, and G_V are presented in Table. The constants F_V and G_V are obtained from the experimental information [10] on the $\rho \to e^+e^-$ and $\rho \to \pi^+\pi^-$ decay widths: $\Gamma(\rho \to e^+e^-) = 7.04 \pm 0.06$ keV and $\Gamma(\rho \to \pi^+\pi^-) = = 146.2 \pm 0.7$ MeV. To find F_V and G_V, we can use the tree-level relations

$$\Gamma(\rho \to e^+e^-) = \frac{e^4 F_V^2}{12\pi m_\rho},$$

$$\Gamma(\rho \to \pi^+\pi^-) = \frac{G_V^2 m_\rho^3}{48\pi F_\pi^4}\left(1 - \frac{4m_\pi^2}{m_\rho^2}\right)^{3/2}.$$

Table. Two sets of the coupling constants. Values for set 1 are calculated from the $\rho \to e^+e^-$, $\rho \to \pi^+\pi^-$ decay widths and $f_A(0)_{exp}$. Values for set 2 are chosen according to Ref. [18], and $F_\pi = 0.0924$ GeV

	F_A	F_V	G_V
Set 1	0.1368 GeV	0.1564 GeV	0.06514 GeV
Set 2	F_π	$\sqrt{2}F_\pi$	$F_\pi/\sqrt{2}$

The constant F_A is then calculated from Eq. (47) using the average value $f_A(0)_{exp} = 0.0119 \pm 0.0001$ measured in the radiative pion decays [10]. The constants F_V, G_V, and F_A calculated for central values of the data are hereafter called "set 1" and are shown in Table.

As another option, we choose theoretically motivated values of the constants from Ref. [18]. In particular, the relations $F_V = 2G_V$ and $F_V G_V = F_\pi^2$ are suggested there. The corresponding parameters are called "set 2" and are also given in Table.

In the calculation of the vector form factor $f_V(t)$, we need the transition $W^\pm \to \rho^\pm \to \pi^\pm\gamma$, which involves an odd-intrinsic-parity vertex. For this last, we use the vector (or Proca) representation for spin-one fields. As shown in Ref. [33] (see also Ref. [19]), the use of the vector field V^μ instead of the antisymmetric tensor field $V^{\mu\nu}$ in the description of spin-one resonances ensures the correct behavior of the Green's functions to the order $\mathcal{O}(p^6)$, while the tensor formulation would require additional local terms (see also the discussion in Appendix F of Ref. [22]).

Thus, we choose the Lagrangian [19, 33]

$$\mathcal{L}_{VP\gamma} = -h_V\epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\,(V^\mu\{u^\nu,\, f_+^{\alpha\beta}\}) = = \frac{4\sqrt{2}\,e\,h_V}{3F_\pi}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha B^\beta\vec{\rho}^{\,\mu}\partial^\nu\vec{\pi} \quad (50)$$

with the coupling constant h_V.

For the $W^\pm \to \rho^\pm\,(K^{*\pm})$ vertex, in the vector formulation, we have

$$\mathcal{L}_{WV} = -\frac{1}{4} g \frac{F_V}{m_\rho} \operatorname{Tr}(W_{\mu\nu}\partial^\mu V^\nu) =$$
$$= -\frac{gF_V}{4m_\rho} W^+_{\mu\nu}(V_{ud}\partial^\mu \rho^{-\nu} + V_{us}\partial^\mu K^{*-\nu}) + \text{H.c.} \quad (51)$$

Using (50) and (51), and adding vertex (A.12), we obtain the vector form factor

$$f_V(t) = \frac{\sqrt{2} m_{\pi\pm}}{F_\pi} \times$$
$$\times \left[\frac{N_C}{24\pi^2} + \frac{4\sqrt{2} h_V F_V}{3m_\rho} \frac{t}{m_\rho^2 - t - i m_\rho \Gamma_\rho(t)} \right]. \quad (52)$$

The width of the off-mass-shell ρ meson can be calculated from the interaction Lagrangian $\mathcal{L}^R_{int}$ in Eqs. (A.9). It is written in the form

$$\Gamma_\rho(t) = \frac{G_V^2 m_\rho t}{48\pi F_\pi^4} \left[\left(1 - \frac{4m_\pi^2}{t}\right)^{3/2} \theta(t - 4m_\pi^2) + \right.$$
$$\left. + \frac{1}{2}\left(1 - \frac{4m_K^2}{t}\right)^{3/2} \theta(t - 4m_K^2) \right], \quad (53)$$

where $m_K = 0.4937$ GeV is the mass of the $K^\pm$ meson. Other contributions to the width, coming, for example, from the four-pion decay of the ρ, are neglected in (53).

The coupling constant h_V can be fixed from the decay width $\Gamma(\rho^\pm \to \pi^\pm\gamma) = 68 \pm 7$ keV [10]. From the equation

$$\Gamma(\rho^\pm \to \pi^\pm\gamma) = \frac{e^2 m_\rho^3 h_V^2}{27\pi F_\pi^2}\left(1 - \frac{m_\pi^2}{m_\rho^2}\right)^3,$$

we then obtain $h_V = 0.036$. Alternatively, h_V can be constrained from the high-energy behavior of the vector form factor. Such constraints have been used in Refs. [8, 12]. According to the asymptotic predictions of the perturbative QCD [34], at $t \to -\infty$ the form factor behaves as $f_V(t) \sim \text{const}/t$. Imposing this condition on $f_V(t)$ in Eq. (52), we obtain

$$h_V = \frac{N_c m_\rho}{32\pi^2\sqrt{2}F_V}. \quad (54)$$

This yields $h_V = 0.033$ (0.040) for F_V from set 1 (set 2) in Table. These values are close to $h_V = 0.036$ derived from the $\rho \to \pi\gamma$ decay width, and that value is used in our calculations.

