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MASS SPECTRUM IN SQCD AND PROBLEMS WITH THE SEIBERG DUALITY. ANOTHER SCENARIO

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The $\mathcal{N}=1$ SQCD with $SU(N_c)$ colors and N_F flavors of light quarks is considered within the dynamical scenario that assumes that quarks can be in the two different phases only: the heavy-quark phase, where they are confined, and the phase of higgsed quarks, at the appropriate values of the Lagrangian parameters. The mass spectra of this (direct) theory and its Seiberg dual are obtained and compared for quarks of small equal or unequal masses. It is shown that in those regions of the parameter space where an additional small parameter exists (it is $0 < b_0/N_F = (3N_c - N_F)/N_F \ll 1$ at the right end of the conformal window, where the direct theory is weakly coupled in the vicinity of its IR fixed point, or its analog $0 < \overline{b}_0/N_F = (2N_F - 3N_c)/N_F \ll 1$ for the dual theory at the left end of the conformal window), the mass spectra of the direct and dual theories are parametrically different. A number of other regimes are also considered.

1. INTRODUCTION

The dynamics of 4-dimensional strongly coupled non-Abelian gauge theories is complicated. It is well known that supersymmetry (SUSY) leads to some simplifications in comparison with the ordinary (i.e., non-SUSY) theories. Besides, it is widely believed that SUSY is relevant to the real world. In any case, it is of great interest to study the dynamics of the nearest SUSY-relative of the ordinary QCD, i.e., the $\mathcal{N}=1$ SQCD. But even with $\mathcal{N}=1$ SQCD, there is currently no proven physical picture of even the main nontrivial features of its dynamics.

The best proposal so far seems to be Seiberg's dual theory [1], which is weakly coupled when the direct theory is strongly coupled, and *vice versa* (see, e.g., reviews [2, 3] and the references therein for $\mathcal{N}=1$ SQCD and [4, 5] for Seiberg's dual theory). The Seiberg duality passed some nontrivial checks (the 't Hooft triangles and the behavior in the conformal regime), but up to now, unfortunately, no proof has been given that the direct and dual theories are (or are not) equivalent. The reason is that such a proof needs real understanding of and the control over the dynamics of both theories.

A definite dynamical scenario for $\mathcal{N}=1$ SQCD was proposed recently [6, 7]. The main idea of this scenario (No. 1) was that the quarks are not higgsed when

$$0 < m_Q = m(\mu = \Lambda_Q) \ll \Lambda_Q,$$

$$m_Q^2 \ll \mathcal{M}_{ch}^2 = \langle \overline{Q}Q(\mu = \Lambda_Q)\rangle \ll \Lambda_Q^2,$$

but form a coherent condensate of colorless chiral pairs $\overline{Q}Q$ in the vacuum (the diquark condensate phase DC). As a result, they acquire the large dynamical constituent mass $\mu_C = \mathcal{M}_{ch}$ and light pseudo-Goldstone mesons π_i^j ("pions") appear, with masses $\mu_\pi \ll \mu_C$. The mass spectra of the direct and dual theories were obtained in [6] within this scenario No. 1, and they appeared to be quite different. Besides, some more general arguments (not related with the use of dynamical scenario No. 1 with the diquark condensate) presented in Sec. 7 in [6] suggest that the direct and dual theories are not equivalent in the region $N_c < N_F < 3N_c/2$.

The purpose of this paper is to consider another dynamical scenario (No. 2) in which it is assumed that quarks can be in two different phases only: either in the heavy quark (HQ) phase and are therefore confined or (instead of forming the diquark condensate) they are higgsed at $\mu = \mu_{gl} \ll \Lambda_Q$, at the appropriate values of the parameters of the theory.

The direct and dual theories for quarks of equal small masses, $0 < m_Q \ll \Lambda_Q$, are considered in Secs. 2–4 and 7. Other sections deal with quarks of

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unequal masses, when there are N_l lighter flavors with masses m_l and $N_h = N_F - N_l$ heavier flavors with masses $0 < m_l < m_h \ll \Lambda_Q$. The mass spectra of both the direct and dual theories in the conformal window are described in Secs. 2–5. It is shown that in all cases where an additional small parameter is available (it is $0 < b_0/N_F = (3N_c - N_F)/N_F \ll 1$ at the right end of the conformal window and its dual analog $0 < \overline{b}_0/N_F = (2N_F - 3N_c)/N_F \ll 1$ at the left end), the parametrical differences in the mass spectra of the direct and dual theories can be traced. Sections 6–9 deal with the direct and dual theories in some special regimes of interest. Finally, some conclusions are presented in Sec. 10.

2. DIRECT THEORY. EQUAL QUARK MASSES, $0 < b_0/N_F \ll 1$

The Lagrangian of the direct theory at scales $\mu > \Lambda_Q$ has the form

$$L = \left[\mathrm{Tr}\left(Q^\dagger e^V Q + \overline{Q}^\dagger e^{-V}\overline{Q}\right)\right]_D + \left[-\frac{2\pi}{\alpha(\mu)}S + \mathrm{Tr}\left(m_Q(\mu)\,\overline{Q}Q\right) + \mathrm{H.c.}\right]_F.$$

Here, $S = W_\alpha^2/32\pi^2$, $\alpha(\mu)$ is the running gauge coupling (with its scale parameter Λ_Q independent of quark masses), W_α is the gluon field strength, and $m_Q(\mu) \ll \Lambda_Q$ is the current quark mass.

The theory is UV free and is in the conformal regime at scales $\mu_H < \mu < \Lambda_Q$[1)], where μ_H is the highest physical mass, and in this case it is the quark pole mass m_Q^{pole}:

$$\frac{m_Q^{pole}}{\Lambda_Q} = \frac{m_Q(\mu = m_Q^{pole})}{\Lambda_Q} = = \frac{m_Q}{\Lambda_Q}\left(\frac{\Lambda_Q}{m_Q^{pole}}\right)^{\gamma_Q} = \left(\frac{m_Q}{\Lambda_Q}\right)^{1/(1+\gamma_Q)},$$
$$m_Q \equiv m_Q(\mu = \Lambda_Q), \quad \gamma_Q = b_0/N_F, \qquad (2.1)$$
$$b_0 = 3N_c - N_F, \quad \frac{1}{1+\gamma_Q} = \frac{N_F}{3N_c},$$
$$\frac{m_Q^{pole}}{\Lambda_Q} = \left(\frac{m_Q}{\Lambda_Q}\right)^{N_F/3N_c},$$

where γ_Q is the quark anomalous dimension [8]. After integrating out all quarks as heavy ones, a pure Yang–Mills (YM) theory remains with N_c colors at lower energies. Its scale parameter Λ_{YM} can be determined from matching the couplings of the higher- and lower-energy theories at $\mu = m_Q^{pole}$. Proceeding as in [6], we obtain

$$3N_c \ln\left(\frac{m_Q^{pole}}{\Lambda_{YM}}\right) \approx \frac{2\pi}{\alpha_*} \approx N_c\frac{N_F}{b_0} \to \Lambda_{YM} = = \exp\left(-\frac{N_c}{b_0}\right) m_Q^{pole} \ll m_Q^{pole}. \qquad (2.2)$$

Therefore, there are two parametrically different scales in the mass spectrum of the direct theory in this case. There is the large number of colorless flavored hadrons made of weakly confined (i. e., with the string tension $\sqrt{\sigma} \sim \Lambda_{YM} \ll m_Q^{pole}$) and weakly interacting nonrelativistic HQs Q, $\overline{Q}$ with masses $m_Q^{pole} \gg \Lambda_{YM}$, and a large number of gluonia with the mass scale $\Lambda_{YM} = \exp(-N_c/b_0) m_Q^{pole} \ll m_Q^{pole}$.

