

THE VARIED NONLINEAR INTERFACE WAVES IN A SCALE-LIMITED TWO-MEDIUM OSCILLATORY SYSTEM

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We investigate spontaneously generated waves around the interfaces between two different media in a system where the domain scales are limited. These two media are carefully selected such that there exists a theoretical interface wave with the frequency and wave number that can be predicted according to the control parameters. We present the rules of how the frequency and wave number vary with reducing the scales of media domains. We find that the frequency decreases with reducing the scale of antiwave (AW) media, but increases with reducing the scale of normal wave (NW) media in both one-dimensional and two-dimensional systems. The wave number always decreases with reducing scales of either NW or AW media. The least scale to generate the theoretical wave is the predicted wavelength. These special phenomena around the interfaces may be applied to detect the limited scale of a system.

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1. INTRODUCTION

The formation, propagation, and interaction of nonlinear oscillation around the interface between different media have long been interesting topics. The results have attracted much attention in recent decades in the area of reaction–diffusion systems, optics, ultrasonic, biological systems, and so on. Numerous characteristic features and complex phenomena have been investigated [1–12]. Together with the normal wave (NW) with positive phase velocity, the recently found antiwave (AW) with negative phase velocity provides even more interesting dynamical behaviors and pattern formations [8–12].

The method for generating an antiwave in a homogeneous oscillatory medium has no essential difference from that for normal waves. A proper initial condition or an external pacing can both produce the desired wave. Researches have mentioned that the control parameters of media determine the kind or kinds of wave that can be produced in it. Pacing can generate NW in NW media, AW in AW media, and different pacings can generate either NW or AW in N–AW media due to the dispersion relation determined by the control parameters [13, 14]. One interesting phenomenon that involves both an NW and an AW is the generation of an interface-selected wave (ISW), first noted in [15]. When a system is constructed by two linked domains of one NW medium and one AW medium, and the dispersion relations of these two media have an intersection point, the ISW can be generated spontaneously from the interface of the two media. The ISW has the frequency and the wave number that can be theoretically calculated by the dispersion relations of two media. Once an ISW is generated, it eventually occupies the whole two-medium system.

Intuitively, the ISW is the result of an interplay of two different media at the interface. The geometrical shape and scale of each medium should have no effect on the dynamical behavior. Also, a system on which a theoretical experiment carried out is normally large enough to avoid the effect of a boundary, if the system can be considered a continuous medium, not a series of

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discrete grids. But by reducing the scale properly, it might become possible to observe the detailed behavior occurring exactly on the interface and reveal the generation process of interface waves.

We first study the interface waves by reducing the scales of media in a one-dimensional ($1D$) two-medium system. It is surprising that the frequency and wave number of the generated ISW both vary continuously with the scale of the media. Analyzing the results, we find that there is a necessary condition for the generated wave to be the theoretically predicted one, i. e., have the same frequency and wave number. That is, the length scale of each participant medium should be at least equal to half the predicted wavelength. Once the condition is fulfilled, the frequency and wave number of the survivor wave are equal to the predicted ones. Otherwise, the two-medium system can still enter a homogeneous dynamics, but with a different oscillating frequency and wave number, which are related to the geometrical scale of the system. Numerical studies in a two-dimensional ($2D$) patched system yield similar results. The variation is the same. However, irrespective of how large each participant domain is, the generated frequency can never reach the theoretical value.

This paper is organized as follows. In Sec. 2, we explain the selected model and our motivation. In Sec. 3, we present series of results for $1D$ and $2D$ two-medium oscillatory systems. The alterations of frequency and wave number are specified in different situations. Section 4 contains a discussion and analysis of the phenomena. Section 5 is our conclusion.

2. MOTIVATION

We construct our system using the complex Ginzburg–Landau equation, which is the commonly used model describing extended systems in the vicinity of a Hopf bifurcation from a homogeneous stationary state [16–19]:

$$\frac{\partial A}{\partial t} = A - (1 + i\alpha)|A|^2 A + (1 + i\beta)\nabla^2 A. \quad (1)$$

For a general reaction–diffusion system, the dynamics of oscillations with the amplitude and phase are scaled to one complex order parameter A. When time is scaled by the characteristic reaction time, space by the characteristic diffusion length, and the modulus of the amplitude by the radius of the limit cycle, the remaining two control parameters α and β govern the universal dynamics around the bifurcation [19].