5. CALCULATION OF DIFFERENTIAL DECAY WIDTH, STOKES PARAMETERS, POLARIZATION ASYMMETRY AND SPIN-CORRELATION PARAMETERS

5.1. The t-dependence in the case of an unpolarized τ^-

Because the vector and axial-vector form factors depend on the invariant mass of the π–γ system, we can integrate the double differential width (both spin-independent and spin-dependent) at fixed values of the variable t, using restrictions (7). For the decay width $d\Gamma_0$, we have

$$\frac{d\Gamma_0}{dt} = P\big[I_0(t) + (|a(t)|^2 + |v(t)|^2)A_0(t) +$$
$$+ \operatorname{Re}(a(t))B_0(t) + \operatorname{Re}(v(t))C_0(t)\big], \quad (55)$$
$$P = \frac{Z^2}{2^8\pi^3 M^2},$$

where $I_0(t)$ is the contribution of the inner bremsstrahlung

$$I_0(t) = \frac{4M}{t - m^2}\left\{\frac{M^2 - t}{t}\big[(t + m^2)^2 - 4M^2 t\big] + \right.$$
$$\left. + \big[2M^2(M^2 + t - m^2) - m^4 - t^2\big]L\right\}, \quad L = \ln\frac{M^2}{t}.$$

As we can see from Eq. (55), the structure-dependent (resonance) contribution to $d\Gamma_0/dt$ does not contain vector–axial-vector interference, but it includes a sum of the squared moduli of the vector and axial-vector form factors. This sum is multiplied by the function

$$A_0(t) = \frac{(t - m^2)^3 (M^2 - t)^2 (M^2 + 2t)}{3M^5 t^2}.$$

The interference of the IB and structural amplitudes includes only real parts of the form factors and

$$B_0(t) = \frac{4(t - m^2)}{Mt} \times$$
$$\times \big[(2t + M^2 - m^2)(M^2 - t) + t(m^2 - 2M^2 - t)L\big],$$
$$C_0(t) = \frac{4(t - m^2)^2}{Mt}\big(t - M^2 + tL\big).$$

As regards the quantity $d\Gamma_1/dt$ connected with the Stokes parameter ξ_1 (see Eq. (14) and formulas just below), it is given by

$$\frac{d\Gamma_1}{dt} = P\big[\operatorname{Im}(a(t)^* v(t))A_1(t) + \operatorname{Im}(v(t))C_1(t)\big], \quad (56)$$

where

$$A_1(t) = -2\frac{(t - m^2)^3(M^2 - t)^3}{3M^5 t^2},$$

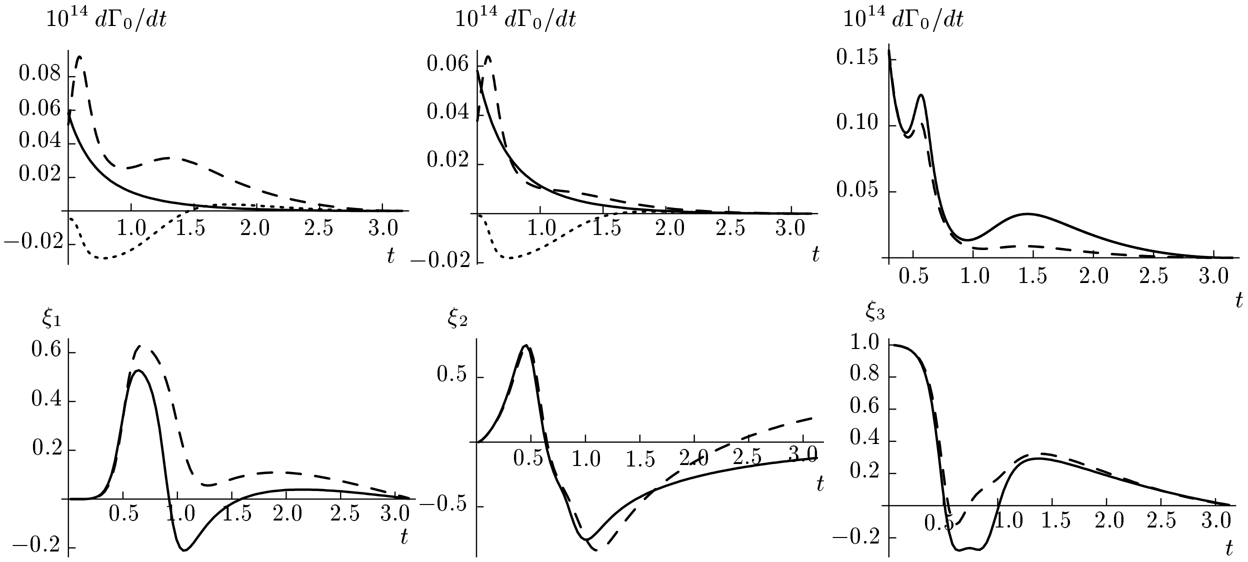

Fig. 4. The t-distribution of the differential decay width, in GeV^{-1}, is shown in the upper row. The first (second) figure corresponds to set 1 (set 2) of the resonance parameters given in Table; the solid curve represents the inner bremsstrahlung contribution, the dashed curve, the resonance contribution, and the dotted curve, their interference. The third figure shows the sum of all contributions: the solid (dashed) curve corresponds to set 1 (set 2). The quantities ξ_i (see Eq. (14)) in the lower row are calculated including all the contributions for two sets of the parameters

$$C_1(t) = \frac{4(t-m^2)}{Mt}\big(t^2 - M^4 + 2M^2 t\,L\big).$$

This quantity, in the case of an unpolarized τ, is the only one that includes the imaginary parts of the vector–axial-vector interference and the imaginary part of the vector form factor. Because it does not contain the pure IB contribution, it may be useful in studying the resonance contribution.

We also have

$$\frac{d\Gamma_2}{dt} = P\big[I_2(t) + \mathrm{Re}(a(t)^* v(t))A_2(t) + \\ + \mathrm{Re}(a(t))B_2(t) + \mathrm{Re}(v(t))C_2(t)\big], \quad (57)$$

where

$$I_2(t) = -\frac{4M}{t}\big[(3t+m^2)(M^2-t) - t(t+m^2+2M^2)L\big],$$

$$A_2 = -2\,A_0(t), \quad B_2(t) = -C_0(t), \quad C_2(t) = -B_0(t)$$

for the part corresponding to the circular polarization of the gamma quantum (the parameter ξ_2) and

$$\frac{d\Gamma_3}{dt} = P\big[I_3(t) + \big(|a(t)|^2 - |v(t)|^2\big)A_3(t) + \\ + \mathrm{Re}(a(t))B_3(t)\big], \quad (58)$$

$$I_3(t) = \frac{8M(M^2-m^2)}{t-m^2}\big(2(t-M^2) + (M^2+t)L\big),$$

$$A_3(t) = -\frac{1}{2}A_1(t), \quad B_3(t) = -C_1(t)$$

for the part connected with the parameter ξ_3.

