ЖЭТФ, 2013, том 144, вып. 4 (10), стр. 756–764

EFFECT OF THE REFRACTIVE INDEX ON THE HAWKING TEMPERATURE: AN APPLICATION OF THE HAMILTON–JACOBI METHOD

*I. Sakalli**, *S. F. Mirekhtiary***

Department of Physics, Eastern Mediterranean University
G. Magosa, North Cyprus, Mersin 10, Turkey

Received April 20, 2013

Hawking radiation of a non-asymptotically flat 4-dimensional spherically symmetric and static dilatonic black hole (BH) via the Hamilton–Jacobi (HJ) method is studied. In addition to the naive coordinates, we use four more different coordinate systems that are well-behaved at the horizon. Except for the isotropic coordinates, direct computation by the HJ method leads to the standard Hawking temperature for all coordinate systems. The isotropic coordinates allow extracting the index of refraction from the Fermat metric. It is explicitly shown that the index of refraction determines the value of the tunneling rate and its natural consequence, the Hawking temperature. The isotropic coordinates in the conventional HJ method produce a wrong result for the temperature of the linear dilaton. Here, we explain how this discrepancy can be resolved by regularizing the integral possessing a pole at the horizon.

DOI: 10.7868/S0044451013100088

1. INTRODUCTION

In 1974, Hawking [1, 2] proved, taking quantum effects into account, that a BH can emit thermal radiation. This means, that each BH has a characteristic temperature and can be regarded as a thermodynamical system. In fact, this discovery broke all taboos that were classically imposed on BHs until that day. Together with Bekenstein's work [3], it led to the birth of a new subject, the so-called quantum gravity theory, which has not yet been completed. After Hawking, there has always been interest in deriving new methods for the Hawking radiation (HR), which can decode the underlying BH spacetime. Today, many methods for the HR have been found in the literature (see [4] and the references therein for a general review). Among them, the most promising one is the tunneling method of Kraus and Wilczek (KW) [5, 6]. KW used the null-geodesic method to develop the action for the tunneling particle that is considered a self-gravitating thin spherical shell and then managed to quantize it. The KW method provides a dynamical model of HR in which the BH shrinks as particles are radiated. In this dynamical model, both energy conservation and back-reaction effects are included, which were not considered in the original derivation of HR. Six years later, their calculations were reinterpreted by Parikh and Wilczek (PW) [7]. They showed that the HR spectrum can deviate from pure thermality, which implies unitarity of the underlying quantum process and the resolution of the information loss paradox [8, 9]. Nowadays, PW's pioneering work is still preserving its popularity. Numerous works for various BHs proves its validity (we refer the reader to [10]). As far as we know, the original PW's tunneling method only suffers from one of the non-asymptoticaly flat (NAF) BHs, which is the so-called linear dilaton BH (LDBH). In contrast to the other well-known BHs, its evaporation does not admit nonthermal radiation, and therefore causes the violation of information conservation. This problem was first unraveled in [11]. Recently, it was shown that the weakness of the PW's method in retrieving the information from the LDBH can be overcome by adding quantum corrections to the entropy [12]. Furthermore, it was proved in another study [13] that the entropy of the LDBH can be tweaked by the quantum effects such that both its temperature and mass simultaneously become zero at the end of the complete evaporation.

*E-mail: izzet.sakalli@emu.edu.tr
**E-mail: fatemeh.mirekhtiary@emu.edu.tr

Based on the complex path analysis in [14–16], the authors of [17] developed an alternative method for calculating the imaginary part of the action belonging to the tunneling particles. For this, they used the relativistic HJ equation. Their method neglects the effects of particle self-gravitation and involves the WKB approximation. In general, the relativistic HJ equation can be solved by substituting a suitable ansatz. The chosen ansatz should respect the symmetries of the spacetime in order to allow for the separability. The resulting equation thus obtained is solved by integrating along the classically forbidden trajectory that starts inside the BH and ends at the outside observer. However, the integral has always a pole located at the horizon. For this reason, the method of complex path analysis must be used to circumvent the pole.