If the frequency ω of the external pacing is close to the natural frequency $\omega_0 = \alpha$ of the media, the dispersion relation

$$\omega = \omega_0 + f_1 k^2 = \alpha + (\beta - \alpha)k^2$$

determines the characteristics of generated waves in the following way under the conditions of a 1:1 pacing-reaction region and $\omega\omega_0 > 0$ [14]:

$$\text{AWs — for } \omega_0 f_1 = \alpha(\beta - \alpha) < 0, \quad (2a)$$

$$\text{NWs — for } \omega_0 f_1 = \alpha(\beta - \alpha) > 0. \quad (2b)$$

We have $|\omega| < |\omega_0|$ for AWs and $|\omega| > |\omega_0|$ for NWs. For two media whose dispersion relation lines have an intersection at one's AW region and one's NW region, ISW trains emerge at the interface of these two media. This means that the ISW trains are always normal waves in the NW domain and antiwaves in the AW domain. Because the frequencies used in this paper are always positive, the wave numbers for different waves then have different signs. For simplicity and convenience, we focus on the absolute value and the square of the wave number in different regions.

Theoretically, the control parameters of two media determine the frequency and wave number of the generated ISW as

$$\omega_I = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\alpha_2 - \alpha_1 + \beta_1 - \beta_2}, \quad (3a)$$

$$k_I^2 = \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_1 + \beta_1 - \beta_2}. \quad (3b)$$

In previous studies, the results are perfectly well consistent with the theoretical values in $1D$ systems [15, 20]. But there are some inconsistent situations in $2D$ systems, for example, in the patching system where one medium is surrounded by another. In that case, the frequency is always slightly different from the theoretical value [20–23]. This has not yet been given a clear explanation.

If the structure of media can alter the dynamical behavior, what happens when the geometrical scale is changed? If the ISW is generated exactly on the interface, the geometrical scale should have no effect on the properties of the waves. However, from another standpoint, if the generation of an ISW requires a range of the medium, a very small system may not be able to produce ISWs. Studies in this paper aim to answer these questions. By reducing the geometrical scale of a two-medium system in several ways and by comparing the results in $1D$ and $2D$ systems, we reveal the exact dynamics occurring around the interface.

In the next section, we first present our experimental results for a $1D$ two-medium system. The results for a $2D$ patched two-medium system are then shown.

3. RESULTS OF MANIPULATING THE SCALES OF MEDIA IN A TWO-MEDIUM SYSTEM

3.1. Reducing one medium in a $1D$ two-medium system

We construct a one-dimensional two-medium system as follows:

$$\frac{\partial A_1}{\partial t} = A_1 - (1+i\alpha_1)|A_1|^2 A_1 + (1+i\beta_1)\nabla^2 A_1, \quad 0 \le x \le L_1, \tag{4a}$$

$$\frac{\partial A_2}{\partial t} = A_2 - (1+i\alpha_2)|A_2|^2 A_2 + (1+i\beta_2)\nabla^2 A_2, \quad L_1 \le x \le L_1 + L_2 + 1, \tag{4b}$$

$$A_{1(I)} = A_{2(I)}, \quad \frac{\partial A_{1(I)}}{\partial n} = \frac{\partial A_{2(I)}}{\partial n}. \tag{4c}$$

The system is divided into two domains. We let the left domain M_1 be an AW media of length L_1, and the right domain M_2 be an NW media of length L_2. Equation (4c) is the continuity condition, where I means the value on the interface and $\partial A_{i(I)}/\partial n$ is the gradient to the normal direction. No-flux boundary conditions are used on all outer boundaries, such that the inner dynamics is not affected. We then set $A(0) = A(1)$ and $A(L_1 + L_2 + 1) = A(L_1 + L_2)$, and $[1, L_1 + L_2]$ is the area that we calculated. The system is integrated using second order Runge–Kutta (RK2) method and the standard three-point approximation for the Laplace operator.