The obtained results are illustrated in Fig. 4, where we show the quantities $d\Gamma_0(t)/dt$ and $\xi_i(t)$ defined as

$$\xi_i(t) = \frac{d\Gamma_i(t)/dt}{d\Gamma_0(t)/dt}, \quad i = 1, 2, 3.$$

5.2. The t-dependence for a polarized τ^-

If τ^- is polarized in decay (1), we can also write analytic expressions for the quantities

$$\frac{d\Gamma_0^S}{c_2\,dc_2\,dt}, \quad \frac{d\Gamma_i^S}{c_2\,dc_2\,dt}, \quad i = 1, 2, 3.$$

They can be obtained from the corresponding fully differential distributions by integrating with respect to c_1 (using relations (9)) and ω at fixed t. We recall that in the rest frame of τ^-, its polarization 3-vector is directed along the z axis and in this subsection, we consider effects caused by the component of this 3-vector that belongs to the decay plane $(\mathbf{q}, \mathbf{k})$.

The quantity that defines the polarization asymmetry of the decay is

$$\frac{d\Gamma_0^S}{c_2\,dc_2\,dt} = \frac{P}{2}\big[I_0^S(t) + \big(|a(t)|^2 + |v(t)|^2\big)A_0^S(t) + \\ + \mathrm{Re}(a(t)^* v(t))B_0^S(t) + \mathrm{Re}(a(t))C_0^S(t) + \\ + \mathrm{Re}(v(t))D_0^S(t)\big], \quad (59)$$

where

$$I_0^S(t) = \frac{4M}{(t-m^2)}\left[-\frac{m^4M^2}{t} + m^4 - 2\,m^2M^2 - \right. \\ - 6M^4 + 2\,m^2t + 3M^2t + 3t^2 + \\ \left. + [(m^2+M^2)^2 + (M^2+t)^2 + 4\,M^2t]L\right],$$

$$A_0^S(t) = \frac{(t-m^2)^3}{3M^5t^2}\big[M^6 - 6M^4t + 3M^2t^2 + 2t^3 + 6M^2t^2L\big],$$

$$B_0^S(t) = \frac{4(t-m^2)^3}{M^3t}\big(M^2 - t - t\,L\big),$$

$$C_0^S(t) = \frac{4(m^2-t)}{M\,t}\big[(t-M^2)(M^2+m^2+4t) + \\ + t(m^2+4M^2+t)L\big],$$

$$D_0^S(t) = \frac{4(m^2-t)}{Mt}\big[(M^2-t)(m^2+3t) - t(m^2+2M^2+t)L\big].$$

The quantities $d\Gamma_i^S$, $i = 1, 2, 3$, describe correlations between the polarization states of τ^- and the photon. For them, we have

$$\frac{d\Gamma_1^S}{c_2\,dc_2\,dt} = \frac{P}{2}\big[\mathrm{Im}(a(t)^* v(t))B_1^S(t) + \\ + \mathrm{Im}(a(t))C_1^S(t) + \mathrm{Im}(v(t))D_1^S(t)\big], \quad (60)$$

$$B_1^S(t) = A_1(t),$$

$$C_1^S(t) = \frac{8(t-m^2)}{M}\big[2(M^2-t) - (M^2+t)L\big],$$

$$D_1^S(t) = \frac{4(t-m^2)}{Mt}\big[(t-M^2)(5t+M^2) + 2t(2M^2+t)L\big];$$

$$\frac{d\Gamma_2^S}{c_2\,dc_2\,dt} = \frac{P}{2}\big[I_2^S(t) + \big(|a(t)|^2 + |v(t)|^2\big)A_2^S(t) + \\ + \mathrm{Re}(a(t)^* v(t))B_2^S(t) + \mathrm{Re}(a(t))C_2^S(t) + \\ + \mathrm{Re}(v(t))D_2^S(t)\big], \quad (61)$$

$$I_2^S(t) = -\frac{M^2}{t-m^2}D_0^S(t),$$

$$A_2^S(t) = \frac{2(t-m^2)^3}{M^3t}\big(t - M^2 + tL\big), \quad B_2^S(t) = -2A_0^S(t),$$

$$C_2^S(t) = -D_0^S(t), \quad D_2^S(t) = -C_0^S(t);$$

$$\frac{d\Gamma_3^S}{c_2\,dc_2\,dt} = \frac{P}{2}\big[I_3^S(t) + \big(|a(t)|^2 - |v(t)|^2\big)A_3^S(t) + \\ + \mathrm{Re}(a(t))C_3^S(t) + \mathrm{Re}(v(t))D_3^S(t)\big], \quad (62)$$

$$I_3^S(t) = \frac{8M}{(t-m^2)t}\big[-(M^2-t)(t+2m^2+3M^2) + \\ + (M^4 + 3M^2t + m^2(t+M^2))L\big],$$

$$A_3^S(t) = -\frac{1}{2}A_1(t), \quad C_3^S(t) = -D_1^S(t),$$

$$D_3^S(t) = -C_1^S(t).$$

It is easy to see that all the quantities $d\Gamma_0^S$ and $d\Gamma_i^S$, vanish under the integration over the full angular region because they are proportional to $c_2\,dc_2$.

In Fig. 5, we show the quantities $d\Gamma_0^S/c_2\,dc_2\,dt$, which is a part of the differential decay width that depends on the τ^- polarization, the ratio

$$A^S(t) = \frac{2d\Gamma_0^S/(c_2\,dc_2\,dt)}{d\Gamma_0/\,dt} \quad (63)$$

about which we can say that $c_2A^S(t)$ is the decay polarization asymmetry at fixed values of c_2 and t, and the parameters

$$\xi_i^S(t) = \frac{2d\Gamma_i^S/(c_2\,dc_2\,dt)}{d\Gamma_0/dt}, \quad i = 1, 2, 3, \quad (64)$$

which characterize different correlations between the polarization states of τ^- and the γ quantum in process (1).