A Friedmann–Robertson–Walker universe — which is assumed to be a good model for our universe — is generally NAF [18]. For this reason, we believe that most of the BHs in the real universe necessarily have NAF geometries. Hence, it is of our special interest to find specific examples of NAF BHs as a test bed for HR problems within the HJ method. Starting from this idea, we consider the LDBHs in this paper. First of all, the eponym of these BHs are Clément and Gal'tsov [19]. Initially, LDBHs were found as a solution of the Einstein–Maxwell–dilaton (EMD) theory [20] in four dimensions. Later on, it was shown that in addition to the EMD theory, $N \geq 4$ dimensional LDBHs (even in the case of higher dimensions) are available in Einstein–Yang–Mills–dilaton (EYMD) and Einstein–Yang–Mills–Born–Infeld–dilaton (EYMBID) theories (see [21] and the references therein). The most intriguing feature of these BHs is that while radiating, they undergo an isothermal process. Namely, their temperature does not alter with the shrinking of the BH horizon or with the mass loss. Our primary concern in this study is to obtain the imaginary part of the action of the tunneling particle through the LDBH horizon. This produces the tunneling rate that yields the Hawking temperature. To test the HJ method on the LDBH, in addition to the naive coordinates, we consider four other coordinate systems (all regular): isotropic, Painlevé–Gullstrand (PG), ingoing Eddington–Finkelstein (IEF), and Kruskal–Szekeres (KS). Especially, we mainly focus on the isotropic coordinates. They require more straightforward calculations compared with the others. Furthermore, as we show in what follows, the use of the standard HJ method with isotropic coordinates reveals a discrepancy in the temperatures. For a more recent account in the same line of thought applied to the Schwarzschild BH within the isotropic coordinates, we refer to [22], where a similar discrepancy problem in HR has been studied. Gaining inspiration from [22], we also discuss how the discrepancy appearing in the LDBH radiation can be removed. Differently from [22], we also present the calculation of the index of refraction of the LDBH medium, its effect on the tunneling rate and consequently on the Hawking temperature. According to our knowledge, such a theoretical observation has not been reported before in the literature. Slightly different from the other coordinate systems, in applying the HJ method in the KS coordinates, we first reduce the LDBH spacetime to Minkowski space and then demonstrate in detail how the Hawking temperature recovered.

The paper uses the signature $(-,+,+,+)$ and units where

$$c = G = \hbar = k_B = 1.$$

The paper is organized as follows. In Sec. 2, we review some of the geometrical and thermodynamical features of the LDBH with naive coordinates and show the separation of variables of the relativistic HJ equation. The calculation of the tunneling rate and henceforth the Hawking temperature via the HJ method is also presented. In Sec. 3, the metric for an LDBH in isotropic coordinates is derived. The effect of the index of refraction on the tunneling rate is explicitly shown. The obtained temperature is half the accepted value of the Hawking temperature. It is demonstrated how the proper regularization of singular integrals resolves the discrepancy in the aforementioned temperatures. Sections 4 and 5 are devoted to the calculation of the Hawking temperature in PG and IEF coordinate systems. In Sec. 6, we apply the HJ method to the KS form of LDBHs. Finally, the conclusion and future directions are given in Sec. 7.

2. LDBH AND HJ METHOD

In general, the metric of a spherically symmetric and static BH in four dimensions is given by

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + R^2 d\Omega^2, \tag{1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2, \tag{2}$$

is the metric on the unit two-sphere S^2. Because we aim to solve the relativistic HJ equation for a massive but uncharged scalar field in the LDBH background, we first analyze the geometry of the LDBH. Whenever the metric functions of line element (1) are given by

$$R^2 = A^2 r, \quad f = \Sigma(r - r_h), \tag{3}$$

we call the metric in (1) the LDBH [19, 21]. In various theories (EMD, EYMD, and EYMBID), metric functions do not alter their forms as seen in Eq. (3). Only nonzero positive constants A and Σ take different values depending on which theory is used [21]. It can be easily deduced from the metric function f that LDBHs have an NAF geometry and r_h represents the horizon. For $r_h \neq 0$, the horizon hides the null singularity at $r = 0$. Even in the extreme case $r_h = 0$ in which the central null singularity at $r = 0$ is marginally trapped, such that outgoing waves are permitted to reach the external observers, the LDBH still sustains its BH property.