We carefully choose the control parameters such that the interface select waves can be generated. For the AW media M_1, $\alpha_1 = 0.4$ and $\beta_1 = -1.0$ ($\alpha_1(\beta_1 - \alpha_1) < 0$). For the NW media M_2, $\alpha_2 = 0.2$ and $\beta_2 = 2.0$ ($\alpha_2(\beta_2 - \alpha_2) > 0$). According to Eqs. (3a) and (3b), the frequency and wave number of the ISW can be theoretically predicted as $\omega_I = 0.3125$ and $k_I^2 = 0.0625$, whence $|k_I| = 0.25$. As shown in Fig. 1*a*, the dispersion relation curves of these two media have an intersection point with the coordinates $\omega_I = 0.3125$ and $k_I^2 = 0.0625$. That implies that if these two media are connected together, the ISWs emerge from the interface spontaneously, with the frequency and wave number equal to the coordinates of the intersection point. In Fig. 1*b*, we show a spatiotemporal pattern of the two-medium system in which the AW media M_1 and the NW media M_2 occupy half the system each. The system is integrated with the RK2 method and the standard three-point approximation for the Laplace operator. The space and time steps are $\Delta x = 0.5$ and $\Delta t = 0.005$. The outer boundaries are set with the no-flux boundary condition, while the interface between different media is set with the continuity condition. All the following patterns utilize the same parameters, integration method, boundary condition, and discrete steps. It is obvious that from the very beginning of evolution, ISWs are generated from the interface and propagate gradually into both domains with different speeds and different directions of the phase velocity. As predicted, eventually, the ISWs occupy the whole system. The interface becomes transparent in the final pattern. The frequency and wave number are $\omega_I = 0.3125$, and $|k_I| = 0.25$, which are equal to the theoretical predictions in Fig. 1*a*.

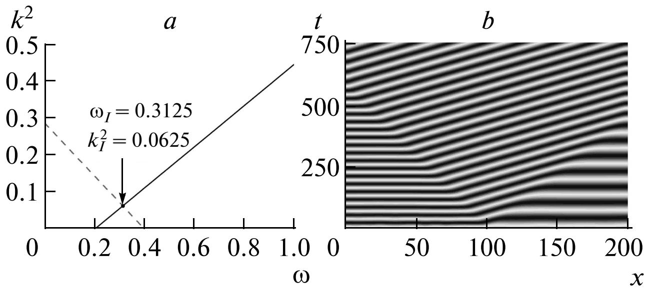


Fig. 1. *a*) Dispersion relation curves of an AW medium (dashed curve) with the control parameters $\alpha_1 = 0.4$ and $\beta_1 = -1.0$, and an NW medium (solid curve) with the control parameters $\alpha_2 = 0.2$ and $\beta_2 = 2.0$. *b*) Spatiotemporal pattern of a $1D$ two-medium system. The left domain of the length $L_1 = 100$ and the right domain of the length $L_2 = 100$ are respectively filled with AW and NW media, whose dispersion curves are both presented in Fig. *a*

If the dynamics of ISWs is determined by grids right on the interface, the characteristics do not change irrespective of how large the media domains are. However, by reducing the geometrical scales of the domains, we do alter the characteristics of ISWs. The systems in Fig. 2 are of the same size as in Fig. 1*b*. The AW domain in Fig. 2*a* is reduced to $L_1 = 2$, while the NW domain is enlarged to $L_2 = 198$. The AW domain in Fig. 2*b* is enlarged to $L_1 = 198$, while the NW domain is reduced to $L_2 = 2$. The systems are finally entirely occupied by ISWs. But the ultimate patterns are different from each other in both the horizontal spatial axis and the vertical time axis. In Fig. 2*a*, the frequency and wave number values are changed to $\omega = 0.2860$ and $|k| = 0.2185$. In Fig. 2*b*, they are $\omega = 0.3758$ and $|k| = 0.1318$. The respective relative errors in frequency compared to ω_I in Fig. 2*a* and 2*b* are 8.5 %