5.3. Dependence on the photon energy

The photon energy distribution involves integration over the pion energy in the limits defined by inequality (6). This integration cannot be performed analytically because of the nontrivial dependence of the vector and axial-vector form factors on the pion energy. In this section, we illustrate the results of our numerical calculations for both the unpolarized (Fig. 6) and polarized (Fig. 7) τ lepton.

In the polarized case, we define the quantities $A^S(\omega)$ and $\xi_i^S(\omega)$ in full analogy with relations (63) and (64) for $A^S(t)$ and $\xi_i^S(t)$.

6. DISCUSSION

To determine the moduli and phases of the form factors, a procedure was suggested in Ref. [9] (see Eq. (66)) that does not require polarization measurements. To

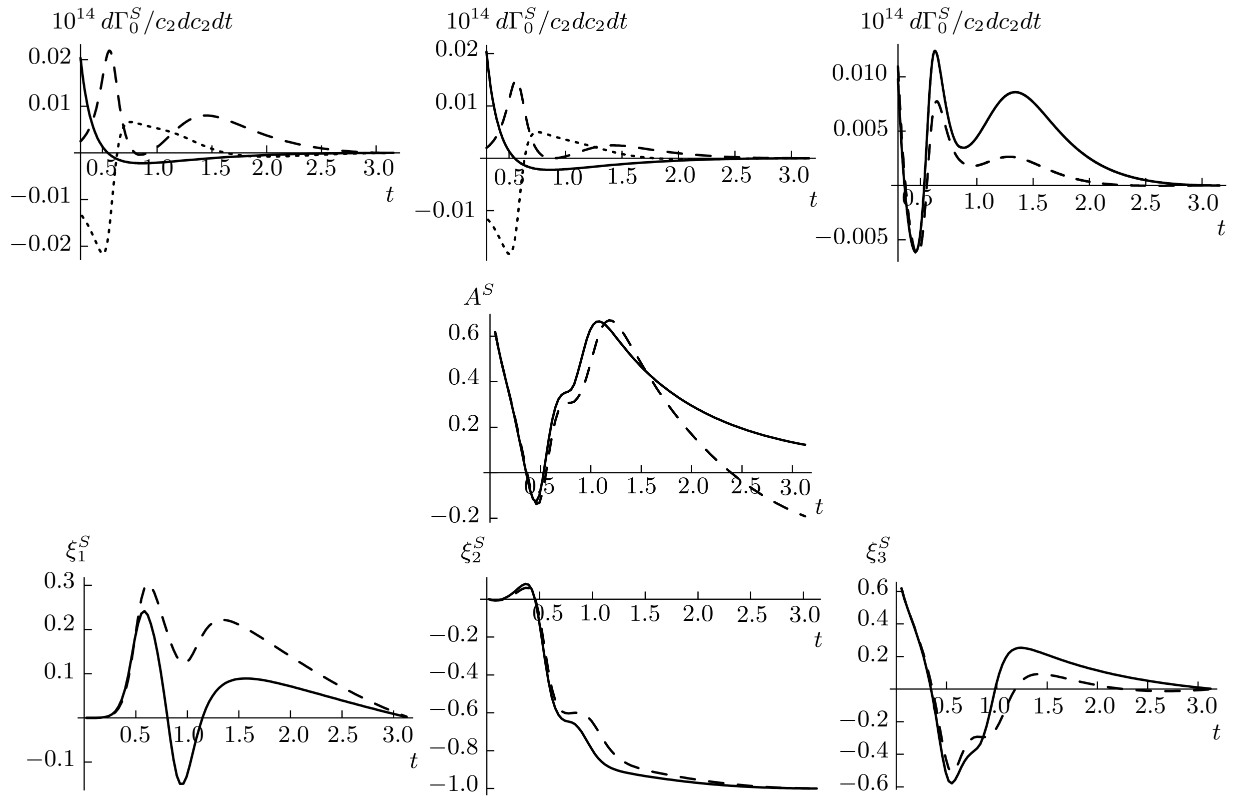

Fig. 5. The notation for the curves in the upper and lower rows is the same as in Fig. 4. The figure in the middle row shows the quantity defined in accordance with Eq. (63) for two sets of the parameters. The ξ_i^S are defined in Eq. (64)

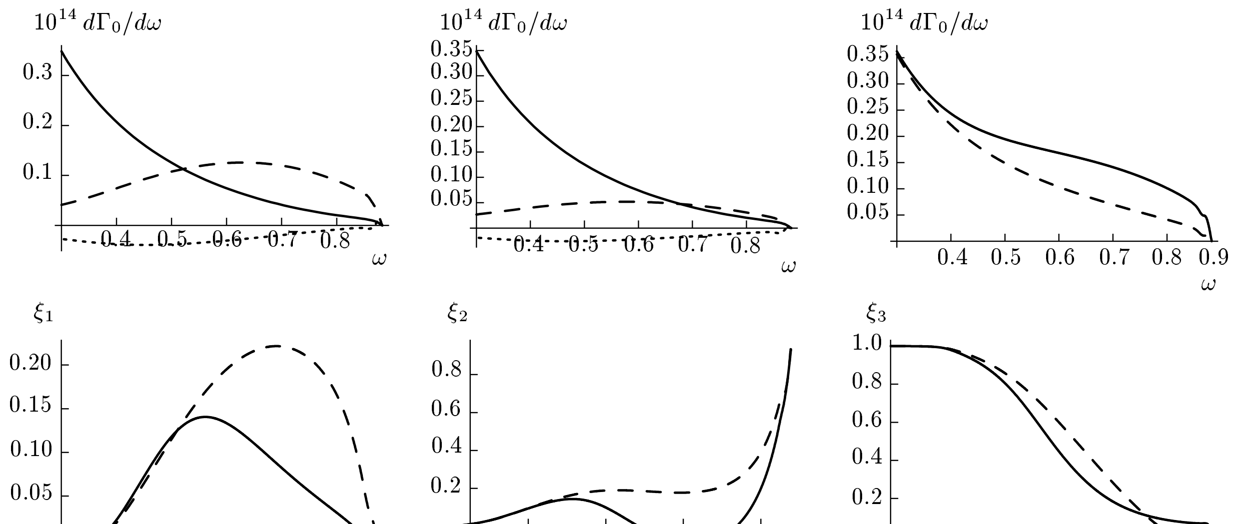

Fig. 6. The photon spectrum in decay (1) and the Stokes parameters versus the photon energy. The notation for the curves is the same as in Fig. 4

