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ELECTROMAGNETIC WAVE PROPAGATION WITH NEGATIVE PHASE VELOCITY IN REGULAR BLACK HOLES

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We discuss the propagation of electromagnetic plane waves with negative phase velocity in regular black holes. For this purpose, we consider the Bardeen model as a nonlinear magnetic monopole and the Bardeen model coupled to nonlinear electrodynamics with a cosmological constant. It turns out that the region outside the event horizon of each regular black hole does not support negative phase velocity propagation, while its possibility in the region inside the event horizon is discussed.

1. INTRODUCTION

The phenomenon of electromagnetic plane wave propagation with negative phase velocity (NPV) in a curved spacetime has gained much attention during the last few years [1–7]. This occurs when the wave vector has a negative projection on the time-average Poynting vector. A wide range of useful phenomena such as the negative Doppler effect, inverse Cerenkov radiation, and negative refraction have been predicted for NPV materials [8].

In [1–5], NPV wave propagation was investigated in different black-hole spacetimes and it was proved that a gravitationally affected vacuum resembling a medium can admit NPV wave propagation in certain curved spacetimes. The same authors [6, 7] showed that the erogosphere of an uncharged rotating and charged rotating black holes admits NPV wave propagation. Plasma wave properties in a Veselago medium for particular black holes were discussed in [9]. In a recent paper [10], we have explored conditions of NPV for static charged black strings.

Here, we extend the above work to investigate propagation of electromagnetic plane waves with NPV in regular black holes. In the next section, we briefly review the relevant formulation. Section 3 explicitly deals with wave propagation and NPV conditions for two types of regular black holes. In Sec. 4, we discuss the results and the possibility of NPV wave propagation in the region inside the event horizon.

2. GENERAL FORMULATION: AN OVERVIEW

We briefly review the formulation in [1–7] to explain the electrodynamics of a gravitationally affected vacuum serving as an equivalent instantaneously responding medium for wave propagation. The general line element of a charged regular black hole is of the form

$$ds^2 = (1-h(r))\,dt^2 - (1-h(r))^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (1)$$

The metric can be expressed in Cartesian coordinates as

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$$g_{ab} = \begin{pmatrix} 1-h(r) & 0 & 0 & 0 \\ 0 & -1-\dfrac{h(r)x^2}{r^2(1-h(r))} & -\dfrac{xyh(r)}{r^2(1-h(r))} & -\dfrac{xzh(r)}{r^2(1-h(r))} \\ 0 & -\dfrac{xyh(r)}{r^2(1-h(r))} & -1-\dfrac{h(r)y^2}{r^2(1-h(r))} & -\dfrac{zyh(r)}{r^2(1-h(r))} \\ 0 & -\dfrac{zyh(r)}{r^2(1-h(r))} & -\dfrac{zyh(r)}{r^2(1-h(r))} & -1-\dfrac{h(r)z^2}{r^2(1-h(r))} \end{pmatrix},$$

whose inverse is

$$g^{ab} = \begin{pmatrix} \dfrac{1}{1-h(r)} & 0 & 0 & 0 \\ 0 & -1+\dfrac{h(r)x^2}{r^2} & \dfrac{xyh(r)}{r^2} & \dfrac{xzh(r)}{r^2} \\ 0 & \dfrac{xyh(r)}{r^2} & -1+\dfrac{h(r)y^2}{r^2} & \dfrac{zyh(r)}{r^2} \\ 0 & \dfrac{zyh(r)}{r^2} & \dfrac{zyh(r)}{r^2} & -1+\dfrac{h(r)z^2}{r^2} \end{pmatrix}.$$

The electromagnetic response of the vacuum in a curved spacetime is described by the constitutive relations

$$\mathbf{D}(\mathbf{r},t) = \epsilon_0 \underline{\underline{\gamma}} \cdot \mathbf{E}(\mathbf{r},t), \quad \mathbf{B}(\mathbf{r},t) = \mu_0 \underline{\underline{\gamma}} \cdot \mathbf{H}(\mathbf{r},t). \quad (2)$$

The dyadic $\underline{\underline{\gamma}}$ is a second-rank Cartesian tensor that can be expressed in the metric $[\gamma_{ab}]$ form with

$$\gamma_{ab} = -\sqrt{-g}\,\frac{g^{ab}}{g_{00}}. \quad (3)$$

We note that only a global observer based on a curved spacetime can observe electromagnetic wave propagation in a gravitationally affected vacuum, and we therefore observe this phenomenon globally [2]. The constitutive relations describe the regular black hole spacetime globally.