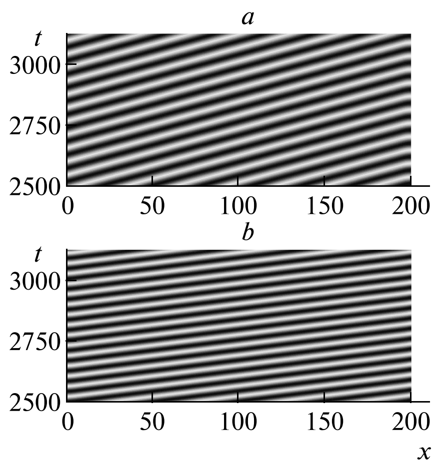


Fig. 2. Ultimate spatiotemporal patterns of a $1D$ two-medium system of the same total length $L = 200$ as in Fig. 1. The control parameters are also the same as Fig. 1. *a*) The left AW medium domain has the length $L_1 = 2$, while the right NW medium domain has the length $L_2 = 198$. The frequency decreases to $\omega = 0.2860$, and the wave number decreases to $|k| = 0.2185$. *b*) The left AW medium has the length $L_1 = 198$, and the right NW medium has the length $L_2 = 2$. The frequency increases to $\omega = 0.3758$, but the wave number decreases to $|k| = 0.1318$

and 20 %. The relative errors in wave number compared to $|k_I|$ are 13 % and 47 %. These changes are purely caused by the geometrical scale reduction of one medium in the system, because the control parameters and boundary conditions are the same. This indicates that the emergence of ISWs may not rely only on the interface grids. A range of both media around the interface produces the wave. Changing the AW and NW media leads to different results. With $\omega = 0.2860$ in the dispersion relation for M_2 (the larger domain in Fig. 2*a*), we obtain

$$|k| = \sqrt{\frac{\omega - \alpha_2}{\beta_2 - \alpha_2}} = 0.2185,$$

which is exactly the practical value of the wave number. With $\omega = 0.3758$ in the dispersion relation for M_1 (the larger domain in Fig. 2*b*), the wave number becomes

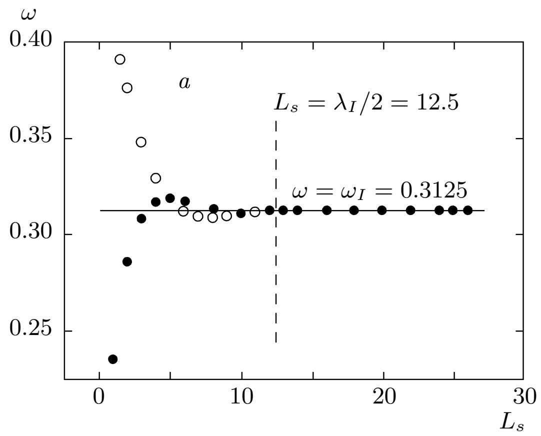
$$|k| = \sqrt{\frac{\omega - \alpha_1}{\beta_1 - \alpha_1}} = 0.1315,$$

which is quite close to the real value. This is logical because the main part of the system can determine the characteristics of the generated ISWs.

To reveal the range of media required for general evolution of ISWs, we gradually increase the geometrical scale of each domain of the different media from the minimum limit. When the domain of one medium is increased, the domain of the other is decreased, such that total geometrical size of the system is kept constant. By measuring the frequency and wave number of the ultimate patterns, we find different phenomena in changing the domains of different media. The results are shown in Fig. 3. The open circles represent the results in the system with a relatively smaller length scale of the NW medium and a larger length scale of the AW medium. The solid circles represent the results in the system with a relatively small AW media and a larger NW media. We let L_s denote the length of the smaller domain in the system. While increasing the AW medium length from $L_1 = L_s = 1$ to $L_1 = L_s = 30$, the domain of the NW medium is decreased from $L_2 = 199$ to $L_2 = 170$. The circles in Fig. 3 illustrate that along with increasing the length scale of the AW medium, both the frequency and the wave number increase continuously. However, when we increase the domain scale of the NW media instead, the frequency turns out to decrease as shown by the solid points in Fig. 3*a*, while the wave number value keeps increasing as shown in Fig. 3*b*. Similarly, around the length $L_s = 12.5$, both the frequency and the wave number approach constant values, which are equal to the theoretical ISW values. Interestingly, the length is equal to half the wavelength of the theoretical ISW for $\lambda_I = 2\pi/k_I \approx 25$. This means that in a two-medium system with one medium smaller than half the wavelength of the theoretical ISW, the generated ISW is not the theoretical predicted one. The exact frequency value is then close to the natural frequency of the medium that occupies the larger domain. The frequency determines the wave number. For simplicity and convenience, the wave numbers we show in Fig. 3*b* are the absolute values, not the exact ones, because the signs for the antiwave and the normal wave are different.