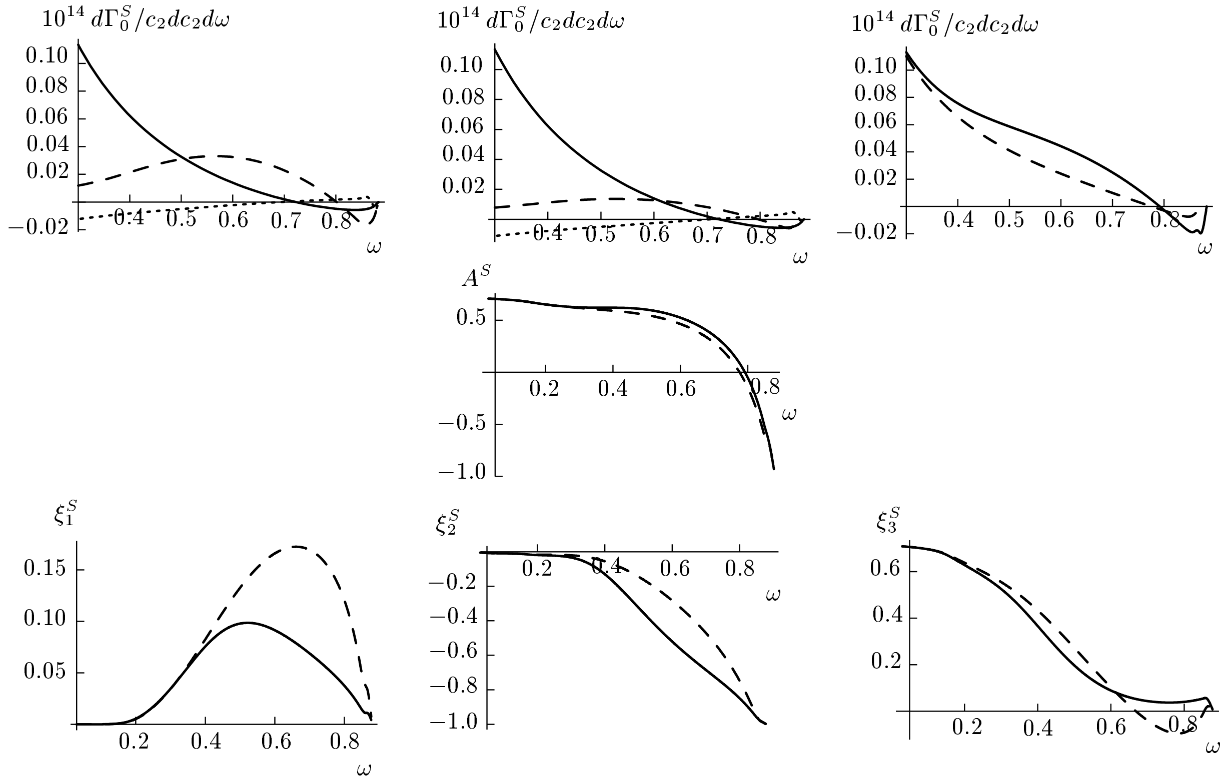

Fig. 7. The spin-dependent part of the photon spectrum, the decay polarization asymmetry, and the spin-correlation parameters as functions of the photon energy. The notation for the curves is the same as in Fig. 5

separate the contributions of different form factor combinations, it was suggested to measure the photon energy dependence of the differential probability $d\Gamma/d\omega\, dt$ at a fixed t value (i. e., at a fixed value of the sum of the photon and pion energies). The obtained expression (in the zero-pion-mass approximation) for this quantity is a sum of terms multiplied by the photon energy to negative and positive powers. Measuring this distribution, in principle, permits finding the coefficients of this power series. The combinations of the form factors $\mathrm{Re}\, v(t)$, $\mathrm{Re}\, a(t)$, $|v(t)|^2 + |v(t)|^2$ and $\mathrm{Re}\, a(t)v^*(t)$ can be determined from these coefficients. We note that our calculations are performed with the pion mass taken into account. This is important for the decay $\tau^- \to K^-\gamma\nu_\tau$ where it is necessary to take the kaon mass into account.

Equation (56) shows that the Stokes parameter ξ_1, as a function of the variable t, is determined by the imaginary parts of the vector and axial-vector form factors. But it follows from the unitarity condition that $\mathrm{Im}\, v(t) \neq 0$ for $t > 4m^2$ and $\mathrm{Im}\, a(t) \neq 0$ for $t > 9m^2$. Hence, this parameter must be zero for $t < 4m^2$. Thus, the value of $\xi_1(t)$ is completely determined by the resonance contribution to the matrix element. Measuring this parameter in the region $t > 4m^2$ can test the validity of this mechanism for the description of decay (1). Of particular interest is the region of high values of t, where it may be necessary to include the contribution of the additional resonances beyond the ρ and a_1 mesons that are included in this paper. We can see from Fig. 4 that the $\xi_1(t)$ parameter is sensitive to the choice of the parameters describing the resonance contribution. In the region 1 GeV^2 $< t <$ 1.5 GeV^2, the parameter $\xi_1(t)$ has opposite signs for the parameter sets 1 and 2. The same conclusions are valid for the spin correlation coefficient $\xi_1^s(t)$ (the Stokes parameter $\xi_1(t)$ that depends on the τ lepton polarization vector), as is seen from Fig. 5. The Stokes parameters $\xi_1(\omega)$ and $\xi_1^s(\omega)$, as functions of the photon energy, are also sensitive to the choice of the parameter sets, but in the region $t >$ 1.5 GeV^2. In this case, the signs of the parameters $\xi_1(\omega)$ and $\xi_1^s(\omega)$ are the same for both parameter sets.