To approximate a nonuniform metric γ_{ab} by a uniform metric $\tilde{\gamma}_{ab}$, we partition the global spacetime into adjoining and sufficiently small neighborhoods $\mathbf{R}$ at an arbitrary location $(\tilde{x},\tilde{y},\tilde{z})$ and formulate a global solution by stitching together the solutions evaluated from the neighborhoods. Also, it is assumed that the wavelength is shorter than the linear dimension of the neighborhood $\mathbf{R}$, which is in turn shorter than the curvature radius of the spacetime. Differential equations with nonhomogeneous coefficients are solved by this method.

The uniform metric is defined as

$$\underline{\underline{\tilde{\gamma}}} = [\tilde{\gamma}_{ab}] = \frac{1}{1-\tilde{h}(\tilde{r})} \times \\ \times \begin{pmatrix} 1-\dfrac{\tilde{h}(\tilde{r})\tilde{x}^2}{\tilde{r}^2} & -\dfrac{\tilde{h}(\tilde{r})\tilde{x}\tilde{y}}{\tilde{r}^2} & -\dfrac{\tilde{h}(\tilde{r})\tilde{x}\tilde{z}}{\tilde{r}^2} \\ -\dfrac{\tilde{h}(\tilde{r})\tilde{x}\tilde{y}}{\tilde{r}^2} & 1-\dfrac{\tilde{h}(\tilde{r})\tilde{y}^2}{\tilde{r}^2} & -\dfrac{\tilde{h}(\tilde{r})\tilde{y}\tilde{z}}{\tilde{r}^2} \\ -\dfrac{\tilde{h}(\tilde{r})\tilde{x}\tilde{z}}{\tilde{r}^2} & -\dfrac{\tilde{h}(\tilde{r})\tilde{y}\tilde{z}}{\tilde{r}^2} & 1-\dfrac{\tilde{h}(\tilde{r})\tilde{z}^2}{\tilde{r}^2} \end{pmatrix},$$

where

$$\tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2$$

and

$$\det\left[\underline{\underline{\tilde{\gamma}}}\right] = (1-h(\tilde{r}))^{-2}.$$

We now find the dispersion relation of the medium and the complex time-average Poynting vector in a time-harmonic electromagnetic field [11, 12]. We consider plane wave solutions

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \operatorname{Re}\mathbf{E}_0 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \\ \mathbf{H}(\mathbf{r},t) &= \operatorname{Re}\mathbf{H}_0 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \end{aligned} \quad (4)$$

where $\mathbf{r}$ is the position vector within the neighborhood $\mathbf{R}$ containing $(\tilde{x},\tilde{y},\tilde{z})$, $\mathbf{k}$ is the wave vector, and ω is the angular frequency. The complex-valued amplitudes are represented by $\mathbf{E}_0$ and $\mathbf{H}_0$.

The source-free Maxwell curl postulates in $\mathbf{R}$ are given by

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r},t) + \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = 0, \\ \nabla \times \mathbf{H}(\mathbf{r},t) - \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} = 0. \end{aligned} \qquad (5)$$

We seek a plane-wave solution of Eq. (5) and obtain an eigenvector equation, after some algebraic manipulations, as

$$\left[\left(k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right] - \mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{k}\right)\underline{\underline{I}} + \mathbf{kk}\cdot\underline{\underline{\tilde{\gamma}}}\right]\cdot\mathbf{E}_0 = 0, \qquad (6)$$

where

$$k_0 = \omega\sqrt{\epsilon_0\mu_0}$$

and $\mathbf{kk}$ represents the tensor product of vectors. The corresponding dispersion relation is obtained by setting

$$\det\left[\left(k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right] - \mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{k}\right)\underline{\underline{I}} + \mathbf{kk}\cdot\underline{\underline{\tilde{\gamma}}}\right] = 0,$$

which is [5]

$$k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right]\left(k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right] - \mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{k}\right)^2 = 0. \qquad (7)$$

Since $\underline{\underline{\tilde{\gamma}}}$ is nonsingular, the above equation leads to

$$\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{k} = k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right]. \qquad (8)$$

Inserting this value in Eq. (6) yields

$$\mathbf{kk}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{E}_0 = 0, \qquad (9)$$

which shows that $\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}$ and $\mathbf{E}_0$ are orthogonal.