3.2. Reducing both parts in a $1D$ two-medium system

In the above study, the total length scale of the system is kept constant, which is much larger than the wavelength of theoretical ISWs. According to the previous results, we are sure that by changing the system scale, the generated ISWs can be altered. We then keep the domain scales of two media to be equal, and gradually increase them from the minimum limit $L = 1$ to



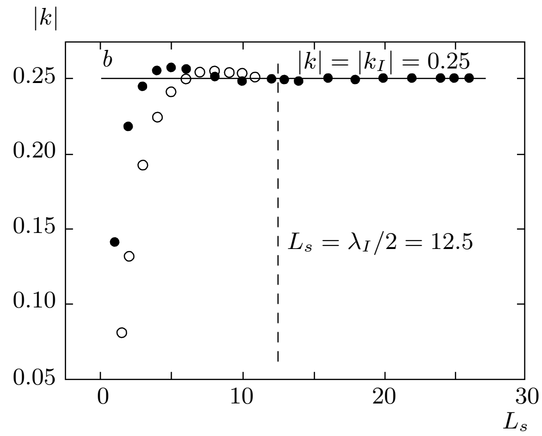


Fig. 3. The frequencies and wave numbers calculated from the ultimate pattern of the two-medium system. L_s is the length of the smaller domain in the system. For open circles, the smaller domain is an NW medium; and for solid circles, the smaller domain is an AW medium. *a*) In systems with the smaller AW medium, the frequency increases with increasing the AW domain length. However, in systems with the smaller NW medium, the frequency decreases. When the length is increased to $\lambda_I/2$, the frequencies reach the value ω_I. *b*) In systems with the smaller AW medium and NW medium, the wave number always increases with increasing the length, and reaches the value $|k_I|$ after the length increases to $\lambda_I/2$

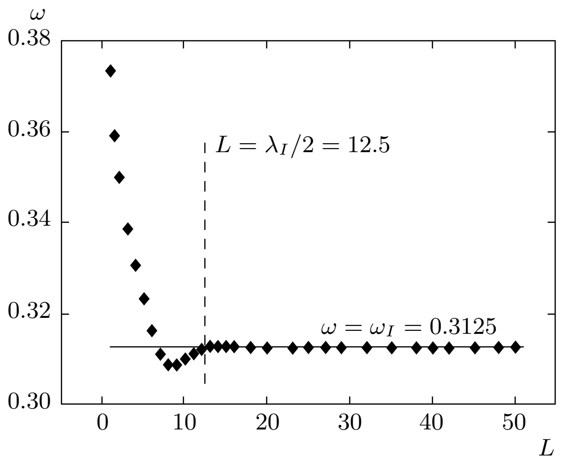


Fig. 4. Frequencies measured in a system of two domains of equal size L but different media, $L_1 = L_2 = L$. The control parameters are the same as in Fig. 1. As the geometrical scale of the system increases, the frequency of the generated ISWs decreases. When the total length reaches λ_I, meaning the length of each domain reaches $\lambda_I/2$, the frequency equals ω_I