To check the self-consistency, we estimate the scale of the gluon masses due to a possible quark higgsing. The quark chiral condensate $\mathcal{M}_{ch}^2$ at $\mu = \Lambda_Q$ is given by [9]

$$\frac{\mathcal{M}_{ch}^2}{\Lambda_Q^2} \equiv \frac{1}{\Lambda_Q^2}\langle 0|\overline{Q}Q(\mu = \Lambda_Q)|0\rangle = \frac{\langle S\rangle = \Lambda_{YM}^3}{m_Q \Lambda_Q^2} = = \exp\left(-\frac{N_F}{b_0}\right)\left(\frac{m_Q}{\Lambda_Q}\right)^{(N_F - N_c)/N_c}. \qquad (2.3)$$

If the gluons acquired masses $\mu_{gl} > m_Q^{pole}$ due to higgsing of the quarks, then the conformal renolmalization group (RG) evolution stops at $\mu = \mu_{gl}$. Hence, μ_{gl} can be estimated from

1) By definition, Λ_Q is a scale such that the coupling $a(\mu = \Lambda_Q) = N_c\alpha(\mu = \Lambda_Q)/2\pi$ is sufficiently close to its fixed-point value a_*, i. e., $a_* - a(\mu = \Lambda_Q) = \delta a_*$, $\delta \ll 1$ (and similarly for the dual theory, $\overline{a}_* - \overline{a}(\mu = \Lambda_q) = \delta\,\overline{a}_*$). The coupling a_* is weak at $b_0/N_F \ll 1$ and $\gamma_Q(a_*) = (b_0/N_F) \approx (1 - 1/N_c^2)a_* \approx a_* \ll 1$. Here and in what follows, we trace only the leading exponential dependence on the small parameter $b_0/N_F \ll 1$ (or $\overline{b}_0/N_F \ll 1$), i.e., factors of the order of $\exp\{-c_0(N_F/b_0)\}$, $c_0 = O(1)$, while the nonleading terms of the order of $\exp\{-c_0\delta(N_F/b_0)\}$ or pre-exponential factors $\sim (N_c/b_0)^\sigma$, $\sigma = O(1)$, are neglected, because this simplifies greatly all expressions. Besides, it is always implied that even when b_0/N_F or $\overline{b}_0/N_F$ are $\ll 1$, these small parameters do not compete in any way with the main small parameter m_Q/Λ_Q (see the Appendix for more details).

$$\mu_{gl}^2 \sim a(\mu=\mu_{gl})\langle 0|\overline{Q}Q(\mu=\mu_{gl})|0\rangle,$$
$$\mu_{gl}^2 \sim \mathcal{M}_{ch}^2\left(\frac{\mu_{gl}}{\Lambda_Q}\right)^{\gamma_Q}, \quad a(\mu)\equiv\frac{N_c\alpha(\mu)}{2\pi},$$
$$\frac{\mu_{gl}}{\Lambda_Q}\sim\left(\frac{\mathcal{M}_{ch}^2}{\Lambda_Q^2}\right)^{1/(2-\gamma_Q)}\sim\exp\left(-\frac{N_c}{2b_0}\right)\times \times\frac{\Lambda_{YM}}{\Lambda_Q}\ll\frac{\Lambda_{YM}}{\Lambda_Q}\ll\frac{m_Q^{pole}}{\Lambda_Q}. \quad (2.4)$$

Therefore, the scale of possible gluon masses $\mu=\mu_{gl}$ is parametrically smaller not only than the quark pole mass but also than Λ_{YM}, $\mu_{gl}\ll\Lambda_{YM}\ll m_Q^{pole}$, and the picture of $Q,\overline{Q}$ being in the HQ phase is self-consistent.

3. DUAL THEORY. EQUAL QUARK MASSES, $0<b_0/N_F\ll 1$

The Lagrangian of the dual theory (at the scale $\mu\sim\Lambda_Q$) is taken in the form [1]

$$\overline{L}=\left[\mathrm{Tr}\left(q^\dagger e^{\overline{V}}q+\overline{q}^\dagger e^{-\overline{V}}\overline{q}\right)+\frac{1}{\mu_2^2}\mathrm{Tr}\left(M^\dagger M\right)\right]_D + \left[-\frac{2\pi}{\overline{\alpha}(\mu)}\overline{s}+\frac{1}{\mu_1}\mathrm{Tr}\left(\overline{q}Mq\right)+ + \mathrm{Tr}\left(\overline{m}_Q(\mu)M\right)+\mathrm{H.c.}\right]_F. \quad (3.1)$$

Here, $\overline{s}=\overline{w}_\alpha^2/32\pi^2$, $\overline{\alpha}(\mu)=\overline{g}^2(\mu)/4\pi$ is the dual running gauge coupling (with its scale parameter Λ_q), $\overline{w}_\alpha$ is the dual gluon field strength, the number of dual colors is $\overline{N}_c=N_F-N_c$, and $M_i^{\overline{j}}$ are the N_F^2 dual meson (mion) fields.

The scale parameter Λ_q of the dual theory can be taken as $|\Lambda_q|\sim\Lambda_Q$, such that both direct and dual theories enter the conformal regime simultaneously at $\mu\sim\Lambda_Q$ (see also the Appendix)[2)], and $\mu_2\lesssim\mu_1$ (the dual theory is considered UV free in the conformal window). Besides, the normalization of $\overline{Q}Q(\mu=\Lambda_Q)$ and $M(\mu=\Lambda_Q)$ can always be matched,

$$M_0\equiv\langle 0|M(\mu=\Lambda_Q)|0\rangle=\langle 0|\overline{Q}Q(\mu=\Lambda_Q)|0\rangle=\mathcal{M}_{ch}^2.$$

Hence, because the gluino condensates are also matched, it follows that [9]

$$\langle S\rangle=\Lambda_{YM}^3=m_Q\mathcal{M}_{ch}^2=|\langle\overline{s}\rangle|= =|\overline{\Lambda}_{YM}|^3=\overline{m}_Q(\mu=\Lambda_Q)M_0,$$
$$\overline{m}_Q(\mu=\Lambda_Q)=m_Q.$$

Therefore, only one free parameter $\mu_1\equiv Z_q\Lambda_Q$ remains in the dual theory. It is to be determined below from the explicit matching of gluino condensates in the direct and dual theories.

The dual theory is also in the conformal regime at $b_0/N_F\ll 1$ and $\overline{\mu}_H<\mu<\Lambda_Q$, where $\overline{\mu}_H$ is the corresponding highest physical mass.

Assuming that the dual quarks $\overline{q}$, q are in the HQ phase and hence $\overline{\mu}_H=\mu_q^{pole}$, we find their pole mass ($\overline{N}_c=N_F-N_c$, $\overline{b}_0=3\overline{N}_c-N_F$, $\gamma_q=\overline{b}_0/N_F$):

$$\frac{\mu_q^{pole}}{\Lambda_Q}=\frac{\mu_q}{\Lambda_Q}\left(\frac{\Lambda_Q}{\mu_q^{pole}}\right)^{\gamma_q}=\left(\frac{\mathcal{M}_{ch}^2}{Z_q\Lambda_Q^2}\right)^{N_F/3\overline{N}_c}, \quad \mu_q=\mu_q(\mu=\Lambda_Q)=\frac{M_0}{\mu_1}=\frac{\mathcal{M}_{ch}^2}{Z_q\Lambda_Q}. \quad (3.2)$$

We now integrate out all dual quarks as heavy ones and determine the scale factor $\overline{\Lambda}_{YM}$ of the remaining dual YM theory (the dual coupling is $\overline{a}_*=\overline{N}_c\overline{\alpha}_*/2\pi= =O(1)$):

$$3\overline{N}_c\ln\left(-\frac{\mu_q^{pole}}{\overline{\Lambda}_{YM}}\right)\sim\frac{\overline{N}_c}{\overline{a}_*}\sim\overline{N}_c\to \to-\overline{\Lambda}_{YM}\sim\mu_q^{pole}. \quad (3.3)$$

Equating the gluino condensates of the direct and dual theories (i. e., $(-\overline{\Lambda}_{YM})^3=\Lambda_{YM}^3$), we obtain

$$-\overline{\Lambda}_{YM}=\Lambda_{YM}\to Z_q=\exp\left(-\frac{N_c}{b_0}\right)\ll 1. \quad (3.4)$$

We now find the mass μ_M of the mions M. It can be found from their effective Lagrangian, obtained by integrating out all dual quarks and gluons. Proceeding as in [6, 7] and integrating out all quarks as heavy ones and all gluons via the Veneziano–Yankielowicz (VY) procedure [10], we obtain the Lagrangian of mions (the fields M are normalized in (3.5) and (3.6) at $\mu=\Lambda_Q$ and $\overline{b}_0=3\overline{N}_c-N_F$)

$$L_M=\left[\frac{z_M(\Lambda_Q,\mu_q^{pole})}{\mu_2^2}\mathrm{Tr}\,M^\dagger M\right]_D+ +\left[-\overline{N}_c\left(\frac{\det M}{Z_q^{N_F}\Lambda_Q^{\overline{b}_0}}\right)^{1/\overline{N}_c}+\mathrm{Tr}\,m_QM\right]_F,$$
$$z_M(\Lambda_Q,\mu_q^{pole})=\left(\frac{\Lambda_Q}{\mu_q^{pole}}\right)^{2\gamma_q}= =\left(\frac{\Lambda_Q}{\Lambda_{YM}}\right)^{2\overline{b}_0/N_F}\gg 1, \quad (3.5)$$
$$m_Q\langle M\rangle=m_Q\mathcal{M}_{ch}^2=\Lambda_{YM}^3.$$

[2)] The phase of Λ_q has to be chosen appropriately [4]. This is always implied without being stipulated explicitly in what follows.