The NAF structure of the LDBH leads us to use the definition of a quasi-local mass M [23] to obtain a relation between the horizon r_h and the mass M of the BH as follows

$$r_h = \frac{4M}{\Sigma A^2}. \tag{4}$$

According to the laws of BH thermodynamics, the conventional definition of the Hawking temperature T_H [24] is expressed in terms of the surface gravity κ. For metric (1), T_H is explicity given as

$$T_H = \frac{\kappa}{2\pi} = \left.\frac{\partial_r f}{4\pi}\right|_{r=r_h}. \tag{5}$$

After substituting metric function f (3) in the above equation, T_H of the LDBH becomes

$$T_H = \frac{\Sigma}{4\pi}. \tag{6}$$

We can immediately observe that the obtained temperature is constant. In general, this typically occurs in an isothermal process of the standard thermodynamics in which $\Delta T = 0$. Therefore, the LDBH's radiation is such a particular process that the energy (mass) transfer out of the BH typically occurs at a slow rate such that thermal equilibrium is maintained.

Here, we consider the problem of a scalar particle that moves in this spacetime, while there is no back-reaction or self-gravitational effect. Within the semi-classical framework, the classical action I of the particle satisfies the relativistic HJ equation

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0, \tag{7}$$

in which m is the mass of the scalar particle and $g^{\mu\nu}$ represents the inverse metric tensors derived from metric (1). By considering Eqs. (1), (3), and (7), we obtain

$$-\frac{1}{f}(\partial_t I)^2 + f(\partial_r I)^2 + \\ + \frac{1}{A^2 r}\left[(\partial_\theta I)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi I)^2\right] + m^2 = 0. \tag{8}$$

For the HJ equation, it is common to use the separation-of-variables method for the action $I = I(t, r, \theta, \varphi)$ as follows:

$$I = -Et + W(r) + J(x^i), \tag{9}$$

where

$$\partial_t I = -E, \quad \partial_r I = \partial_r W(r), \quad \partial_i I = J_i, \tag{10}$$

and the J_i are constants in which $i = 1, 2$ label respectively angular coordinates θ and φ. Since the norm of the timelike Killing vector ∂_t is (negative) unity at a particular location

$$r \equiv \check{r} = \frac{1}{\Sigma} + r_h,$$

E is the energy of the particle detected by an observer at $\check{r}$, which is outside the horizon. Solving for $W(r)$ yields

$$W(r) = \\ = \pm\int \frac{\sqrt{E^2 - \frac{f}{A^2 r}\left[J_\theta^2 + \frac{J_\varphi^2}{\sin^2\theta} + (mA)^2 r\right]}}{f}\, dr, \tag{11}$$

where $\pm$ occurs naturally because Eq. (8) is quadratic in $W(r)$. The solution of Eq. (11) with the plus sign corresponds to scalar particles moving away from the BH (outgoing) and the solution with the minus sign represents particles that move toward the BH (incoming). After evaluating the above integral around the pole at the horizon (adhering to Feynman's prescription [25]), we arrives at

$$W_{(\pm)} = \pm\frac{i\pi E}{\Sigma} + c, \tag{12}$$

where c is a complex integration constant. Thus, we can deduce that imaginary parts of the action can arise due to the pole at the horizon and from the complex constant c. Hence, we can determine the probabilities of incoming and outgoing particles while crossing the horizon as

$$P_{out} = \exp(-2\,\mathrm{Im}\, I) = \\ = \exp\left[-2(\mathrm{Im}\, W_{(+)} + \mathrm{Im}\, c)\right], \tag{13}$$

$$P_{in} = \exp(-2\,\mathrm{Im}\,I) = \\ = \exp\left[-2(\mathrm{Im}\,W_{(-)} + \mathrm{Im}\,c)\right]. \quad (14)$$

