

RECIPROCAL RELATIONS IN DISSIPATIONLESS HYDRODYNAMICS

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Hidden symmetry in dissipationless terms of arbitrary hydrodynamics equations is recognized. We demonstrate that all fluxes are generated by a single function and derive conventional Euler equations using the proposed formalism.

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I am enjoying the privilege to learn from Alexander Fedorovich Andreev for about 20 years. One of the doors he opened for me gives access to the power of hydrodynamics. The universal derivation procedure for its equations (seemingly pioneered by Landau) is based purely on the conservation laws and symmetry considerations. In this paper, I would like to paraphrase this procedure in a formal way without specifying a particular system.

Generic equations of hydrodynamics are contained [1] in local conservation laws

$$\dot{y}_\alpha + \frac{\partial j_\alpha^k}{\partial x^k} = 0. \quad (1)$$

Here and below, Latin superscripts are used for 3D space coordinates and Greek subscripts enumerate the integrals of motion; summation over repeated indices is assumed. Densities of these conserved quantities y_α form a complete set of proper thermodynamic variables and uniquely specify all local equilibrium properties.

In the dissipationless approximation, an additional conservation law exists:

$$\dot{\sigma} + \frac{\partial f^k}{\partial x^k} = 0, \quad (2)$$

where σ is the entropy density,

$$S = \int \sigma dV.$$

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The entropy density is a function of state,

$$d\sigma = \frac{\partial \sigma}{\partial y_\alpha} dy_\alpha. \quad (3)$$

Equations (1) have internal symmetry that is revealed if expressed in terms of the thermodynamically conjugate quantities (see [2])

$$Y_\alpha = -\frac{\partial \sigma}{\partial y_\alpha}.$$

The transformation from y_α to Y_α is invertible and we can use either set of variables to characterize the state.

The fluxes j_α^k and f^k generally depend on Y_α (or, equivalently, on y_α) and their spatial derivatives. In the dissipationless approximation, the dependence on spatial derivatives is neglected. We can therefore substitute (3) in (2) and transform as follows:

$$0 = \frac{\partial \sigma}{\partial y_\alpha} \dot{y}_\alpha + \frac{\partial f^k}{\partial x^k} = \left(Y_\alpha \frac{\partial j_\alpha^k}{\partial Y_\beta} + \frac{\partial f^k}{\partial Y_\beta} \right) \frac{\partial Y_\beta}{\partial x^k}.$$

This equation holds for arbitrary $\partial Y_\beta / \partial x^k$ and the expression in parentheses is identically zero:

$$0 = Y_\alpha \frac{\partial \mathbf{j}_\alpha}{\partial Y_\beta} + \frac{\partial \mathbf{f}}{\partial Y_\beta} = \frac{\partial}{\partial Y_\beta} (Y_\alpha \mathbf{j}_\alpha + \mathbf{f}) - \mathbf{j}_\beta.$$

In other words, all fluxes \mathbf{j}_α are generated by a single function \mathbf{g} ,

$$\mathbf{j}_\alpha = \frac{\partial \mathbf{g}}{\partial Y_\alpha}, \quad (4)$$

and the entropy flux \mathbf{f} is its Legendre transform:

$$\mathbf{f} = \mathbf{g} - Y_\alpha \mathbf{j}_\alpha. \quad (5)$$

This completes the proof that the matrix of derivatives $\partial j_\alpha^k / \partial Y_\beta$ is symmetric,

$$\frac{\partial \mathbf{j}_\alpha}{\partial Y_\beta} = \frac{\partial \mathbf{j}_\beta}{\partial Y_\alpha}, \quad (6)$$

in agreement¹⁾ with Onsager's principle [3].

The proof does not rely upon any specific property of the entropy itself. In fact, Eq. (2) is just one of local conservation laws and "thermodynamically conjugate quantities" Y_α could have been defined with respect to any other integral of motion. In practice, it might be more convenient to use energy rather than entropy for this.

To illustrate this formalism, we consider a classical ideal fluid. Energy per unit volume E is a function of other conserved quantities:

$$dE = T d\sigma + \left(\mu - \frac{v^2}{2} \right) d\rho + \mathbf{v} d\mathbf{j},$$

where μ is the chemical potential, ρ is the mass density, \mathbf{v} is the fluid velocity, and \mathbf{j} is the momentum density. The conjugate variables are therefore

$$Y_\sigma = -T, \quad Y_\rho = v^2/2 - \mu, \quad Y_{\mathbf{j}} = -\mathbf{v}.$$

Due to the fluid isotropy, the generating function is

$$\mathbf{g} = \mathbf{v}h(T, Y_\rho, v),$$

where the scalar h can be obtained from the identity $\mathbf{j}_\rho = \mathbf{j} = \rho \mathbf{v}$:

$$\rho \mathbf{v} = \left(\frac{\partial \mathbf{g}}{\partial Y_\rho} \right)_{T, \mathbf{v}} = -\mathbf{v} \left(\frac{\partial h}{\partial \mu} \right)_{T, \mathbf{v}}.$$

Recalling the expression for the pressure differential

$$dp = \sigma dT + \rho d\mu,$$

we obtain $h = -p$. It can be easily verified that the function $\mathbf{g} = -p\mathbf{v}$ generates conventional Euler fluxes:

$$\mathbf{j}_\sigma = \left(\frac{\partial \mathbf{g}}{\partial Y_\sigma} \right)_{Y_\rho, Y_{\mathbf{j}}} = \sigma \mathbf{v},$$

$$\Pi^{ik} = j_{j^k}^i = \left(\frac{\partial g^i}{\partial Y_{j^k}} \right)_{Y_\sigma, Y_\rho} = p\delta^{ik} + \rho v^i v^k.$$

REFERENCES

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2. L. D. Landau and E. M. Lifshitz, *Statistical Physics*, part 1, Pergamon Press, Oxford (1980).
3. L. Onsager, Phys. Rev. **37**, 405 (1931).

¹⁾ Macroscopic reversibility implies that \mathbf{j}_α and Y_α (or y_α) behave oppositely under time reversal.