In a number of cases, it is also convenient to write the Lagrangian of mions in the form

$$L_M = \left[\frac{z_M(\Lambda_Q, \mu_q^{pole})}{\mu_2^2}\,\mathrm{Tr}\, M^\dagger M\right]_D + \\ + \left[-\overline{N}_c\,\Lambda_{YM}^3\left(\det\frac{M}{\langle M\rangle}\right)^{1/\overline{N}_c} + \mathrm{Tr}\, m_Q M\right]_F. \quad (3.6)$$

It follows from (3.5) that

$$\mu_M = m_Q\frac{\mu_2^2}{z_M M_0} \sim m_Q\frac{Z_q^2\Lambda_Q^2}{z_M\mathcal{M}_{ch}^2} \sim \Lambda_{YM}. \quad (3.7)$$

To check that the above assumption that the dual quarks q, $\overline{q}$ are in the HQ phase (i. e., are not higgsed) is not self-contradictory, we estimate the mass $\overline{\mu}_{gl}$ of dual gluons due to possible higgsing of the dual quarks. The condensate of the dual quarks at $\mu = \Lambda_Q$ is [9]

$$\mu_0^2 \equiv \mu_C^2(\mu = \Lambda_Q) = |\langle\overline{q}q(\mu = \Lambda_Q)\rangle| = \mu_1 m_Q = \\ = Z_q\Lambda_Q m_Q, \quad \mu_C^2(\mu < \Lambda_Q) = \mu_0^2\left(\frac{\mu}{\Lambda_Q}\right)^{\gamma_q}. \quad (3.8)$$

Therefore, the mass of dual gluons due to possible higgsing of dual quarks can be estimated as

$$\overline{\mu}_{gl}^2 \sim (\overline{a}_* \sim 1)\, Z_q\Lambda_Q m_Q\left(\frac{\overline{\mu}_{gl}}{\Lambda_Q}\right)^{\overline{b}_0/N_F} \to \\ \to \overline{\mu}_{gl} \sim \Lambda_{YM}. \quad (3.9)$$

It can be seen that this is, at least, not parametrically higher than the pole mass of the dual quark, $\overline{\mu}_{gl} \sim \mu_q^{pole}$. Hence, it may actually be even somewhat smaller, $\overline{\mu}_{gl} = \mu_q^{pole}/(\text{several})$ (because we have no real control over possible nonparametrical factors $O(1)$), and the dual quarks are indeed in the HQ phase. In similar situations here and below, we assume that this is indeed the case[3)].

On the whole, there is only one scale $\sim \Lambda_{YM}$ in the mass spectrum of the dual theory in this case. The masses of the dual quarks q, $\overline{q}$, mions M, and dual gluonia are all $\sim \Lambda_{YM}$.

Comparing the mass spectra of the direct and dual theories shows that they are parametrically different. The direct quarks have large pole masses $m_Q^{pole}/\Lambda_{YM} \sim \sim \exp(N_c/b_0) \gg 1$ and are parametrically weakly coupled and nonrelativistic inside hadrons (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \ll m_Q^{pole}$), and therefore the mass spectrum of low-lying mesons is Coulomb-like, with small mass differences $\Delta\mu_H/\mu_H = O(b_0^2/N_F^2) \ll 1$ between nearest hadrons. All low-lying gluonia have masses $\sim \Lambda_{YM}$, i. e., are parametrically smaller than the masses of flavored hadrons. At the same time, all hadron masses in the dual theory are of the same scale $\sim \Lambda_{YM}$, and all couplings are $O(1)$, and hence there is no reason for parametrically small mass differences between hadrons. We conclude that the direct and dual theories are not equivalent.

4. DIRECT AND DUAL THEORIES. EQUAL QUARK MASSES, $0 < \overline{b}_0/N_F \ll 1$

This case is considered analogously to those in Secs. 2 and 3, and we therefore highlight only on differences from Secs. 2 and 3. The main difference is that at $\mu \sim \Lambda_Q$, the direct coupling is now not small, $a_* = O(1)$, while both the gauge and Yukawa couplings of the dual theory are parametrically small,

$$\overline{a}_* \sim a_f^* \sim \overline{b}_0/N_F \ll 1, \quad a_f^* \equiv N_F(f^*)^2/8\pi^2, \\ f(\mu = \Lambda_Q) = \mu_2/\mu_1 \approx f^*.$$

The pole mass m_Q^{pole} of direct quarks is the same as in (2.1). But Λ_{YM} is now parametrically the same, $\Lambda_{YM} \sim m_Q^{pole}$ (as is the scale of the gluon mass due to possible quark higgsing, $\mu_{gl} \sim m_Q^{pole} \sim \Lambda_{YM}$). Assuming that quarks are not higgsed, but are in the HQ phase (see footnote 3), we obtain that there is only one mass scale $\mu_H \sim \Lambda_{YM} \sim \Lambda_Q(m_Q/\Lambda_Q)^{N_F/3N_c}$ in the mass spectrum of hadrons.

We now consider the weakly coupled dual theory. It already entered the conformal regime at $\mu < \Lambda_Q$ (see footnote 1), and therefore the pole mass of dual quarks is now given by

$$\frac{\mu_q^{pole}}{\Lambda_Q} = \left(\frac{\mu_q}{\Lambda_Q}\right)^{N_F/3\overline{N}_c}, \\ \frac{\mu_q = \mu_q(\mu = \Lambda_Q)}{\Lambda_Q} = \frac{\mathcal{M}_{ch}^2}{Z_q\Lambda_Q^2}, \quad (4.1) \\ \frac{\mathcal{M}_{ch}^2}{\Lambda_Q^2} = \frac{\Lambda_{YM}^3}{m_Q\Lambda_Q^2} = \left(\frac{m_Q}{\Lambda_Q}\right)^{\overline{N}_c/N_c}.$$

After integrating out all dual quarks as heavy ones at $\mu = \mu_q^{pole}$, the dual YM theory remains (together with

[3)] The opposite case, where quarks are actually higgsed, would result in a genuine spontaneous breaking of the diagonal flavor symmetry $SU(N_F)_{L+R}$ (and breaking of the gauge group) and the appearance of a large number of strictly massless Nambu–Goldstone particles. In what follows, it is assumed that this variant is not realized (at least, in the cases considered in this paper). Besides, if there are only N_c isolated supersymmetric vacua in this theory, then either this opposite case is excluded, or this exclude the whole scenario No. 2.

the mions M). The scale parameter $\overline{\Lambda} = \langle \overline{\Lambda}_L(M) \rangle$ of its gauge coupling can be found from [11]

$$3\overline{N}_c \ln\left(-\frac{\mu_q^{pole}}{\overline{\Lambda}}\right) \approx \frac{2\pi}{\overline{\alpha}_*} \approx \frac{N_F}{\overline{b}_0}\frac{\overline{N}_c^2 - 1}{2N_F + \overline{N}_c} \approx \approx \frac{3\overline{N}_c^2}{7\,\overline{b}_0}. \quad (4.2)$$

Now, from matching the gluino condensates in the direct and dual theories, we obtain

$$|\overline{\Lambda}| = \Lambda_{YM} \to Z_q \sim \exp\left(-\frac{\overline{N}_c}{7\,\overline{b}_0}\right) \ll 1, \qquad \mu_q^{pole} \sim \exp\left(\frac{\overline{N}_c}{7\,\overline{b}_0}\right)\Lambda_{YM} \gg \Lambda_{YM}. \quad (4.3)$$

The expressions for Λ_{YM} and Z_q can be written in the general case where $b_0 > 0$ and $\overline{b}_0 > 0$ as

$$\frac{\Lambda_{YM}}{\Lambda_Q} \sim \exp\left(-\frac{N_c}{b_0}\right)\left(\frac{\det m_Q}{\Lambda_Q^{N_F}}\right)^{1/3N_c}, \qquad Z_q \sim \exp\left[-\left(\frac{N_c}{b_0} + \frac{\overline{N}_c}{7\,\overline{b}_0}\right)\right]. \quad (4.4)$$

Here, the symbol "$\sim$" denotes exponential accuracy in dependence on the large parameters $N_c/b_0 \gg 1$ or $\overline{N}_c/\overline{b}_0 \gg 1$ (see footnote 1). Hence, if N_c/b_0 or $\overline{N}_c/\overline{b}_0$ are $O(1)$, then the dependence on these has to be omitted from (4.4). For our purposes, this exponential accuracy in (4.4) is sufficient.

Therefore, similarly to the case of the weakly coupled direct theory in Sec. 2, the dual quark pole mass μ_q^{pole} is parametrically larger than Λ_{YM}. Then, proceeding as in [6, 7] and integrating out the dual gluons via the VY procedure yields the Lagrangian of mions M, which can be written in form (3.6). The mion masses are therefore given by

$$\mu_M = m_Q \frac{\mu_2^2}{z_M \mathcal{M}_{ch}^2} \sim m_Q \frac{{Z_q}^2 \Lambda_Q^2}{z_M \mathcal{M}_{ch}^2}, \qquad z_M = z_M(\Lambda_Q, \mu_q^{pole}) = \left(\frac{\Lambda_Q}{\mu_q^{pole}}\right)^{2\overline{b}_0/N_F} \gg 1. \quad (4.5)$$

It follows from (4.5) that

$$\mu_M \sim {Z_q}^2 \Lambda_{YM} = \exp\left(-\frac{2\overline{N}_c}{7\,\overline{b}_0}\right)\Lambda_{YM} \ll \Lambda_{YM}. \quad (4.6)$$

Therefore, the mion masses are parametrically smaller than Λ_{YM}.