The Stokes parameter $\xi_3(t)$ contains the contributions of the IB, the interference between the IB and resonance terms (which is determined by $\mathrm{Re}\, a(t)$), and a resonance term that depends on the combination $|a(t)|^2 - |v(t)|^2$. Therefore, this parameter is less sensitive to the choice of the parameter sets than the Stokes parameter $\xi_1(t)$. Nevertheless, a sizeable sensitivity exists in the region 0.5 GeV^2 $< t <$ 1 GeV^2. The Stokes parameter $\xi_3^s(t)$ is appreciably less sensitive to the choice of the parameter sets than $\xi_3(t)$ (Fig. 5). The Stokes parameter $\xi_3(\omega)$ ($\xi_3^s(\omega)$), as a function of the photon energy, is also less sensitive than the parameter $\xi_1(\omega)$ ($\xi_1^s(\omega)$).

We recall that the meaning of the parameters ξ_1 and ξ_3 requires knowing the photon polarization vectors e_1 and e_2. But that requires knowing the photon and pion momenta. As regards the parameter ξ_2, in this case it suffices to know only the photon momentum.

The Stokes parameter $\xi_2(t)$ contains the contributions of the IB, the interference between the IB and resonance terms (which is determined by the $\mathrm{Re}\, a(t)$ and $\mathrm{Re}\, v(t)$), and a resonance term that depends on $\mathrm{Re}(a(t)v^*(t))$. From Fig. 4, we can see that this parameter is sensitive to the choice of the parameter sets in the region of high values of the variable t ($t > 1$ GeV^2). The Stokes parameter $\xi_2^s(t)$ is weakly sensitive to that choice. The corresponding parameters, as functions of the photon energy, show a greater sensitivity to this choice in comparison with the same parameters as functions of the t variable.

The photon energy has to be large enough to study the sensitivity to the choice of the model parameters. Although the number of events in this region is an order of magnitude smaller than in the region of low photon energies, where the IB-contribution dominates, we can expect the high-statistics precision measurements at the Super c–τ and SuperKEKB factories to improve some model resonance parameters used in this paper and a number of other papers.

We also note that in Refs. [9, 35], the authors suggested studying the resonance mechanism in the radiative τ decay by selecting events (in the rest frame) at the maximal possible pion energy $\varepsilon = (M^2 + m^2)/2M$, where the IB contribution to the decay width vanishes due to the radiative zeros of electromagnetic amplitudes for point-like particles [36]. But the corresponding number of events decreases due to the essential decrease of the kinematic region. On the other hand, one can be sure that at a chosen direction of the axes, the IB contribution to the Stokes parameter $\xi_1(\varepsilon, \omega)$ vanishes in the whole kinematic region.

Most of the analytic calculations presented in this paper can also be used for analysis of the decay $\tau^- \to \nu_\tau K^- \gamma$. All the results in Secs. 2 and 3 will then remain the same, apart from trivial changes of the masses, form factors, and CKM matrix elements. Moreover, the t-distributions obtained in Sec. 5 will retain their form, but will be expressed in terms of the kaon mass and the corresponding vector and axial-vector form factors. The latter can be derived in the RχT framework following the procedure in Sec. 4. We plan to perform calculations for the decay $\tau^- \to \nu_\tau K^- \gamma$ in the future.

7. CONCLUSION

We have investigated the radiative one-meson decay of the τ lepton, $\tau^- \to \pi^- \gamma \nu_\tau$. The photon energy spectrum and the t-distribution (t is the square of the invariant mass of the pion–photon system) of the unpolarized τ-lepton decay were calculated. We also studied the polarization effects in this decay. The following polarization observables have been calculated: the asymmetry caused by the τ-lepton polarization, the Stokes parameters of the emitted photon, and spin correlation coefficients that describe the influence of the τ-lepton polarization on the photon Stokes parameters.

The amplitude of the τ-lepton decay $\tau^- \to \pi^- \gamma \nu_\tau$ has two contributions: the inner bremsstrahlung, which does not contain any free parameters, and a structure-dependent term, which is parameterized in terms of the vector and axial-vector form factors. We note that in our case, these form factors are functions of the t variable and $t > 0$, i. e., we are in the time-like region. The form factors in this region are complex functions, and finding not only their moduli but also their phases is non-trivial in this case. This requires performing polarization measurements.

We calculated the unpolarized and polarized observables for two sets of the parameters describing the vector and axial-vector form factors. A numerical estimation shows that some polarization observables (the asymmetry and the Stokes parameters, especially ξ_1) can be effectively used for the discrimination between two parameter sets because these observables have opposite signs in some regions of the variable t or the photon energy.

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APPENDIX A

Interactions in the framework of chiral theory with resonances

In this Appendix, we outline the framework for the calculation of the form factors for the decay $\tau^- \to \nu_\tau \pi^- \gamma$. We use the formalism of chiral theory with resonances (RχT) suggested in Refs. [17, 18]. The corresponding Lagrangian can be written as

$$\mathcal{L}_{R\chi T} = \mathcal{L}_\chi^{(2)} + \mathcal{L}_R, \tag{A.1}$$

where the $\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$ chiral Lagrangian in the order $\mathcal{O}(p^2)$ is

$$\mathcal{L}_\chi^{(2)} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle, \tag{A.2}$$

$$\begin{aligned} u_\mu &= i[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger], \\ \chi_+ &= u^\dagger\chi u^\dagger + u\chi^\dagger u\,, \quad \chi = 2B_0(s+ip), \end{aligned} \tag{A.3}$$

where $\langle\ldots\rangle$ denotes the trace in the flavor space and F is the pion weak decay constant in the chiral limit.