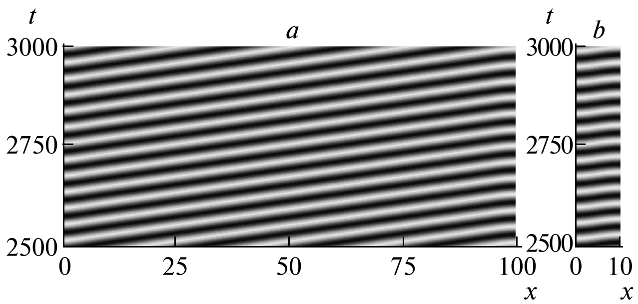


Fig. 5. Spatiotemporal pattern of a two-medium system with each domain of equal size. $L_1 = L_2 = 50$ (*a*), 5 (*b*). In both systems, the interface-generated waves propagate gradually into two domains, and eventually occupy the whole system. But in the smaller system, the frequency is larger than in the other one

twice the theoretical ISW wavelength $L = 2\lambda_I = 50$. We note that $L = L_1 = L_2$ is half the length of the whole system. We see from Fig. 4 that as the length of the system increases to λ_I, the frequency value approaches ω_I. For a smaller system, the frequency is larger. This is consistent with the previous result, because each domain then has the length $\lambda_I/2$. Along with increasing the system length, the frequency decreases.

In Fig. 5, we present two spatiotemporal pattern examples: one with $L = 50$ and the other with $L = 5$. The respective output frequencies are $\omega = 0.3125$ and $\omega = 0.3232$. It is clear that in a small system, the interface can still generate a wave with the same frequency in two different media. Although it is different from the theoretical prediction, the generated wave can still occupy the whole system. In extreme cases, the system is smaller than the wave length. It is then impossible to measure the wavelength and the wave number precisely. In fact, the wave number has no significant physical meaning because the system with few grids cannot be considered a continuous medium.

3.3. Reducing the center domain of a 2D two-medium system

The study of a 2D two-medium system with two parallel domains yields the same results as in 1D system. This confirms the effect of changing the geometrical scale of the system.

However, in 2D two-medium systems, there do exist different structures, for example, one medium can be surrounded by the other. This is what we called a patched system. The patch is typically used to control the patched medium, so as to obtain the desired pattern. For the patched system containing both NW and AW media, a perfect target wave can be generated, which propagates in the whole system as previous researches have mentioned. But the generated frequencies and wave numbers are not exactly equal to the theoretical prediction [20].

We then change the size of the patch. Similar changes of the frequency related to the size of the patch are observed. In previous studies, we have found that the generated wave eventually evolves into a perfect target wave, irrespective of whether the patch is round or square. We therefore take the round patch as a general example.

The patched 2D two-medium system is constructed as follows:

$$\frac{\partial A_1}{\partial t} = A_1-(1+i\alpha_1)|A_1|^2A_1+(1+i\beta_1)\nabla^2A_1, \quad 0 \le r \le R, \tag{5a}$$

$$\frac{\partial A_2}{\partial t} = A_2-(1+i\alpha_2)|A_2|^2A_2+(1+i\beta_2)\nabla^2A_2, \quad r \ge R, \tag{5b}$$

$$A_{1(I)} = A_{2(I)}, \quad \frac{\partial A_{1(I)}}{\partial n} = \frac{\partial A_{2(I)}}{\partial n}, \tag{5c}$$

where r is the distance from grids to the center point of the system. The main system is square with the length $L \times L$, the round inner patch medium with a radius R is denoted by M_1, and the outer medium is denoted by M_2. Continuity condition (5c) is the same as in 1D systems. We apply the same time and space step and the same numerical method and boundary conditions as in 1D systems.

Firstly, the AW medium is placed outside, and the NW medium is placed inside as a patch. For the solid points shown in Fig. 6, the radius of NW patch is increased from $R = 5$ to $R = 95$, while the system remains 200×200. As the radius increases, the frequency decreases. When the radius approaches $\lambda_I/2$, meaning that the diameter approaches λ_I, the frequency reaches $\omega = 0.3250$. It is very close to the theoretical ω_I with a relative error of 4.0 %. But with $R = 5$, the frequency is $\omega = 0.3774$, and the relative error becomes 21 %. The change of frequency in the patching 2D system is much more obvious than in the parallel 2D two-medium system, or the 1D two-medium system. With a larger patch, the frequency approaches the theoretical ω_I. But it cannot reach the exact value, even if the total size of the system is increased. It is tested in a system with twice the total size and the patch of the same size, where the frequency remains the same. We note that when the radius is larger than $R = 50$, the range of the NW patch is actually larger than the outside AW media. Yet the frequency is still in the upper zone.