To check that there are no self-contradictions, it remains to estimate the gluon masses due to possible higgsing of dual quarks. We have

$$\frac{\overline{\mu}_{gl}^2}{\Lambda_Q^2} \sim Z_q \frac{m_Q}{\Lambda_Q}\left(\frac{\overline{\mu}_{gl}}{\Lambda_Q}\right)^{\overline{b}_0/N_F} \to \overline{\mu}_{gl} \sim \sim \exp\left(-\frac{\overline{N}_c}{14\,\overline{b}_0}\right)\Lambda_{YM} \ll \Lambda_{YM} \ll \mu_q^{pole}. \quad (4.7)$$

Therefore, there are three parametrically different mass scales in the dual theory in this case.

a) A large number of flavored hadrons made of weakly coupled nonrelativistic (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mu_q^{pole}$) dual quarks with the pole masses $\mu_q^{pole}/\Lambda_{YM} = \exp(\overline{N}_c/7\,\overline{b}_0) \gg 1$. The mass spectrum of low-lying flavored mesons is Coulomb-like, with parametrically small mass differences $\Delta\mu_H/\mu_H = O(\overline{b}_0^2/N_F^2) \ll 1$.

b) A large number of gluonia with the mass scale $\sim \Lambda_{YM}$.

c) N_F^2 lightest mions M with parametrically smaller masses, $\mu_M/\Lambda_{YM} \sim \exp(-2\overline{N}_c/7\,\overline{b}_0) \ll 1$.

At the same time, there is only one mass scale $\sim \Lambda_{YM}$ of all hadron masses in the direct theory, which is strongly coupled here, $a_* = O(1)$. Clearly, the mass spectra of the direct and dual theories are parametrically different.

On the whole, it follows that when an appropriate small parameter is available ($0 < b_0/N_F \ll 1$ when the direct theory is weakly coupled, or $0 < \overline{b}_0/N_F \ll 1$ when weakly coupled is the dual theory), the mass spectra of the direct and dual theories are parametrically different. Therefore, there are no reasons for these mass spectra to become exactly the same when b_0/N_F and $\overline{b}_0/N_F$ become $O(1)$.

5. DIRECT AND DUAL THEORIES. UNEQUAL QUARK MASSES, $0 < \overline{b}_0/N_F \ll 1$, $N_c < N_l < 3N_c/2$

The standard consideration that allows "verifying" that the duality works properly for quarks of unequal masses is as follows [1, 4]. For instance, we take N_h quarks of the direct theory to be heavier than the other $N_c < N_l = N_F - N_h$ quarks. Then, after integrating out these h-flavored quarks as heavy ones, the direct theory with N_c colors and N_l of l-flavors remains at lower energies. By duality, it is equivalent to the dual theory with $N_l - N_c$ colors and N_l flavors. On the other hand, the original theory is equivalent to the dual theory with $\overline{N}_c = N_F - N_c$ colors and N_F flavors.

In this dual theory, the h-flavored dual quarks are assumed to be higgsed, and hence the dual theory with $N_F - N_c - N_h = N_l - N_c$ colors and $N_F - N_h = N_l$ of l-flavors remains at lower energies. Therefore, all looks self-consistent.

But we consider this variant in more details, starting with the left end of the conformal window, $0 < \overline{b}_0/N_F \ll 1$. At $\mu < \Lambda_Q$, the dual theory has already entered the conformal regime, and hence both its gauge and Yukawa couplings are close to their parametrically small frozen values, $\overline{a}_* \sim a^*_f \sim \overline{b}_0/N_F \ll 1$. We take $N_c < N_l < 3N_c/2$ direct quarks $Q_l, \overline{Q}^{\,l}$ to have smaller masses m_l at $\mu = \Lambda_Q$, and the other $N_h = N_F - N_l$ quarks $Q_h, \overline{Q}^{\,h}$ to have larger masses $m_h, r \equiv m_l/m_h < 1$.

5.1. Direct theory

We start with the direct theory because, in a sense, its mass spectrum is easier to calculate. The theory is in the conformal regime at $\mu < \Lambda_Q$, and the highest physical mass scale μ_H is equal to the pole mass of the heavier $Q_h, \overline{Q}^{\,h}$ quarks:

$$\frac{m_h^{pole}}{\Lambda_Q} = \frac{m_h}{\Lambda_Q}\left(\frac{\Lambda_Q}{m_h^{pole}}\right)^{\gamma_Q = b_0/N_F} = \left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c}. \quad (5.1)$$

After integrating these quarks out, the lower-energy direct theory with N_c colors and $N_c < N_l < 3N_c/2$ flavors of lighter $Q_l, \overline{Q}^{\,l}$ quarks remains[4]. From matching with the coupling $a^* = O(1)$ of the higher-energy theory, its gauge coupling is also $O(1)$ at $\mu = m_h^{pole}$ and hence the scale parameter of this gauge coupling is $\Lambda'_Q \sim m_h^{pole}$. The current masses of $Q_l, \overline{Q}^{\,l}$ quarks at $\mu = m_h^{pole}$ are

$$\hat{m}_l \equiv m_l(\mu = m_h^{pole}) = r m_h^{pole} \ll \Lambda'_Q = m_h^{pole}, \quad r \ll 1.$$

Now, how to deal further with this theory at lower energies? As was described in [6] (see Sec. 7), there are two variants, "a" and "b". All hadron masses μ_H are much smaller than Λ'_Q in variant "a", $\mu_H \ll \Lambda'_Q$, and the direct theory and its dual are not equivalent [6]. We therefore do not consider this variant "a" in this section, because our purpose here is to check the duality in the variant most favorable for its validity. This is variant "b", i. e., "confinement without chiral symmetry breaking" (although some general arguments have been presented in [6] that this variant cannot be realized). This amounts to assuming (because all the original direct degrees of freedom cannot "dissolve in the air") that due to strong nonperturbative confining effects, the direct $Q_l, \overline{Q}^{\,l}$ quarks and gluons form a large number of heavy hadrons with masses $\mu_H \sim \Lambda'_Q$.

Instead, new light composite particles (special solitons) appear, whose masses are parametrically smaller than Λ'_Q. These are the dual quarks $\hat{q}^l$, $\hat{\overline{q}}_{\bar{l}}$, the dual gluons with $\overline{N}'_c = N_l - N_c$ dual colors, and the mions $\hat{M}_l \equiv \hat{M}^{\bar{l}}_l$. Their couplings at scales $\mu < \Lambda'_Q$ are described by the Seiberg dual Lagrangian. Their dual gauge coupling $\overline{a}(\mu)$ at $\mu \lesssim \Lambda'_Q \sim m_h^{pole}$ is $\overline{a}(\mu = \Lambda'_Q/(\text{several})) \sim a^* = O(1)$, and therefore the scale parameter of this dual gauge coupling is $\sim \Lambda'_Q \sim m_h^{pole}$.

The dual Lagrangian (at the scale $\mu \sim \Lambda'_Q$, $(\mathcal{M}^l_{ch})^2 \equiv \langle \overline{Q}^{\,l} Q_l(\mu = \Lambda_Q)\rangle = \langle S\rangle/m_l = \Lambda^3_{YM}/m_l$) is

$$\overline{L} = \left[\mathrm{Tr}_{\,l}\left(\hat{q}^\dagger e^{\overline{V}}\hat{q} + \hat{\overline{q}}^\dagger e^{-\overline{V}}\hat{\overline{q}}\right) + \frac{1}{(\Lambda'_Q)^2}\mathrm{Tr}\left(\hat{M}^\dagger_l \hat{M}_l\right)\right]_D + \left[-\frac{2\pi}{\overline{a}(\Lambda'_Q)}\,\overline{s} + \frac{1}{\Lambda'_Q}\,\mathrm{Tr}_{\,l}\left(\hat{\overline{q}}\,\hat{M}_l\,\hat{q}\right) + \hat{m}_l\,\mathrm{Tr}\,\hat{M}_l\right]_F, \quad (5.2)$$

where

$$\overline{s} = \overline{w}^2_\alpha/32\pi^2, \quad \langle \hat{\overline{q}}_{\bar{l}}\,\hat{q}^l\rangle = -\delta^l_{\bar{l}}\,\hat{m}_l\,\Lambda'_Q,$$
$$\hat{m}_l = m_l\, z_Q^{-1}, \quad \langle \hat{M}_l\rangle = (\mathcal{M}^l_{ch})^2\, z_Q,$$
$$z_Q = \left(\frac{m_h^{pole}}{\Lambda_Q}\right)^{\gamma_Q = b_0/N_F} = \left(\frac{m_h}{\Lambda_Q}\right)^{b_0/3N_c} \ll 1. \quad (5.3)$$

This dual theory is in the HQ phase [6] (see Sec. 7). At lower scales $\mu^{pole}_{q,l} < \mu < \Lambda'_Q$, both its gauge and Yukawa couplings decrease logarithmically ($\overline{b}'_0 = 3\overline{N}'_c - N_l < 0$) and become much less than unity at $\mu = \mu^{pole}_{q,l} \ll \Lambda'_Q$ and $r \ll 1$. The pole mass $\mu^{pole}_{q,l}$ of the $\hat{q}^l$, $\hat{\overline{q}}_{\bar{l}}$ quarks is

$$\mu^{pole}_{q,l} = \frac{\mu_{q,l}}{z'_q} = \frac{\langle \hat{M}_l\rangle}{z'_q \Lambda'_Q} = \left(r = \frac{m_l}{m_h}\right)^{(N_l - N_c)/N_c} \frac{m_h^{pole}}{z'_q} \ll m_h^{pole}, \quad (5.4)$$