Because spacetime (1) is spherically symmetric, we can take

$$\mathbf{k} = k\hat{\mathbf{k}} = k\hat{\mathbf{u}}_z$$

without any loss of generality. Here, $\hat{\mathbf{u}}_z$ is a unit vector along the $\tilde{z}$ axis and k is the magnitude of the wave vector. Then

$$\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}} = k\left[\tilde{\gamma}_{13}\hat{\mathbf{u}}_x + \tilde{\gamma}_{23}\hat{\mathbf{u}}_y + \tilde{\gamma}_{33}\hat{\mathbf{u}}_z\right].$$

The terms $\tilde{\gamma}_{13}$, $\tilde{\gamma}_{23}$, and $\tilde{\gamma}_{33}$ are given by

$$\tilde{\gamma}_{13} = -\frac{\tilde{h(\tilde{r})}\tilde{x}\tilde{z}}{(1-\tilde{h}(\tilde{r}))\tilde{r}^2}, \quad \tilde{\gamma}_{23} = -\frac{\tilde{h(\tilde{r})}\tilde{y}\tilde{z}}{(1-\tilde{h}(\tilde{r}))\tilde{r}^2},$$

$$\tilde{\gamma}_{33} = \frac{(\tilde{h}(\tilde{r}))_z}{(1-\tilde{h}(\tilde{r}))},$$

where

$$(\tilde{h}(\tilde{r}))_z = 1 - \frac{\tilde{h}(\tilde{r})\tilde{z}^2}{\tilde{r}^2}$$

and $\hat{\mathbf{u}}_x$ and $\hat{\mathbf{u}}_y$ are unit vectors along the $\tilde{x}$ and $\tilde{y}$ axes. Equation (8) yields

$$k^2\frac{(\tilde{h}(\tilde{r}))_z}{1-\tilde{h}(\tilde{r})} - \frac{k_0^2}{(1-\tilde{h}(\tilde{r}))^2} = 0,$$

whose roots are

$$k = \pm k_0\left[(1-\tilde{h}(\tilde{r}))(\tilde{h}(\tilde{r}))_z\right]^{-1/2}. \qquad (10)$$

For $k^{\pm}$ to be real, we must have

$$(1-\tilde{h}(\tilde{r}))(\tilde{h}(\tilde{r}))_z > 0 \qquad (11)$$

which is possible if both terms $(1-\tilde{h}(\tilde{r}))$ and $(\tilde{h}(\tilde{r}))_z$ have the same sign. Equation (9) is satisfied if two linearly independent eigenvectors are

$$\mathbf{e}_1 = \tilde{\gamma}_{23}\hat{\mathbf{u}}_x - \tilde{\gamma}_{13}\hat{\mathbf{u}}_y,$$

$$\mathbf{e}_2 = \tilde{\gamma}_{33}\tilde{\gamma}_{13}\hat{\mathbf{u}}_x + \tilde{\gamma}_{33}\tilde{\gamma}_{23}\hat{\mathbf{u}}_y - \left[(\tilde{\gamma}_{13})^2 + (\tilde{\gamma}_{23})^2\right]\hat{\mathbf{u}}_z.$$

The general solution of Eq. (6) can be written as

$$\mathbf{E}_0 = c_1\mathbf{e}_1 + c_2\mathbf{e}_2, \qquad (12)$$

where c_1 and c_2 are complex constants. Combining Eqs. (2), (4), (5), and (12), we obtain

$$\mathbf{H}_0 = \frac{k\left[c_1(1-\tilde{h}(\tilde{r}))\mathbf{e}_2 - c_2(\tilde{h}(\tilde{r}))_z\mathbf{e}_1\right]}{\omega\mu_0}. \qquad (13)$$

The NPV wave propagation is defined by

$$\mathbf{k}\cdot\langle\mathbf{P}\rangle_t < 0, \qquad (14)$$

where

$$\langle\mathbf{P}\rangle_t = \frac{1}{2}\mathrm{Re}\{\mathbf{E}_0\times\mathbf{H}_0^*\}$$

is the time-average complex Poynting vector and $\mathbf{H}_0^*$ is the complex conjugate of $\mathbf{H}_0$. Using Eqs. (12) and (13) in the above equation, we have

$$\mathbf{k}\cdot\langle\mathbf{P}\rangle_t = \frac{1}{2\omega\mu_0}\left[\frac{k(\tilde{h}(\tilde{r}))_z}{\tilde{r}^2(1-\tilde{h}(\tilde{r}))}\right]^2(\tilde{x}^2+\tilde{y}^2)\times \\ \times\left(|c_1|^2 + |c_2|^2\frac{(\tilde{h}(\tilde{r}))_z}{1-\tilde{h}(\tilde{r})}\right)(\tilde{h}(\tilde{r}))_z.$$

Using this value in Eq. (14) along with (11), we see that NPV exists if

$$(\tilde{h}(\tilde{r}))_z < 0. \qquad (15)$$

Because this condition is derived for the neighborhood $\mathbf{R}$ on an arbitrary location within the spacetime, it holds generally.