From the classical standpoint, a BH absorbs any incoming particles crossing its horizon. In other words, there is no reflection for the incoming waves, which corresponds to $P_{in} = 1$. This is enabled by setting

$$\mathrm{Im}\,c = \frac{\pi E}{\Sigma}.$$

This choice also implies that the imaginary part of the action I for a tunneling particle can only originate in W_+. Namely, we obtain

$$\mathrm{Im}\,I = \mathrm{Im}\,W_+ = \frac{2\pi E}{\Sigma}, \quad (15)$$

which is independent of the horizon radius r_h. Therefore, the tunneling rate for the LDBH can be obtained as

$$\Gamma = P_{out} = \exp\left(\frac{-4\pi E}{\Sigma}\right), \quad (16)$$

and since [7]

$$\Gamma = \exp(-\beta E), \quad (17)$$

where β denotes the Boltzmann factor and $T = 1/\beta$, we can easily find the horizon temperature of the LDBH as

$$\check{T}_H = \frac{\Sigma}{4\pi}, \quad (18)$$

which means that the Hawking temperature T_H in (6) is impeccably recovered.

3. ISOTROPIC COORDINATES

In general, when metric (1) is transformed to the isotropic coordinates, the resulting line element admits a BH spacetime in which the metric functions are nonsingular at the horizon, the time direction is a Killing vector, and the three-dimensional subspace of the spatial part of the line element (known as a time slice) appears as Euclidean with a conformal factor. Furthermore, using these coordinates renders the calculation of the index of refraction of light rays (a subject of gravitational lensing) around a BH possible. Therefore, the light propagation of a BH can be mimicked by the index of refraction. In this way, an observer can identify the type of the BH [26].

In this section, we first transform the LDBH to the isotropic coordinates and then analyze the HJ equation. Next, we examine the horizon temperature and see whether it agrees with T_H. At the final stage, we discuss the discrepancy in the temperatures and its resolution.

The LDBH solution in isotropic coordinates can be found by the transformation

$$\frac{d\rho}{\rho} = \frac{dr}{A\sqrt{\Sigma(r^2 - rr_h)}}, \quad (19)$$

whence we obtain

$$\rho = \left(2r - r_h + 2\sqrt{r(r-r_h)}\right)^{1/\gamma}, \quad (20)$$

and conversely

$$r = \frac{1}{4\rho^\gamma}\left(\rho^\gamma + r_h\right)^2, \quad (21)$$

where

$$\gamma = A\sqrt{\Sigma}.$$

This transformation takes metric (1) to the form

$$ds^2 = -F\,dt^2 + G(d\rho^2 + \rho^2 d\Omega^2), \quad (22)$$

with

$$F = \frac{\Sigma}{4\rho^\gamma}\left(\rho^\gamma - r_h\right)^2, \quad G = \frac{A^2}{4\rho^{\gamma+2}}\left(\rho^\gamma + r_h\right)^2. \quad (23)$$

In this coordinate system, the event horizon is located at

$$\rho_h = (r_h)^{1/\gamma}$$

and the region $\rho > \rho_h$ covers the exterior region of the LDBH, which is static. In the naive coordinates (1) of the LDBH, all Killing vectors are spacelike in the interior region, and we deduce that the interior of the LDBH is nonstationary. On the other hand, when we consider the interior region $\rho < \rho_h$ of metric (22), it admits a hypersurface-orthogonal timelike Killing vector, which implies a static region. Namely, the region $\rho < \rho_h$ does not cover the interior of the LDBH. Instead, it again covers the exterior region such that metric (22) is a double cover of the LDBH exterior [27].