[4] To simplify all formulas, the value $(3N_c - 2N_l)/N_l$ is considered as $O(1)$ quantity in this section.

where $z'_q = z'_q(\Lambda'_Q, \mu^{pole}_{q,l}) \ll 1$ is the perturbative logarithmic renormalization factor of dual l-quarks:

$$z'_q = z'_q(\Lambda'_Q, \mu^{pole}_{q,l}) \sim \left(\frac{\overline{\alpha}(\mu^{pole}_{q,l})}{\overline{\alpha}(\Lambda'_Q)}\right)^{\overline{N}'_c/|\overline{b}'_0|} \sim \\ \sim \left(\frac{1}{\ln(\Lambda'_Q/\mu^{pole}_{q,l})}\right)^{(N_l-N_c)/(3N_c-2N_l)} \ll 1.$$

At scales $\mu < \mu^{pole}_{q,l}$, all quarks $\hat{q}^l, \hat{\overline{q}}_{\overline{l}}$ can be integrated out as heavy ones, and there remains the dual YM theory with $\overline{N}'_c = N_l - N_c$ colors and mions $\hat{M}_l$. The scale factor $\overline{\Lambda}_{YM} = \langle \overline{\Lambda}_L(\hat{M}_l)\rangle$ (with mions $\hat{M}_l$ sitting down on $\overline{\Lambda}_L(\hat{M}_l)$) of its gauge coupling is determined from the matching [6, 7]

$$3\overline{N}'_c \ln\left(\frac{\mu^{pole}_{q,l}}{-\overline{\Lambda}_{YM}}\right) = \overline{b}'_0 \ln\left(\frac{\mu^{pole}_{q,l}}{\Lambda'_Q}\right) + \\ + N_l \ln\left(\frac{\mu^{pole}_{q,l}}{\mu_{q,l}}\right) \to -\overline{\Lambda}_{YM} = -\langle \overline{\Lambda}_L(\hat{M}_l)\rangle = \\ = \Lambda_{YM} = \left(\Lambda^{b_0}_Q m^{N_l}_l m^{N_h}_h\right)^{1/3N_c}. \quad (5.5)$$

Proceeding as in [6, 7] and applying the VY procedure to dual gluons, we obtain the lowest-energy Lagrangian of mions $\hat{M}_l$:

$$L_M = \left[\frac{z'_M}{(\Lambda'_Q)^2}\mathrm{Tr}\left(\hat{M}^{\dagger}_l \hat{M}_l\right)\right]_D + \left[-(N_l - N_c) \times \\ \times \left(\frac{\det \hat{M}_l}{(\Lambda'_Q)^{(3N_c-N_l)}}\right)^{1/(N_l-N_c)} + \hat{m}_l\,\mathrm{Tr}\hat{M}_l\right]_F, \quad (5.6)$$

where $z'_M = z'_M(\Lambda'_Q, \mu^{pole}_{q,l}) \gg 1$ is the perturbative logarithmic renormalization factor of mions. From (5.6), the masses of mions $\hat{M}_l$ are

$$\mu(\hat{M}_l) \sim \hat{m}_l \frac{(\Lambda'_Q)^2}{z'_M \langle \hat{M}_l\rangle} \sim \\ \sim \left(r = \frac{m_l}{m_h}\right)^{(2N_c-N_l)/N_c} \frac{m^{pole}_h}{z'_M}, \quad r \ll 1, \\ \frac{\Lambda_{YM}}{\mu^{pole}_{q,l}} \sim z'_q\, r^{\Delta} \ll 1, \quad \frac{\mu(\hat{M}_l)}{\Lambda_{YM}} \sim \frac{r^{2\Delta}}{z'_M} \ll 1, \\ 0 < \Delta = \frac{3N_c - 2N_l}{3N_c} < \frac{1}{3}. \quad (5.7)$$

Therefore, on the whole for Sec. 5.1, when we started with the direct theory, then integrated out the heaviest $Q_h, \overline{Q}^h$ quarks at $\mu = m^{pole}_h = \Lambda_{YM}(1/r)^{N_l/3N_c} \gg \Lambda_{YM}$ and, in variant "b" ("confinement without chiral symmetry breaking"), dualized the remaining theory of the direct l flavors and direct gluons, the mass spectrum is as follows.

1. There is a large number of heavy direct h mesons M^{dir}_h, made of the $Q_h, \overline{Q}^h$ quarks (and/or antiquarks and direct gluons) with largest masses $\sim m^{pole}_h = (1/r)^{N_l/3N_c}\Lambda_{YM} \ll \Lambda_Q$.

1′. In variant "b" considered here, the dualization of $Q_l, \overline{Q}^{\overline{l}}$ quarks and direct gluons leaves behind a large number of heavy direct l mesons M^{dir}_l with different spins, made of the $Q_l, \overline{Q}^{\overline{l}}$ quarks (and direct gluons), also with masses $\sim m^{pole}_h$. Besides, there is also a large number of heavy direct hybrid mesons M^{dir}_{lh}, baryons B_l, B_{lh}, and gluonia, all also with masses $\sim m^{pole}_h$. All these heavy particles are strongly coupled, with couplings $O(1)$.

All other particles are parametrically lighter and originate from the dual quarks $\hat{q}^l, \hat{\overline{q}}_{\overline{l}}$, the dual gluons, and mions $\hat{M}_l$.

2. There is a large number of l-flavored dual mesons and b_l baryons made of nonrelativistic (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mu^{pole}_{q,l}$) $\hat{q}^l, \hat{\overline{q}}_{\overline{l}}$ quarks, with their pole masses

$$\mu^{pole}_{q,l} \sim r^{(N_l-N_c)/N_c}\frac{m^{pole}_h}{z'_q} \sim \\ \sim \left(\frac{1}{r}\right)^{(3N_c-2N_l)/3N_c} \frac{\Lambda_{YM}}{z'_q} \gg \Lambda_{YM}.$$

3. There is a large number of gluonia with masses $\sim \Lambda_{YM} \ll \mu^{pole}_{q,l}$.

4. The lightest are N^2_l scalar mions $\hat{M}_l$ with masses $\mu(\hat{M}_l) \sim (r)^{2\Delta}\,\Lambda_{YM}/z'_M \ll \Lambda_{YM}$.

5.2. Dual theory

We return to the beginning of this section and start directly with the dual theory with $\overline{N}_c$ dual colors, N_F dual quarks q and $\overline{q}$, and N^2_F mions $M^{\overline{j}}_i$. At the scale $\mu < \Lambda_Q$, the theory is already in the weak-coupling conformal regime, i.e., its coupling $\overline{a}(\mu)$ is close to $\overline{a}_* = \overline{N}_c\,\overline{\alpha}_*/2\pi \approx 7\overline{b}_0/3\overline{N}_c \ll 1$ (see footnote 1 and formula (4.2)). We first consider the case most favorable for the dual theory, where the parameter $r = m_l/m_h$ is already taken to be sufficiently small (see below). Then in scenario No. 2 considered in this paper, the highest physical mass scale μ_H in the dual theory is determined by masses of dual gluons due to higgsing of the $\overline{q}_h$, q^h quarks:

$$\mu^2_H = \overline{\mu}^2_{ql,h} \sim \overline{a}_*|\langle \overline{q}_h q^h(\mu = \overline{\mu}_{ql,h})\rangle|.$$

The mass spectrum of the dual theory in this phase can be obtained in a relatively standard way, similarly to [6, 7]. We therefore skip some intermediate relations in similar situations in what follows. The emphasis is instead on new elements that have not appeared before.