3. NPV AND REGULAR BLACK HOLES

The concept of a regular black hole has played an important role in understanding the hidden interior of black holes. These are more general black-hole solutions in which true singularities of the black hole such as the Schwarzschild, Reissner–Nordstrom, Kerr and Kerr–Newmann black holes would be replaced by a special matter core. The work on NPV propagation has been done on black-hole solutions experiencing true singularities and therefore the conditions of NPV wave propagation are restricted to the outside of the event horizon. We are interested in extending this study to the interior of the black hole.

In this section, we apply the above formalism to two types of regular black holes. In Refs. [13, 14], the Bardeen model was described as a nonlinear magnetic monopole. This is obtained for

$$h(r) = \frac{2mr^2}{(r^2+q^2)^{3/2}}$$

in Eq. (1), where m is the mass and q is the monopole charge of a self-gravitating magnetic field of a nonlinear electrodynamic source. We note that when

$$q^2 < \frac{16}{27}m^2,$$

there is an event horizon. For $q = 0$, this reduces to the Schwarzschild black hole.

It follows from Eq. (15) that NPV wave propagation is possible if the condition

$$(\tilde{h}(\tilde{r}))_z = \left[1 - \frac{\tilde{h}(\tilde{r})\tilde{z}^2}{\tilde{r}^2}\right] < 0$$

holds, which implies that

$$\frac{2m\tilde{r}^2}{(\tilde{r}^2+q^2)^{3/2}} > \frac{\tilde{r}^2}{\tilde{z}^2}. \tag{16}$$

The event horizon for this regular black hole lies at r_+ that is obtained from

$$1 - \frac{2mr_+^2}{(r_+^2+q^2)^{3/2}} = 0.$$

It is difficult to find an explicit expression for r_+ from here, and hence the regions lying outside or inside the event horizon could not be explored. However, we use the following inequality to investigate NPV wave propagation in these regions. The regions $r < r_+$ and $r > r_+$ can be determined by respectively taking

$$1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} < 0, \quad 1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} > 0.$$

It follows from (16) that

$$1 - \frac{2m\tilde{r}^2}{(\tilde{r}^2+q^2)^{3/2}} < 1 - \frac{\tilde{r}^2}{\tilde{z}^2}. \tag{17}$$

Because

$$\tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2,$$

it follows that $\tilde{r}^2 > \tilde{z}^2$ for nonzero $\tilde{x}^2$ and $\tilde{y}^2$. For $\tilde{r} > r_+$, Eq. (17) does not hold because the term $(1 - \tilde{r}^2/\tilde{z}^2)$ is negative and cannot be greater than any positive term. On the other hand, for $\tilde{r} < r_+$, Eq. (17) holds.

The line element of a regular black hole coupled to nonlinear electrodynamics with a cosmological constant [15, 16] is obtained for

$$h(r) = \frac{2mr^2}{(r^2+q^2)^{3/2}} - \frac{q^2r^2}{(r^2+q^2)^2} + \frac{\Lambda r^2}{3}, \tag{18}$$

where m, q, and Λ are the mass, the electric charge, and the cosmological constant. For $\Lambda > 0$ and $\Lambda < 0$, Eq. (18) respectively represents de Sitter and anti-de Sitter type spacetimes. Furthermore, if $m = q = 0$, this becomes the de Sitter spacetime, and if $q = \Lambda = 0$, it reduces to the Schwarzschild spacetime. The event horizon is obtained from

$$1 - \frac{2mr_+^2}{(r_+^2+q^2)^{3/2}} + \frac{q^2r_+^2}{(r_+^2+q^2)^2} - \frac{\Lambda r_+^2}{3} = 0.$$

As in the preceding case, it is difficult to evaluate the explicit function of r_+ and the regions $r < r_+$ and $r > r_+$. But we can obtain the regions $r < r_+$ and $r > r_+$ as

$$1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2} - \frac{\Lambda r^2}{3} < 0,$$

$$1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2} - \frac{\Lambda r^2}{3} > 0.$$