We can easily rewrite metric (22) as

$$ds^2 = F(-dt^2 + \hat{g}), \quad (24)$$

and obtain the Fermat metric [26] form that of the LDBH as

$$\hat{g} = n(\rho)^2(d\rho^2 + \rho^2 d\Omega^2), \quad (25)$$

where $n(\rho)$ is known as the index of refraction. For the LDBH medium, it is calculated as

$$n(\rho) = \sqrt{\frac{G}{F}} = \frac{A}{\sqrt{\Sigma}\rho}\,\frac{\rho^\gamma + r_h}{\rho^\gamma - r_h}. \quad (26)$$

Hamilton–Jacobi equation (7) on background (22) corresponds to

$$-\frac{1}{F}(\partial_t I)^2 + F(\partial_\rho I)^2 + \frac{1}{G\rho^2} \times \times \left[(\partial_\theta I)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi I)^2\right] + m^2 = 0. \quad (27)$$

There exists a solution of the form

$$I = -Et + W_{iso}(\rho) + J(x^i). \quad (28)$$

Solving for $W_{iso}(\rho)$ yields

$$W_{iso}(\rho) = \pm \int n(\rho) \times \times \sqrt{E^2 - \frac{F}{G\rho^2}\left(J_\theta^2 + \frac{J_\varphi^2}{\sin^2\theta}\right) - m^2 F d\rho}, \quad (29)$$

which can be written near the horizon

$$\rho \approx (r_h)^{1/\gamma}$$

as

$$W_{iso(\pm)} = \pm E \int n(\rho)\, d\rho. \quad (30)$$

Here, it is clear that $W_{iso(\pm)}$ is governed by the index of refraction of the LDBH. Applying Feynman's prescription to the above integral, we obtain

$$W_{iso(\pm)} = \pm \frac{i2\pi E}{\Sigma} + c_2, \quad (31)$$

where c_2 is another integration constant. Similarly to the procedure followed in the previous section i. e., setting $P_{in} = 1$, which yields

$$\operatorname{Im} c_2 = \frac{2\pi E}{\Sigma},$$

we obtain the imaginary part of the action I of the tunneling particle as

$$\operatorname{Im} I = \operatorname{Im} W_{iso(+)} = \frac{4\pi E}{\Sigma}. \quad (32)$$

Thus, by using the tunneling rate formulation (16) we obtain the horizon temperature of the LDBH as

$$\check{T}_H = \frac{\Sigma}{8\pi}. \quad (33)$$

But the obtained temperature is half the conventional Hawking temperature,

$$\check{T}_H = \frac{1}{2} T_H.$$

Hence, the result in (33) shows that transforming the naive coordinates to the isotropic coordinates yields an apparent temperature of the BH that is less than the true temperature T_H. This is analogous to the apparent depth q of a fish swimming at a depth d below the surface of a pond being less than the true depth d, $q < d$. This illusion is due to the difference of the indices of refraction between the media. Particularly, such a case occurs when

$$n_{observer} < n_{object},$$

as is the case here. It is obvious from Eq. (26) that the index of refraction of the medium of an observer located at the outer region is less than the index of refraction of the medium near the horizon. Since the value of $W_{iso(\pm)}$ in (30) acts as a decision-maker on the value of the Hawking temperature T_H of the BH, we can deduce that the index of refraction in (26), and consequently the gravitational lensing effect, play an important role in the observation of the true T_H.

On the other hand, we admittedly know that the coordinate transformation of the naive coordinates to the isotropic coordinates should not alter the true temperature of the BH. Since the appearances are deceptive, we should make a deeper analysis. Very recently, this problem was throughly discussed by Chatterjee and Mitra [22]. Since the isotropic coordinate ρ becomes complex inside the horizon ($r < r_h$), they have proven that while evaluating the integral (30) around the horizon, the path across the horizon involves a change of $\pi/2$ instead of π in the phase of the complex variable $(\rho^\gamma - r_h)$. This can best be seen from Eq. (21), which is rewritten as

$$r = r_h + \frac{(\rho^\gamma - r_h)^2}{\rho^\gamma}, \quad (34)$$

and implies that

$$\frac{dr}{r - r_h} = -\gamma\frac{d\rho}{\rho} + \frac{2\gamma d\rho}{\rho^{1-\gamma}(\rho^\gamma - r_h)} = = -\gamma\frac{d\rho}{\rho} + \frac{2dz}{z - r_h}, \quad (35)$$

where $z = \rho^\gamma$. The first term does not admit any imaginary part at the horizon. Hence, any imaginary contribution coming from $2dz/(z - r_h)$ must be a half of $dr/(r - r_h)$. The last remark produces a factor $i\pi/2$ for the integral in (30) and subsequently it yields

$$\operatorname{Im} W_{iso(+)} = \frac{2\pi E}{\Sigma}$$

as obtained in the previous section. Thus, we obtain the horizon temperature as

$$\check{T}_H = \frac{\Sigma}{4\pi},$$

which is again T_H.