1) The masses of $2\overline{N}_c N_h - N_h^2$ massive dual gluons and their scalar superpartners are

$$\overline{\mu}^2_{ql,h} \sim |\langle \overline{q}_h q^h \rangle| \left(\frac{\overline{\mu}_{ql,h}}{\Lambda_Q}\right)^{\gamma_q} \sim \sim \mu_1 m_h \left(\frac{\overline{\mu}_{ql,h}}{\Lambda_Q}\right)^{\gamma_q} \sim Z_q \Lambda_Q m_h \left(\frac{\overline{\mu}_{ql,h}}{\Lambda_Q}\right)^{\overline{b}_0/N_F}, \quad (5.8)$$

$$\frac{\overline{\mu}_{ql,h}}{\Lambda_Q} \sim \exp\left(-\frac{\overline{N}_c}{14\overline{b}_0}\right)\left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c} \ll \ll \left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c}, \quad Z_q \sim \exp\left(-\frac{\overline{N}_c}{7\overline{b}_0}\right) \ll 1. \quad (5.9)$$

2) The $N_l N_h$ hybrid mions M_{hl} and the $N_l N_h$ nions N_{lh} (these are those dual l quarks that have higgsed colors, their partners M_{lh} and N_{hl} are understood and are not shown explicitly) can be treated independently of other degrees of freedom, and their masses are determined mainly by their common mass term in the superpotential:

$$L_{hybr} \approx \left[\hat{z}_M \operatorname{Tr}\left(\frac{M^\dagger_{hl} M_{hl}}{Z_q^2 \Lambda_Q^2}\right) + \hat{z}_q \operatorname{Tr}\left(N^\dagger_{lh} N_{lh}\right)\right]_D + + \left[\sqrt{Z_q m_h \Lambda_Q} \operatorname{Tr}\left(\frac{M_{hl} N_{lh}}{Z_q \Lambda_Q}\right)\right]_F, \quad (5.10)$$

where $\hat{z}_q$ and $\hat{z}_M$ are the perturbative renormalization factors of dual quarks and mions,

$$\hat{z}_q = \hat{z}_q(\Lambda_Q, \overline{\mu}_{ql,h}) = \left(\frac{\overline{\mu}_{ql,h}}{\Lambda_Q}\right)^{\overline{b}_0/N_F} \sim \sim \left(\frac{m_h}{\Lambda_Q}\right)^{\overline{b}_0/3N_c} \ll 1,$$

$$\hat{z}_M = \hat{z}_M(\Lambda_Q, \overline{\mu}_{ql,h}) = 1/\hat{z}_q^2 \gg 1.$$

Therefore,

$$\frac{\mu(M_{hl})}{\Lambda_Q} \sim \frac{\mu(N_{lh})}{\Lambda_Q} \sim \exp\left(-\frac{\overline{N}_c}{14\overline{b}_0}\right) \times \times \left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c} \sim \frac{\overline{\mu}_{ql,h}}{\Lambda_Q}. \quad (5.11)$$

3) Because the $\overline{q}_h$ and q^h quarks are higgsed, N_h^2 pseudo-Goldstone mesons N_{hh} (nions) appear. After integrating out the heavy gluons and their superpartners, the Lagrangian of remained degrees of freedom takes the form

$$L \approx \left[\hat{z}_M \operatorname{Tr}\left(\frac{M^\dagger_{hh} M_{hh} + M^\dagger_{ll} M_{ll}}{Z_q^2 \Lambda_Q^2}\right) + + \hat{z}_q\, 2 \operatorname{Tr}\sqrt{N^\dagger_{hh} N_{hh}} + \hat{z}_q \operatorname{Tr}_l \left(q^\dagger e^{\overline{V}'} q + \overline{q}^\dagger e^{-\overline{V}'} \overline{q}\right)\right]_D + + \left[-\frac{2\pi}{\overline{\alpha}'(\mu)} \overline{s}' + \operatorname{Tr}\left(\frac{M_{hh} N_{hh}}{Z_q \Lambda_Q}\right) + + \operatorname{Tr}_l \left(\frac{\overline{q}\, M_{ll}\, q}{Z_q \Lambda_Q}\right) + \operatorname{Tr}\left(m_l M_{ll} + m_h M_{hh}\right)\right]_F, \quad (5.12)$$

where $\overline{s}'$ includes the field strengths of the remaining $SU(\overline{N}'_c)$ dual gluons with $\overline{N}'_c = \overline{N}_c - N_h = N_l - N_c$ dual colors, and q^l, $\overline{q}_{\overline{l}}$ are the l-flavored dual quarks with unhiggsed dual colors and, finally, the nions N_{hh} are "sitting down" inside $\overline{\alpha}'(\mu)$.

At lower scales $\mu < \overline{\mu}_{ql,h}$, the mions M_{hh} and nions N_{hh} are frozen and do not evolve, while the gauge coupling decreases logarithmically in the interval $\overline{\mu}^{pole}_{q,l} < \mu < \overline{\mu}_{ql,h}$. The numerical value of the pole mass of the $\overline{q}_{\overline{l}}$, q^l quarks is

$$\frac{\overline{\mu}^{pole}_{q,l}}{\Lambda_Q} = \frac{\langle M_{ll}\rangle}{Z_q \Lambda_Q^2} \frac{1}{\hat{z}_q\, z''_q} \sim \exp\left(\frac{\overline{N}_c}{7\overline{b}_0}\right) \times \times \left[r^{(N_l - N_c)/N_c}\left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c}\right] \frac{1}{z''_q},$$
$$\overline{\mu}^{pole}_{q,l} \sim \exp\left(\frac{\overline{N}_c}{7\overline{b}_0}\right) \mu^{pole}_{q,l} \gg \mu^{pole}_{q,l},$$
$$z''_q = z'_q(\overline{\mu}_{ql,h}, \overline{\mu}^{pole}_{q,l}) \approx z'_q(m^{pole}_h, \mu^{pole}_{q,l}) = z'_q, \quad (5.13)$$

where $z''_q \ll 1$ is the logarithmic renormalization factor of the q^l, $\overline{q}_{\overline{l}}$ quarks.

After integrating out the $\overline{q}_{\overline{l}}$, q^l quarks as heavy ones at $\mu = \overline{\mu}^{pole}_{q,l}$, the dual $SU(N_l - N_c)$ YM theory remains with the scale factor of its gauge coupling $\langle -\overline{\Lambda}_L \rangle = \Lambda_{YM}$ (and with nions N_{hh} and mions M_{ll} "sitting down" on $\overline{\Lambda}_L$). Finally, integrating the dual gluons via the VY procedure, we obtain (recalling that all fields in (5.12) and (5.14) are normalized at $\mu = \Lambda_Q$, and $\Lambda_{YM}/\Lambda_Q = r^{N_l/3N_c}(m_h/\Lambda_Q)^{N_F/3N_c}$,

$\langle N_{hh}\rangle = -Z_q m_h \Lambda_Q)$

$$L \approx \left[\hat{z}_M \operatorname{Tr}\left(\frac{M^{\dagger}_{hh}M_{hh} + z''_M M^{\dagger}_{ll}M_{ll}}{Z_q^2\Lambda_Q^2}\right) + \hat{z}_q\, 2\operatorname{Tr}\sqrt{N^{\dagger}_{hh}N_{hh}}\right]_D + \left[\operatorname{Tr}\left(\frac{M_{hh}N_{hh}}{Z_q\Lambda_Q}\right) - (N_l - N_c)\Lambda^3_{YM}\left(\det\frac{\langle N_{hh}\rangle}{N_{hh}}\det\frac{M_{ll}}{\langle M_{ll}\rangle}\right)^{1/(N_l-N_c)} + \operatorname{Tr}\left(m_l M_{ll} + m_h M_{hh}\right)\right]_F, \quad (5.14)$$

where

$$z''_M = z''_M(\overline{\mu}_{ql,h}, \overline{\mu}^{pole}_{q,l}) \approx z'_M(\mu_{gl,h}, \mu^{pole}_{q,l}) = z'_M \gg 1$$

is the logarithmic renormalization factor of M_{ll} mions[5)].

The masses obtained from (5.14) are then as follows:

$$\frac{\mu(M_{hh})}{\Lambda_Q} \sim \frac{\mu(N_{hh})}{\Lambda_Q} \sim \sqrt{\hat{z}_q\frac{|\langle N_{hh}\rangle|}{\Lambda_Q^2}} \sim \exp\left(-\frac{\overline{N}_c}{14\overline{b}_0}\right)\left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c} \sim \frac{\overline{\mu}_{ql,h}}{\Lambda_Q},$$
$$\frac{\mu(M_{ll})}{\Lambda_Q} \sim \frac{Z_q^2}{z''_M}\frac{m_l\,\Lambda_Q}{\hat{z}_M\langle M_{ll}\rangle} \sim \exp\left(-\frac{2\overline{N}_c}{7\overline{b}_0}\right) \times r^{(2N_c-N_l)/N_c}\left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c}\frac{1}{z''_M}. \quad (5.15)$$

On the whole for Sec. 5.2, when we started directly with the dual theory with $\overline{N}_c = N_F - N_c$ colors, $N_F = N_l + N_h$ quarks $\overline{q}, q$, and N_F^2 mions M, its mass spectrum is as follows.

1. The sector of heavy masses includes: a) $2\overline{N}_c N_h - N_h^2$ massive dual gluons and their scalar superpartners; b) $2N_lN_h$ hybrid scalar mions $M_{hl} + M_{lh}$; c) $2N_lN_h$ hybrid scalar nions $N_{lh} + N_{hl}$ (these are the $q^l, \overline{q}_{\overline{l}}$ quarks with higgsed colors); d) N_h^2 scalar mions M_{hh} and N_h^2 scalar nions N_{hh}. All these particles, with specific numbers of each type, definite spins, and other quantum numbers, have definite masses of the same scale of the order of

$$\overline{\mu}_{ql,h} \sim \exp\left(-\frac{\overline{N}_c}{14\overline{b}_0}\right)\left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c}\Lambda_Q = \exp\left(-\frac{\overline{N}_c}{14\overline{b}_0}\right) m_h^{pole} \ll m_h^{pole} = (1/r)^{N_l/3N_c}\Lambda_{YM}.$$

All these particles are coupled only weakly, with both their gauge and Yukawa couplings $\overline{a}_* \sim a_f^* \sim \overline{b}_0/N_F \ll 1$.