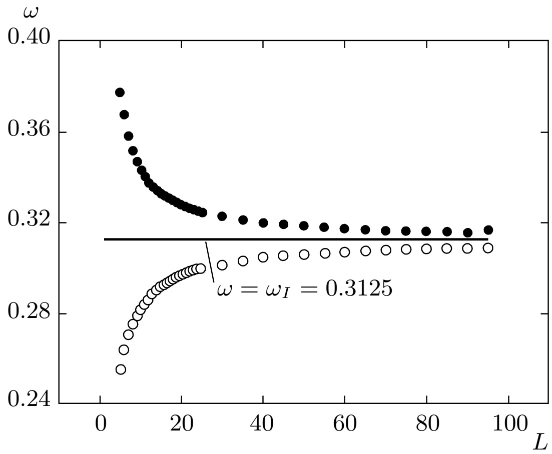


Fig. 6. The alteration of frequencies with changing the radius of a round patch in a 2D two-medium system. The patched system consists of an AW medium and an NW medium, whose parameters are the same as in Fig. 1. For the solid circles, the patch is the NW medium. The frequency decreases from 0.3774 to 0.3173 as the patch radius increases from 5 to 95. For the open circles, the patch is the AW medium. The frequency increases from 0.2558 to 0.3094. In both cases, the frequencies are approaching the theoretical value

If we exchange the media of the inside patch and the outside domain, the changing tendency of the frequency becomes opposite to the above rule. As shown in Fig. 6 with open circles, it is then smaller than the theoretical prediction, and increases as the domain of the inside AW patch increases, just as for the 1D two-medium system.

4. DISCUSSION

In both $1D$ and $2D$ two-medium systems with relatively altered length scales, we find the frequency and wave number being not consistent with the theoretical predictions.

If and only if every part of the system is equal to or larger than half the wavelength of the theoretical ISW can the generated wave train have the same frequency and wave number as the predicted values. This implies that the least scale to generate the predicted ISW is the theoretical wavelength, and every medium should occupy a half of the system. If the total size is fixed, either one being smaller than half the wavelength, the absolute value of the produced wave number decreases with reducing the geometrical scale, for both the NW and AW media. But the frequency decreases with reducing the AW medium and increases with reducing the NW medium. As a result, the frequency is always larger than ω_I when AW medium occupies the larger domain; and it is smaller than ω_I if the NW medium occupies the larger domain. The alterations of the wave number and frequency indicate that the generation of an ISW involves not only the interface but also the dynamics of a certain domain around the interface. The exact frequency and wave number are then related to the limited size of the media.

In each case that we studied, the systems always approach the same vibration frequency. This confirms that linked media with different control parameters tend to find a wave with a particular frequency that can propagate in both domains. The competition occurs around the interface. The dynamics of every grid is influenced by the grids next to it. When grids of the NW medium and the AW medium are linked, they approach a frequency between the larger natural frequency of the AW medium and the smaller one of the NW medium. The range of area that affects the interplay result is the wavelength scale.

We have confirmed that all the discrete time and space steps applied have a good fit in our numerical programs. The number of grids increases from small to large to simulate the behaviors of a bi-domain oscillatory system. The free boundary condition assures that the dynamics of oscillations is not affected by the limited area. And the continuity condition is used between the media. In the cases of an extremely small system, the dynamics exhibited is an interplay of a few grids. Then the system can be considered not a continuous medium, but a series of scattered grids. Although the wave number has no essential physical meaning, the frequency still reflects the cooperation mechanism of the different media.

We choose the complex Ginzburg–Landau equation to construct our numerical experimental system because it may be the most generally applied nonlinear equation to demonstrate the oscillatory dynamics in physics. Reaction–diffusion systems, superconductivity, fluid dynamics, and many other physical phenomena at different scales can be mapped to the complex Ginzburg–Landau equation [1, 16–19]. The results that we obtain may be tested and applied to realistic systems. There might be more complicated phenomena in unscaled systems. That requires further numerical and experimental research.