4. PG COORDINATES

Generally, we use the PG coordinates [28, 29] to describe the spacetime on either side of the event horizon of a static BH. In this coordinate system, an observer does not consider the surface of the horizon to be in any way special. In this section, we use the PG coordinates as another regular coordinate system in the HJ equation and examine whether they yield the correct calculation of T_H.

We can pass to the PG coordinates by applying the following transformation [30] to metric (1):

$$dT = dt + \frac{\sqrt{1-f}}{f}\,dr, \tag{36}$$

where T is our new time coordinate (which we call the PG time). Substituting this in metric (1) gives

$$ds^2 = -f dT^2 + 2\sqrt{1-f}\,dT\,dr + dr^2 + R^2 d\Omega^2. \tag{37}$$

One of the main features of these coordinates is that the PG time concurrently corresponds to the proper time. For metric (37), HJ equation (7) takes the form

$$-(\partial_T I)^2 + 2\sqrt{1-f}(\partial_T I)(\partial_r I) + f(\partial_r I)^2 + \\ + \frac{1}{R^2}(\partial_\theta I)^2 + \frac{1}{R^2\sin^2\theta}(\partial_\varphi I)^2 + m^2 = 0. \tag{38}$$

Letting

$$I = -ET + W_{PG}(r) + J(x^i), \tag{39}$$

and substituting Eqs. (39) and (3) in Eq. (38), we find

$$W_{PG}(r) = \int \frac{E}{\Sigma(r-r_h)}\left(\sqrt{1-\Sigma(r-r_h)} \pm \right. \\ \left. \pm\sqrt{1-\Sigma(r-r_h) - \frac{\lambda\Sigma(r-r_h)}{E^2}}\right)dr, \tag{40}$$

where

$$\lambda = m^2 - E^2 + \frac{J_\theta^2}{R^2} + \frac{J_\varphi^2}{R^2\sin^2\theta}. \tag{41}$$

Near the horizon, Eq. (40) in turn implies that

$$W_{PG(\pm)} = \frac{E}{\Sigma}\int \frac{1}{r-r_h}(1\pm 1)\,dr. \tag{42}$$

Therefore, imposing the condition

$$W_{PG(-)} = 0,$$

which ensures that there is no reflection for the incoming particle, we have

$$W_{PG(+)} = \frac{i2\pi E}{\Sigma}. \tag{43}$$

Thus, we obtain the imaginary part of the action I as

$$\operatorname{Im} I = \operatorname{Im} W_{PG(+)} = \frac{2\pi E}{\Sigma}. \tag{44}$$

With the aid of Eqs. (16) and (17), we can readily find the horizon temperature of the LDBH in the PG coordinates as

$$\check{T}_H = \frac{\Sigma}{4\pi}. \tag{45}$$

This result fully agrees with the standard value of the Hawking temperature (6).

5. IEF COORDINATES

Another useful coordinate system, which is also regular at the event horizon, was originally constructed by Eddington [31] and Finkelstein [32]. These coordinates are fixed to radially moving photons. Line element (1) takes the following form in the IEF coordinates (see, e. g., [33]):

$$ds^2 = -f dv^2 + 2\sqrt{1-f}\,dv\,dr + dr^2 + \\ + R^2(d\theta^2 + \sin^2\theta\,d\varphi^2), \tag{46}$$

where v is a new null coordinate, the so-called advanced time. It is given by

$$v = t + r_*, \tag{47}$$

where r_* is known as the Regger–Wheeler coordinate or the tortoise coordinate. For the outer region of the LDBH, it is found to be