1′. Because the dual quarks $q^h, \overline{q}_h$ are higgsed, one can imagine that solitonic excitations also appear in the form of monopoles of the dual group (its broken part) (see, e. g., Sec. 3 in [12] and footnote 6 therein). These dual monopoles are then confined and can, in principle, form a number of additional hadrons H'_h. Because the dual theory is weakly coupled at the scale of higgsing, $\mu \sim \overline{\mu}_{ql,h} \sim \exp(-\overline{N}_c/14\overline{b}_0)m_h^{pole}$, the masses of these monopoles, as well as the tension $\sqrt{\overline{\sigma}}$ of strings confining them (with our exponential accuracy in parametrical dependence on $\overline{N}_c/\overline{b}_0 \gg 1$), are also $\sim \overline{\mu}_{ql,h}$. Therefore, the mass scale of these hadrons H'_h is also $\sim \overline{\mu}_{ql,h} \ll m_h^{pole}$. We even assume (in favor of the duality) that, with respect to their quantum numbers, these hadrons H'_h can be identified in some way with the direct hadrons made of the $\overline{Q}^h, Q_h$ quarks. But even then, the masses of H'_h are parametrically smaller than those of various direct hadrons made of the $\overline{Q}^h$, Q_h quarks, $\mu(H'_h) \sim \exp(-\overline{N}_c/14\overline{b}_0)m_h^{pole} \ll m_h^{pole}$.

Besides, because the chiral symmetry of $\overline{l}, l$ flavors and the R-charges of the lower-energy theory at $\overline{\mu}^{pole}_{q,l} < \mu < \overline{\mu}_{ql,h}$ remain unbroken and the l-flavored dual quarks are not higgsed and remain effectively massless in this interval of scales, no possibility is seen in this dual theory in Sec. 5.2 for the appearance of $\overline{l}, l$-flavored chiral hadrons H'_l (mesons and baryons) with the heavy masses $\sim \overline{\mu}_{ql,h}$ that could be identified with a large number of various direct $\overline{l}, l$-flavored chiral hadrons (mesons and baryons) made of $\overline{Q}^l, Q_l$ quarks (and W_α) in the theory in Sec. 5.1 dualized in variant "b" (not to mention their parametrically different mass scales, $\overline{\mu}_{ql,h} \ll m_h^{pole}$; see point **1′** in Sec. 5.1). This fact shows the self-contradictory character of duality in variant "b" ("confinement without chiral flavor symmetry breaking").

All other particles in the mass spectrum of the theory described in Sec. 5.2 constitute the sector of lighter particles, with their masses being parametrically smaller than $\overline{\mu}_{ql,h}$.

2. The next mass scale is formed by a large number of l-flavored dual mesons and $b_l, \overline{b}_l$ baryons made of nonrelativistic (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \overline{\mu}^{pole}_{q,l}$) dual $q^l, \overline{q}_{\overline{l}}$ quarks with $\overline{N}'_c = N_l - N_c$ unhiggsed colors. The pole masses of these $q^l, \overline{q}_{\overline{l}}$ quarks are

$$\overline{\mu}^{pole}_{q,l} \sim \exp\left(\frac{3\overline{N}_c}{14\overline{b}_0}\right) r^{(N_l-N_c)/N_c}\frac{\overline{\mu}_{ql,h}}{z'_q} \ll \overline{\mu}_{ql,h}$$

[5)] The second term of the superpotential in (5.14) can equivalently be written as

$$-(N_l - N_c)\left(\frac{\det M_{ll}}{\Lambda_Q^{b_0}\det\left(-N_{hh}/Z_q\Lambda_Q\right)}\right)^{1/(N_l-N_c)}.$$

(at $r \ll r_l$; see (5.16) below).

3. Next, there is a large number of gluonia with the mass scale $\sim \Lambda_{YM} \ll \overline{\mu}^{pole}_{q,l}$.

4. Finally, the lightest are N_l^2 scalar mions M_{ll} with masses

$$\mu(M_{ll}) \sim \exp\left(-\frac{2\overline{N}_c}{7\overline{b}_0}\right) r^{2\Delta} \frac{\Lambda_{YM}}{z'_M} \ll \Lambda_{YM}.$$

Comparing the mass spectra of two supposedly equivalent descriptions in Secs. 5.1 and 5.2 above, we see that the masses are clearly different parametrically, in powers of the parameter $Z_q = \exp(-\overline{N}_c/7\overline{b}_0) \ll 1$. Besides, the theory described in Sec. 5.2 contains very specific definite numbers of fields with fixed quantum numbers and spins and with the definite masses $\sim \overline{\mu}_{ql,h}$, all parametrically weakly coupled (see point **1** in Sec. 5.2). No analog of these specific particles is seen in Sec. 5.1. Instead, there is a large number of h-flavored hadrons with various spins, all strongly coupled, with the coupling $a(\mu \sim m_h^{pole}) \sim 1$.

Finally (and we consider this to be of special importance), there are no heavy $\overline{l}, l$-flavored hadrons H'_l in the theory described in Sec. 5.2, which have the appropriate conserved (in the interval of scales $\overline{\mu}^{pole}_{q,l} < \mu < \overline{\mu}_{ql,h}$) chiral flavors $\overline{l}, l$ and R-charges, such that they could be associated with a large number of various heavy flavored chiral hadrons made of the $\overline{Q}^{\overline{l}}, Q_l$ quarks (and W_α), which are present in Sec. 5.1 dualized in variant "b". This shows that the duality in variant "b" ("confinement without chiral flavor symmetry breaking") is not self-consistent[6]. This agrees with some general arguments presented in [6] (see Sec. 7; it is also worth recalling that these arguments were not related with the use of scenario No. 1 with the diquark condensate) that the duality in variant "b" cannot be realized because, in the theory with unbroken chiral flavor symmetries and R-charges and effectively massless quarks, the masses of flavored and R-charged chiral hadron superfields with various spins cannot be made "of nothing".

This is not the whole story, however. For the mass $\overline{\mu}_{ql,h}$ of gluons in the dual theory to be the largest physical mass μ_H (as was used in Sec. 5.2 above), the parameter $r = m_l/m_h$ has to be taken sufficiently small (see (5.9) and (5.13); from now on, the nonleading effects due to logarithmic factors like $z'_q \approx z''_q$ are ignored):

$$\frac{\overline{\mu}^{pole}_{q,l}}{\overline{\mu}_{ql,h}} \ll 1 \to r \ll r_l = \left[z'_q \exp\left(-\frac{3\overline{N}_c}{14\overline{b}_0}\right)\right]^{N_c/(N_l-N_c)} \sim \exp\left(-\frac{3\overline{N}_c}{14\overline{b}_0}\frac{N_c}{N_l-N_c}\right) \ll 1. \quad (5.16)$$

We trace the behavior of the direct and dual theories in the whole interval $r_l < r < 1$[7].

As regards the direct theory (see Sec. 5.1), its regime and all hierarchies in the mass spectrum remain the same for any value of $r < 1$, i. e., the pole mass m_h^{pole} of the $Q_h, \overline{Q}^{\overline{h}}$ quarks becomes the largest physical mass μ_H already at $r < 1/(\text{several})$, and so on.

But this is not the case for the dual theory (see Sec. 5.2). At $r_l < r < 1/(\text{several})$, the pole mass $\hat{\mu}^{pole}_{q,l}$ of $q^l, \overline{q}_{\overline{l}}$ quarks remains the largest physical mass:

$$\frac{\hat{\mu}^{pole}_{q,l}}{\Lambda_Q} = \frac{\langle M_{ll}\rangle}{Z_q\Lambda_Q^2}\left(\frac{\Lambda_Q}{\hat{\mu}^{pole}_{q,l}}\right)^{\overline{b}_0/N_F} \approx \exp\left(\frac{\overline{N}_c}{7\overline{b}_0}\right) \times r^{\frac{N_l-N_c}{N_c}\frac{N_F}{3\overline{N}_c}}\left(\frac{m_h}{\Lambda_Q}\right)^{N_F/3N_c} \sim \frac{\overline{\mu}^{pole}_{q,l}}{\Lambda_Q}. \quad (5.17)$$

Already this is sufficient to see a qualitative difference between the direct and dual theories. The hh-flavored hadrons in the direct theory have the largest masses, while the ll-flavored hadrons are the heaviest ones in the dual theory.