The octet of pseudoscalar mesons P with $J^P = 0^-$ is included in the matrix

$$u(\Phi) = \exp\left(\frac{i\Phi}{\sqrt{2}F}\right) = 1 + \frac{i\Phi}{\sqrt{2}F} - \frac{\Phi^2}{4F^2} + \ldots\,,$$

where

$$\Phi = \begin{pmatrix} \pi^0/\sqrt{2} + \eta_8/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta_8/\sqrt{6} & K^0 \\ K^- & \bar{K}^0 & -2\eta_8/\sqrt{6} \end{pmatrix}. \tag{A.4}$$

The masses of the pseudoscalar mesons enter Eq. (A.2) via the quark mass matrix $\mathcal{M}$:

$$\chi = 2B_0\mathcal{M} + \ldots\,, \quad \mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)\,, \tag{A.5}$$

and the constant B_0 is expressed through the quark condensate $\langle\bar{q}q\rangle$ as

$$\begin{aligned} B_0 = -\frac{\langle\bar{q}q\rangle}{F^2} &= \frac{M^2_{\pi^0,\pm}}{m_u + m_d} = \frac{M^2_{K^0}}{m_d + m_s} = \\ &= \frac{M^2_{K^\pm}}{m_u + m_s} = \frac{3M^2_\eta}{m_u + m_d + 4m_s}. \end{aligned} \tag{A.6}$$

The condensate value is $\langle\bar{q}q\rangle \approx (-240 \pm 10\ \mathrm{MeV})^3$ (at the energy scale $\mu = 1$ GeV). In the limit of exact isospin symmetry, $\chi = \mathrm{diag}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2)$ in terms of the π- and K-meson masses m_π and m_K.

The interaction of the pseudoscalar mesons with the $W_\mu^\pm$ and Z_μ bosons and the electromagnetic field B_μ can be included via the external fields r_μ and l_μ as

$$\begin{aligned} r_\mu &= -eQB_\mu + g\frac{\sin^2\theta_W}{\cos\theta_W}QZ_\mu, \\ l_\mu &= -eQB_\mu - \frac{g}{2\sqrt{2}}W_\mu + \\ &+ \frac{g}{\cos\theta_W}\left(Q\sin^2\theta_W + \frac{1}{6} - Q\right)Z_\mu. \end{aligned} \tag{A.7}$$

Here, the quark charge matrix is

$$Q = \mathrm{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right),$$

$e = \sqrt{4\pi\alpha}$ is the positron charge, $g = e/\sin\theta_W$ is the $\mathrm{SU}(2)_L$ coupling constant, and θ_W is the weak angle. We also introduce the notation

$$W_\mu \equiv W_\mu^+ T_+ + W_\mu^- T_-, \tag{A.8}$$

where $W_\mu^\pm = (W_1 \mp W_2)_\mu$ is the field of the charged weak bosons, and the matrices T_+ and T_- are defined as

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_- = T_+^\dagger,$$

with V_{ud} and V_{us} being the CKM-matrix elements [26].

The lowest-order even-intrinsic-parity Lagrangian $\mathcal{L}_R$, describing the coupling of the resonance fields to the pseudoscalars, has been suggested in Ref. [17]. It is linear in the resonance fields. In this Lagrangian, we keep the contributions from the vector and axial-vector mesons relevant for the process of τ decay to π and γ:

$$\begin{aligned} \mathcal{L}_R &= \mathcal{L}^R_{kin} + \mathcal{L}^R_{int}, \\ \mathcal{L}^R_{kin} &= -\frac{1}{2}\times \\ &\times \sum_{R=V,A}\left\langle \nabla^\lambda R_{\lambda\mu}\,\nabla_\nu R^{\nu\mu} - \frac{M_R^2}{2}R_{\nu\mu}R^{\nu\mu}\right\rangle, \\ \mathcal{L}^R_{int} &= \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle + \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle + \\ &+ \frac{iF_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle\,, \end{aligned} \tag{A.9}$$

where F_V, G_V, and F_A are the coupling constants, and the antisymmetric tensor representation for the spin-1 fields $V_{\mu\nu}$ and $A_{\mu\nu}$ is used [17]. Also,

$$\begin{aligned} f_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u, & F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu], \\ & & F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu], \\ \nabla_\mu X &= \partial_\mu X + [\Gamma_\mu, X], & \Gamma_\mu &= 1/2\quad \{u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\}. \end{aligned} \tag{A.10}$$

The fields $R = \{V, A\}$ represent nonets (an octet and a singlet) of the vector and axial-vector resonances with the lowest masses. In general, the higher-lying multiplets can be added if needed. However, already the low-lying resonances saturate the low-energy constants of chiral perturbation theory [17]. We do not include interactions nonlinear in the resonance fields; the structure of such terms has been investigated in Refs. [19, 37, 38].

To describe the vector form factor, we need the interaction Lagrangian in the odd-intrinsic-parity sector. The lowest-order $\mathcal{O}(p^4)$ interactions follow from the Wess–Zumino–Witten (WZW) functional [39]. In it, suffices to keep only the terms that are linear in the pseudoscalar fields Φ and contain two external fields l_μ and r_μ. Thus, we retain

$$\mathcal{L}^{(2)}_{WZW} = -\frac{\sqrt{2}N_C}{48\pi^2 F}\,\epsilon^{\mu\nu\rho\sigma} \times \\ \times \operatorname{Tr}\left[\partial_\mu\Phi\left(\frac{1}{2}l_\sigma\partial_\nu r_\rho + \frac{1}{2}r_\sigma\partial_\nu l_\rho + \frac{1}{2}r_\nu\partial_\rho l_\sigma + \right.\right. \\ + \frac{1}{2}l_\nu\partial_\rho r_\sigma + l_\nu\partial_\rho l_\sigma + r_\nu\partial_\rho r_\sigma + l_\sigma\partial_\nu l_\rho + \\ \left.\left. + r_\sigma\partial_\nu r_\rho\right)\right], \quad \text{(A.11)}$$

where $N_C = 3$ is the number of quark colors, and $\epsilon^{0123} = +1$.

The external fields r_μ and l_μ are further expressed in terms of the electromagnetic field and the field of the W boson in Eq. (A.7). Choosing only the electromagnetic field in Eq. (A.11), we would obtain $\pi^0\gamma\gamma$, $\eta\gamma\gamma$, and $\eta'\gamma\gamma$ couplings. Here, we are interested in the terms proportional to both fields, the photon and the W boson:

$$\mathcal{L}_{\gamma W\Phi} = \frac{3egN_C}{48\pi^2 F}\,\epsilon^{\mu\nu\rho\sigma} \times \\ \times \operatorname{Tr}\left[\Phi\left(\partial_\mu B_\nu Q\partial_\rho W_\sigma + \partial_\rho W_\sigma Q\partial_\mu B_\nu\right)\right] = \\ = \frac{egN_C}{48\pi^2 F}\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu B_\nu\left[V_{ud}(\pi^-\partial_\rho W^+_\sigma + \pi^+\partial_\rho W^-_\sigma) + \right. \\ \left. + V_{us}(K^-\partial_\rho W^+_\sigma + K^+\partial_\rho W^-_\sigma)\right]. \quad \text{(A.12)}$$

An additional odd-intrinsic-parity interaction relevant for the transition $W \to V \to P\gamma$ is considered in Sec. 4.