$$r_* = \frac{1}{\Sigma}\ln\left(\frac{r}{r_h} - 1\right). \tag{48}$$

Since the metric (46) has the Killing vector field $\xi^\mu = \partial_v$, in this coordinate system an observer measures the scalar particle energy as

$$E = -\partial_v I.$$

In this regard, the action is assumed to be of the form

$$I = -Ev + W_{EF}(r) + J(x^i). \tag{49}$$

Using HJ equation (7) in metric (46), we find the final result

$$W_{EF}(r) = \int \frac{E}{\Sigma(r - r_h)} \times \times \left(1 \pm \sqrt{1 - \frac{\varkappa\Sigma(r - r_h)}{E^2}}\right) dr, \quad (50)$$

where

$$\varkappa = m^2 + \frac{J_\theta^2}{R^2} + \frac{J_\varphi^2}{R^2 \sin^2\theta}. \quad (51)$$

In the vicinity of the event horizon, $W_{EF}(r)$ reduces to the expression

$$W_{EF(\pm)} = \frac{E}{\Sigma} \int \frac{1}{r - r_h}(1 \pm 1)\, dr, \quad (52)$$

which is identical to (42). Hence,

$$W_{EF(-)} = 0, \quad W_{EF(+)} = \frac{i2\pi E}{\Sigma} \rightarrow \rightarrow \operatorname{Im} I = \operatorname{Im} W_{EF(+)} = \frac{2\pi E}{\Sigma}, \quad (53)$$

and similarly to the PG coordinates, the use of the EF coordinates in the HJ equation allows reproducing the standard Hawking temperature from the horizon temperature of the LDBH:

$$\check{T}_H = \frac{\Sigma}{4\pi} = T_H. \quad (54)$$

6. KS COORDINATES

Another well-behaved coordinate system, which covers the entire spacetime manifold of the maximally extended BH solution, is the so-called KS coordinates [34, 35]. These coordinates have the effect of squeezing infinity into a finite distance and hence the entire spacetime can be displayed on a stamp-like diagram. In this section, we apply the HJ equation to the KS metric of the LDBH in order to verify that $\check{T}_H$ is equal to T_H.

Metric (1) can be rewritten as [33]

$$ds^2 = -f\, du\, dv + R^2 d\Omega^2, \quad (55)$$

where

$$du = dt - dr_*, \quad dv = dt + dr_*. \quad (56)$$

Furthermore, if we define new coordinates (U, V) in terms of the surface gravity κ in (5) as

$$U = -e^{-\kappa u}, \quad V = e^{\kappa v}, \quad (57)$$

metric (55) transforms to the KS metric as

$$ds^2 = \frac{f}{\kappa^2} \frac{dU\, dV}{UV} + R^2 d\Omega^2. \quad (58)$$

Recalling the definitions given in Eqs. (3)–(5) and (57), it is then straightforward to obtain the KS metric of the LDBH. It is given by

$$ds^2 = -\frac{16M}{\Sigma^2 A^2} dU\, dV + R^2 d\Omega^2. \quad (59)$$

This metric is well-behaved everywhere outside the physical singularity $r = 0$. Alternatively, metric (59) can be rewritten as

$$ds^2 = -dT^2 + dX^2 + R^2 d\Omega^2. \quad (60)$$

This is possible with the transformation

$$T = \frac{4\sqrt{M}}{\Sigma A}(V + U) = \frac{4\sqrt{M}}{\Sigma A}\sqrt{\frac{r}{r_h} - 1}\, \operatorname{sh}\left(\frac{\Sigma t}{2}\right), \quad (61)$$

$$X = \frac{4\sqrt{M}}{\Sigma A}(V - U) = \frac{4\sqrt{M}}{\Sigma A}\sqrt{\frac{r}{r_h} - 1}\, \operatorname{ch}\left(\frac{\Sigma t}{2}\right). \quad (62)$$