Equation (15) yields the NPV condition in the form

$$\frac{2m\tilde{r}^2}{(\tilde{r}^2+q^2)^{3/2}} - \frac{q^2r^2}{(\tilde{r}^2+q^2)^2} + \frac{\Lambda\tilde{r}^2}{3} > \frac{\tilde{r}^2}{\tilde{z}^2}, \tag{19}$$

which implies that

$$1 - \frac{2m\tilde{r}^2}{(\tilde{r}^2+q^2)^{3/2}} + \frac{q^2r^2}{(\tilde{r}^2+q^2)^2} - \frac{\Lambda\tilde{r}^2}{3} < 1 - \frac{\tilde{r}^2}{\tilde{z}^2}. \tag{20}$$

Here, also

$$\tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2$$

and $\tilde{r}^2 > \tilde{z}^2$ for nonzero $\tilde{x}^2$ and $\tilde{y}^2$. For $\tilde{r} > r_+$, Eq. (20) does not hold because the term $(1 - \tilde{r}^2/\tilde{z}^2)$ cannot be positive and the left-hand side of (20) is positive irrespective of whether Λ is negative or nonnegative. It may be noted that (20) holds for $\tilde{r} < r_+$.

We note that the regular black hole solution given in [15] asymptotically approaches the Reissner–Nordstrom black hole in the de Sitter or anti-de Sitter spacetime according to the sign of the cosmological constant for $r \to \infty$. This black hole solution reduces to a black hole solution with

$$h(r) = \frac{2mr^2}{q^3} - \frac{r^2}{q^2} + \frac{\Lambda r^2}{3}, \qquad (21)$$

near the origin, which is the de Sitter or anti-de Sitter spacetime depending on the sign of

$$\frac{6m}{q^3} - \frac{3}{q^2} + \Lambda.$$

Accordingly, the inequality (15) yields

$$\frac{2m\tilde{r}^2}{q^3} - \frac{\tilde{r}^2}{q^2} + \frac{\Lambda\tilde{r}^2}{3} > \frac{\tilde{r}^2}{\tilde{z}^2}.$$

4. CONCLUSIONS

We have derived a general condition (15) for wave propagation with a NPV in regular black holes. For the first type of regular black holes, we deduce that the NPV propagation is not possible outside the event horizon. For $q = 0$, Eq. (17) reduces to

$$\frac{r_{sch}}{\tilde{r}} > \frac{\tilde{r}^2}{\tilde{z}^2} \qquad (22)$$

which is the NPV condition for the Schwarzschild black hole [5]. This does not support the NPV wave propagation outside the event horizon. For a regular black hole with a cosmological constant coupled to nonlinear electrodynamics, we have seen that the region outside the event horizon does not support NPV wave propagation. Also, the positive value of the cosmological constant does not support NPV wave propagation in contrast to the Schwarzschild–de Sitter and de Sitter spacetimes [4, 5]. For $\Lambda = 0$, it reduces to a regular black hole coupled to nonlinear electrodynamics [16] and NPV condition (19) becomes

$$\frac{2m\tilde{r}^2}{(\tilde{r}^2+q^2)^{3/2}} - \frac{q^2}{(\tilde{r}^2+q^2)^2} > \frac{\tilde{r}^2}{\tilde{z}^2},$$

by the same reasoning as in the previous cases, this implies that NPV wave propagation is not possible in the region outside the event horizon. Moreover, if $q = \Lambda = 0$, the inequality reduces to (22) and if $m = q = 0$, then Eq. (19) yields

$$\frac{\Lambda}{3} < \frac{1}{\tilde{z}^2},$$

which is the NPV condition for the cosmological spacetime with $c = 1$ and is invalid for $\Lambda < 0$.

It is obvious that nobody can see the inside of a black hole because the light is not allowed to escape or reflect back from the region inside the event horizon due to the high gravitational field. However, based on some mathematical calculations, one can try to guess some possible phenomena for the region inside the event horizon of a regular black hole, because it has no singularity. Consequently, there is a possibility of implication of some physical law.

We conclude that the possibility of NPV wave propagation in the region inside the event horizon of regular black holes is given by (17) and (20). Consequently, many new unusual phenomena may occur, such as negative refraction, which occurs when light passes from a PPV (positive phase velocity) medium to an NPV medium and there is a chance of negative electromagnetic energy density [2]. This would help investigate the interior of black holes via the NPV analysis.

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