From Lagrangians (A.1), (A.11), and (A.12), we can obtain the necessary terms describing interactions of the pseudoscalar mesons, the resonances, the W boson, and the photon.

APPENDIX B

Polarization of the τ^- lepton in electron–positron annihilation

We consider the polarization of the τ^- lepton in the annihilation process

$$e^-(p_1) + e^+(p_2) \to \tau^-(q_1) + \tau^+(q_2)$$

in the case where the electron has nonzero longitudinal polarization. The effect can be understood by means of the corresponding matrix element squared

$$|M|^2 = E^{\mu\nu}T_{\mu\nu},$$

$$\frac{1}{2}E^{\mu\nu} = -\frac{s}{2}g^{\mu\nu} + (p_1p_2)^{\mu\nu} + i\lambda(\mu\nu p_1p_2),$$

$$\frac{1}{2}\,T_{\mu\nu} = -\frac{s}{2}g^{\mu\nu} + (q_1q_2)^{\mu\nu} + iM(\mu\nu qS), \\ q = p_1 + p_2 = q_1 + q_2, \quad q^2 = s, \quad \text{(B.1)}$$

where S is the τ-polarization 4-vector and λ is the electron polarization degree.

The contraction of the tensors in Eq. (B.1) is given by (neglecting the electron mass)

$$|M|^2 = A + B(S), \\ A = s^2\left[1 + \cos^2\theta + \frac{4M^2}{s}\sin^2\theta\right], \\ B(S) = -4\lambda Ms[(p_1S) - (p_2S)], \quad \text{(B.2)}$$

where θ is the angle between the momenta of the electron and the τ^- lepton.

The polarizations of τ^- are defined as [29]

$$P^L = \frac{B(S^L)}{A}, \quad P^T = \frac{B(S^T)}{A}. \quad \text{(B.3)}$$

If we choose the 4-vectors S^L and S^T in such a way that in the rest frame of the τ^- lepton they are

$$S^L = (0, \mathbf{n_1}), \quad S^T = (0, \mathbf{n_2}), \\ \mathbf{n}_1\cdot\mathbf{n}_2 = 0, \quad \mathbf{n}_1^2 = \mathbf{n}_2^2 = 1, \quad \text{(B.4)}$$

where $\mathbf{n_1}$ is in the direction of the 3-vector $\mathbf{q_1}$ in the center-of-mass system and $\mathbf{n}_2$ belongs to the scattering plane, then we can write the covariant form of the 4-vectors S^L and S^T in terms of the momentum 4-vectors, namely (neglecting the electron mass)

$$S^L = \frac{(q_1q_2)q_1 - M^2q_2}{M\sqrt{(q_1q_2)^2 - M^4}},$$

$$S^T = \frac{1}{N}\{[M^2-(p_2q_1)]q_1 + + [M^2-(p_1q_1)]q_2 + [(p_1p_2)-2M^2]p_1\}, \quad (B.5)$$

$$N^2 = [(p_1p_2)-2M^2][2(q_1p_1)(q_1p_2) - M^2(p_1p_2)].$$

It is easy to show that in the rest frame of τ^-, the covariant forms in Eq. (B5) coincide with relations given in Eq. (B4). Using the above formulas, we find

$$B(S^L) = 2\lambda s^2\cos\theta, \quad B(S^T) = 4\lambda M s\sqrt{s}\sin\theta, \quad (B.6)$$

and the corresponding results for the polarizations P^L and P^T are given in the text.

At the Super c–τ factory planned in Novosibirsk, τ-pairs will be created near the threshold, where the directions of their 3-momenta are not determined, and therefore the above evaluations are not convenient. The only preferred direction for the considered reaction in the center-of-mass system is then the direction of the colliding beams. This means that choosing the unit 3-vector $\mathbf{n}_1$ in Eq. (B.4) to lie along the electron beam direction $\mathbf{p}_1$ and, as before, having $\mathbf{n}_2$ in the reaction plane, we are able to go to the threshold limit and clearly interpret the τ^--polarization states. This requires a modification of the corresponding covariant forms for the polarization 4-vectors as follows:

$$S^l = \frac{1}{M}\left(\frac{M^2}{(p_1q_1)}p_1 - q_1\right),$$

$$S^t = \frac{1}{N_1}\left[\left(\frac{M^2}{(p_1q_1)} - 1 + \frac{(p_1q_1)}{(p_1p_2)}\right)p_1 + + \frac{(p_1q_1)}{(p_1p_2)}p_2 - q_1\right], \quad (B.7)$$

$$N_1^2 = 2(p_1q_1) - M^2 - \frac{2(p_1q_1)^2}{(p_1p_2)}.$$

In this case, we have

$$B(S^{l,t}) = 2\lambda s^2 D^{l,t},$$

$$D^l = \frac{4M^2}{s}\left[1-\sqrt{1-\frac{4M^2}{s}}\cos\theta\right]^{-1} - \sqrt{1-\frac{4M^2}{s}}\cos\theta,$$

$$D^t = -\frac{2M}{\sqrt{s}}\sqrt{1-\frac{4M^2}{s}}\sin\theta,$$

and near the threshold, when

$$1-\frac{4M^2}{s} \ll 1,$$

we have

$$P^l = \lambda\left[1-\frac{1}{2}\left(1-\frac{4M^2}{s}\right)\sin^2\theta\right], \quad P^t = -\frac{2\lambda M\sin\theta}{\sqrt{s}}\sqrt{1-\frac{4M^2}{s}}. \quad (B.8)$$

Exactly at the threshold ($s = 4M^2$), $P^l = \lambda$ and $P^t = 0$.

Thus, we see that near the threshold, the τ lepton practically keeps the longitudinal polarization of the electron beam.

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