The geometrical magnitude of a medium is of great importance especially in biological systems. Because the biological tissue may not be as large as we expected, the limited size may have a great effect on the behavior of signal transportation. The varied frequencies and wave numbers are related to the structure of the system. Our results may be helpful in understanding wave propagation in systems with a limited scale, such as the superficial soft tissue interfaces.

5. CONCLUSION

We find up–down mirror images of varied frequencies in changing the scales of NW and AW media in both $1D$ and $2D$ two-medium systems, and an increasing rule of wave numbers in all cases. The least scale of the domain to produce the theoretical ISW is the predicted wavelength. If the length scale of every part of a $1D$ system is equal to or larger than the wavelength, the generated wave has the theoretical frequency and the wave number. Reducing the NW medium causes an increase in frequency, but reducing the AW medium causes a decrease in frequency. For a system containing NW and AW media with the same scale, if the total scale is smaller than the wavelength, the frequency is larger than the theoretical one. And the frequency increases basically as the scale decreases.

All the above results show that the generation of an interface wave involves a certain range of media around the interface rather than the grids right on it. The interplay of different media with different scales yields various patterns. A crucial condition to predict them is the relation between the wavelength and the system scale.

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REFERENCES

1. M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
2. D. Donskoy, A. Sutin, and A. Ekimov, NDT and E Int. **34**, 231 (2001).
3. C. Pecorari, J. Acoust. Soc. Amer. **113**, 3065 (2003).
4. V. F. Nesterenko, C. Daraio, E. B. Herbold, and S. Jin, Phys. Rev. Lett. **95**, 158702 (2005).
5. Jun Zhu, Weiqiu Chen, and Guiru Ye, Ultrasonics **52**, 125 (2012).
6. D. V. Kulagin, G. G. Levchenko, A. S. Savchenko et al., Zh. Eksp. Teor. Fiz. **141**, 540 (2012) [JETP **114**, 474 (2012)].
7. R. Kumar and V. Chawla, Int. Comm. Heat Mass Transfer **48**, 53 (2013).
8. E. Cubukcu, K. Aydin, E. Ozbay, S. Foteinopoulou, and C. M. Soukoulis, Nature **423**, 604 (2003).
9. G. Dolling, M. Wegener, C. M. Soukoulis, and S. Linden, Opt. Lett. **32**, 53 (2007).
10. V. K. Vanag and I. R. Epstein, Int. J. Dev. Biol. **53**, 673 (2009).
11. Xujin Yuan, Hongli Wang, and Qi Ouyang, Phys. Rev. Lett. **106**, 188303 (2011).
12. A. V. Amirkhizi and S. Nemat-Nasser, Theor. Appl. Mech. **38**, 299 (2011).
13. Z. Cao, P. Li, H. Zhang, and G. Hu, Int. J. Mod. Phys. B **21**, 4170 (2007).
14. X. Huang, X. Liao, X. Cui, H. Zhang, and G. Hu, Phys. Rev. E **80**, 036211 (2009).
15. X. Cui, X. Huang, Z. Cao, H. Zhang, and G. Hu, Phys. Rev. E **78**, 026202 (2008).
16. Y. Kuramoto, *Chemical Oscillation, Waves and Turbulence*, Springer-Verlag, Berlin (1984).
17. M. Ipsen, L. Kramer, and P. G. Sorensen, Phys. Rep. **337**, 193 (2000).
18. I. S. Aranson and L. Kramer, Rev. Mod. Phys. **74**, 99 (2002).
19. V. García-Morales and K. Krischer, Contemp. Phys. **53**(2), 79 (2012).
20. X. Huang, X. Cui, X. Liao, and G. Hu, Europhys. Lett. **95**, 24001 (2011).
21. H. Nagashima, J. Phys. Soc. Jpn. **60**, 2797 (1991).
22. M. Stich and A. S. Mikhailov, Physica D **215**, 38 (2006).
23. R. Manor, A. Hagberg, and E. Meron, New J. Phys. **11**, 063016 (2009).