It is easy to see that

$$X^2 - T^2 = \frac{16M}{\Sigma^2 A^2}\left(\frac{r}{r_h} - 1\right), \quad (63)$$

which means that $X = \pm T$ corresponds to the future and past horizons. On the other hand, ∂_T is not a timelike Killing vector anymore for metric (60); instead, we should consider the timelike Killing vector

$$\partial_{\widehat{T}} = N(X\partial_T + T\partial_X), \quad (64)$$

where N denotes the normalization constant. It can admit a special value such that the norm of the Killing vector becomes negative unity at a specific location in the outer region of the LDBH where

$$r = \frac{1}{\Sigma} + r_h.$$

This implies that

$$N = \frac{\Sigma}{2}. \quad (65)$$

Since the energy is defined by

$$\partial_{\widehat{T}} I = -E, \quad (66)$$

it follows that

$$\frac{\Sigma}{2}(X\partial_T I + T\partial_X I) = -E. \quad (67)$$

Without loss of generality, we can consider only the (1+1)-dimensional form of KS metric (60), which now appears as Minkowskian:

$$ds^2 = -dT^2 + dX^2. \tag{68}$$

The calculation by the HJ method is more straightforward in this case. Hamilton–Jacobi equation (7) for the above metric is given by

$$-\left(\partial_T I\right)^2 + \left(\partial_X I\right)^2 + m^2 = 0. \tag{69}$$

This equation implies that the action I to be used in HJ equation (7) for metric (68) can be

$$I = g(X - T) + J(x^i). \tag{70}$$

For simplicity, we can further set

$$J(x^i) = 0, \quad m = 0.$$

Using Eq. (67) with ansatz (70), we derive the expression

$$g(u) = \int \frac{2E}{\Sigma u}\, du, \tag{71}$$

where $u = X - T$. This expression diverges at the horizon $u = 0$, namely, $X = T$. This leads to a pole at the horizon (going over a semi-circular contour of integration in the complex plane) and the result is found to be

$$\operatorname{Im} I = \frac{2\pi E}{\Sigma}. \tag{72}$$

Therefore, referring to tunneling probability (7), we obtain

$$\Gamma = \exp\left(-\frac{4\pi E}{\Sigma}\right), \tag{73}$$

which means that the correct Hawking temperature

$$T_H = \frac{\Sigma}{4\pi}$$

is recovered in the background of the KS metric of the LDBH.

7. CONCLUSION

In this study, the Hawking radiation of the LDBH in four dimensions is studied by the HJ method. To our knowledge, the LDBH is the only BH whose radiation obeys an isothermal process, which corresponds to no change in the temperature during its evaporation, $\Delta T = 0$. This can be easily deduced from its Hawking temperature, which yields a constant value. Namely, it is independent of the mass M (or the horizon radius r_+) of the BH. In addition to the naive coordinates, four different regular coordinate systems are examined in this study. It is shown that the horizon temperatures computed in the naive, PG, IEF, and KS coordinates by the HJ method exactly match the conventional Hawking temperature. Here, we note that in Sec. 6, which considers the KS coordinates, the way that we followed was slightly different than in other sections. In that section, without loss of generality, we discarded the mass of the scalar field and neglected the angular dependence of the HJ equation. This turned out to be the application of the HJ method for the Minkowski metric. As a result, the match of the temperatures was successfully shown.

We believe that the most interesting part of the present paper is Sec. 3, where the LDBH metric is expressed in terms of the isotropic coordinates. Using the Fermat metric enabled us to determine the index of refraction of the LDBH. In particular, it is proved that the index of refraction plays a decisive role on the tunneling rate. Unlike in the other coordinate systems, the standard integration around the pole at the horizon in the isotropic coordinates produced an unacceptable value of the temperature, half the standard T_H. To overcome this discrepancy, we were inspired by recent study [22], which has demonstrated how the proper regularization of singular integrals leads to the standard Hawking temperature for the isotropic coordinates. As a result, it is clarified that the path across the horizon results in the value $i\pi/2$ on integration instead of $i\pi$. The underlying reason of this is that the isotropic radial coordinate ρ in (20) is real outside the BH, but becomes complex inside the BH.

Finally, it would be interesting to extend our analysis to other BHs, which could be BHs with multiple horizons, multi-BHs, higher-dimensional BHs, etc. This will be considered in the near future.

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