We make some rough estimates in Sec. 5.2 at $r > r_l$. After integrating out the heaviest quarks $q^l, \overline{q}_{\overline{l}}$ at the scale $\mu \sim \hat{\mu}^{pole}_{q,l}$, the dual theory remains with $\overline{N}_c$ colors and $N_h < \overline{N}_c$ dual quarks $\overline{q}_{\overline{h}}$, q^h (and mions M), and with $\overline{b}''_0 = 3\overline{N}_c - N_h > 0$. It is in the weak-coupling logarithmic regime at $\mu'_H < \mu < \hat{\mu}^{pole}_{q,l}$, where μ'_H is the highest mass scale in the remaining theory. The new scale factor Λ'_q of its gauge coupling can be found from

$$\overline{b}''_0 \ln\left(\frac{\hat{\mu}^{pole}_{q,l}}{\Lambda'_q}\right) \approx \frac{2\pi}{\overline{\alpha}_*} = \frac{3\overline{N}_c^2}{7\overline{b}_0} \to \frac{\Lambda'_q}{\hat{\mu}^{pole}_{q,l}} \sim \exp\left(-\frac{3\overline{N}_c^2}{7\overline{b}_0\overline{b}''_0}\right) \ll 1. \quad (5.18)$$

[6] It is not difficult to see that a similar situation with the $\overline{l}, l$-flavored chiral hadrons that can be made of the $\overline{Q}^{\overline{l}}, Q_l$ quarks (and W_α) occurs in variant "b" also in scenario No. 1 considered in [6, 7] (see Sec. 3b in [7] for a similar regime). Section 5.1 therein is exactly the same as in this paper. Section 5.2 is different, because the h-flavors are not higgsed, but instead form the diquark condensate; qualitatively, however, the situation is the same, and the difference between Secs. 5.1 and 5.2 is only more prominent in scenario No. 1.

[7] For this, it is convenient to keep m_h intact while m_l is decreased, starting with $m_l = m_h$.

If (see below) $r_l \ll r_h < r < 1/(\text{several})$, then $\mu'_H = \hat{\mu}^{pole}_{q,h} > \Lambda'_q$, where $\hat{\mu}^{pole}_{q,h}$ is the pole mass of $q^h, \overline{q}_h$ quarks, which are in the HQ phase (i.e., not yet higgsed). Roughly, $\hat{\mu}^{pole}_{q,h} \sim r\,\hat{\mu}^{pole}_{q,l}$, whence

$$\frac{\hat{\mu}^{pole}_{q,h}}{\Lambda'_q} > 1 \to r > r_h \sim \exp\left(-\frac{3\overline{N}_c^2}{7\overline{b}_0\overline{b}''_0}\right) \gg r_l. \quad (5.19)$$

In the interval $r_h < r < 1/(\text{several})$, the mass spectrum of the dual theory is qualitatively not much different from the case $r = 1$ (see Sec. 4). The heaviest are the ll-hadrons, then the hl-hadrons, then the hh-hadrons, then gluonia, and, in addition, there are the mions M_{hh}, M_{hl}, M_{ll} with their masses

$$\mu(M_{hh}) \sim \frac{m_h^2}{\mu_0}, \quad \mu(M_{hl}) \sim \frac{m_h m_l}{\mu_0}, \quad \mu(M_{ll}) \sim \frac{m_l^2}{\mu_0},$$

where

$$\mu_0 = z_M(\Lambda_Q, \hat{\mu}^{pole}_{q,l})\Lambda^3_{YM}/Z_q^2\Lambda_Q^2,$$

$$z_M(\Lambda_Q, \hat{\mu}^{pole}_{q,l}) \sim (\Lambda_Q^2/m_h m_l)^{\overline{b}_0/3N_c} \gg 1.$$

As r decreases further, a phase transition occurs from the HQ_h phase to the Higgs_h phase at $r \sim r_h$, i.e., after $\hat{\mu}^{pole}_{q,h} > \Lambda'_q$ becomes $\hat{\mu}^{pole}_{q,h} < \Lambda'_q$ and the quarks $q^h, \overline{q}_h$ are higgsed[8)]. But even then the $q^l, \overline{q}_{\overline{l}}$ quarks remain the heaviest ones. And only when r becomes $r < r_l \ll r_h \ll 1$, the gluon mass $\overline{\mu}_{ql,h}$ becomes the largest and the mass spectrum of the dual theory becomes that described above in this section.

6. DIRECT THEORY. UNEQUAL QUARK MASSES, $3N_c/2 < N_F < 3N_c$, $b_0/N_F = O(1)$, $N_l > N_c$

This section continues the preceding one, but we forget about any dualizations in what follows and we deal with the direct theory as it is, i.e., in variant a) (see [6], Sec. 7). This means that after the heavy quarks $Q_h, \overline{Q}^h$ have been integrated out at $\mu = m_h^{pole}$, all particle masses in the lower-energy theory with N_c colors and $N_c < N_l < 3N_c/2$ lighter $Q_l, \overline{Q}^{\overline{l}}$ quarks are parametrically smaller than the scale $\Lambda'_Q \sim m_h^{pole}$ (see (5.1)), and at $\mu < \Lambda'_Q$, the lower-energy theory enters the strong-coupling regime with $a(\mu \ll \Lambda'_Q) \gg 1$[9)].

[8)] As was argued in [12] (see Sec. 3 therein), the transition proceeds through the formation of a mixed phase in the threshold region $\Lambda'_q/(\text{several}) < \hat{\mu}^{pole}_{q,h} < (\text{several})\Lambda'_q$ (i.e., $r_h/(\text{several}) < r < (\text{several})r_h$ here).

[9)] Hereafter, to have definite answers, we use the anomalous quark dimension $1 + \gamma_Q(N_F, N_c, a(\mu) \gg 1) = N_c/(N_F - N_c)$, and the strong coupling $a(\mu) \gg 1$ given in Eq. (7.4) in [6].

This lower-energy theory is assumed to be in the HQ phase (see footnote 3), and the pole mass of $Q_l, \overline{Q}^{\overline{l}}$ quarks is

$$\begin{gathered} \frac{m_l^{pole}}{\Lambda'_Q} = \left(\frac{m_l(\mu = \Lambda'_Q)}{\Lambda'_Q} = r\right)\left(\frac{\Lambda'_Q}{m_l^{pole}}\right)^{\gamma'_Q} \to \\ \to \frac{m_l^{pole}}{\Lambda'_Q} = r^{(N_l - N_c)/N_c} \ll 1, \quad r \ll 1, \\ b'_0 = 3N_c - N_l, \quad 1 + \gamma'_Q = \frac{N_c}{N_l - N_c}, \\ \nu = \frac{N_F\gamma'_Q - b'_0}{N_c} = \frac{3N_c - 2N_l}{N_l - N_c}. \end{gathered} \quad (6.1)$$

The coupling $a_+(\mu = m_l^{pole})$ is large:

$$a_+(\mu = m_l^{pole}) = \left(\frac{\Lambda'_Q}{\mu = m_l^{pole}}\right)^{\nu} = \left(\frac{1}{r}\right)^{(3N_c - 2N_l)/N_c} \gg 1. \quad (6.2)$$

Hence, after integrating out all the $Q_l, \overline{Q}^{\overline{l}}$ quarks as heavy ones at $\mu = m_l^{pole}$, we are left with the pure $SU(N_c)$ YM theory, but in the strong-coupling regime. This is somewhat unusual, but there is no contradiction because this perturbative strong coupling regime with $a_{YM}(\mu) \gg 1$ is realized in a restricted interval of scales only, $\Lambda_{YM} \ll \mu < m_l^{pole} \ll \Lambda'_Q$. It follows from the NSVZ β-function [8] that the coupling $a_{YM}(\mu \gg \Lambda_{YM}) \gg 1$ is then given by

$$\begin{gathered} a_{YM}(\mu = m_l^{pole}) = \left(\frac{\mu = m_l^{pole}}{\lambda_{YM}}\right)^3 = \\ = a_+(\mu = m_l^{pole}) \to \frac{\lambda_{YM}}{\Lambda_Q} = \frac{\Lambda_{YM}}{\Lambda_Q} = \\ = \left[\left(\frac{m_l}{\Lambda_Q}\right)^{N_l}\left(\frac{m_h}{\Lambda_Q}\right)^{N_h}\right]^{1/3N_c}, \\ \frac{\Lambda_{YM}}{m_l^{pole}} = r^{(3N_c - 2N_l)/3N_c} \ll 1, \\ a_{YM}(\Lambda_{YM} \ll \mu < m_l^{pole}) = \\ = a_{YM}(\mu = m_l^{pole})\left(\frac{\mu}{m_l^{pole}}\right)^3 = \left(\frac{\mu}{\Lambda_{YM}}\right)^3, \end{gathered} \quad (6.3)$$

and it now decreases from $a_{YM}(\mu = m_l^{pole}) \gg 1$ to $a_{YM}(\mu \sim \Lambda_{YM}) \sim 1$ as μ decreases, after which the nonperturbative effects become essential.

Therefore, decreasing the scale μ from $\mu = m_l^{pole}$ to $\mu < \Lambda_{YM}$, integrating out all gauge degrees of freedom except the one whole field $S \sim W_\alpha^2$, and using the

VY form for the superpotential of S [10], we obtain the standard gluino condensate, $\langle S\rangle = \Lambda_{YM}^3$.

To verify the self-consistency, we have to estimate the scale μ_{gl} of the possible higgsing of $Q_l, \overline{Q}^{\,l}$ quarks. This estimate is given by

$$\mu_{gl}^2 \sim a_+(\mu=\mu_{gl})\langle \overline{Q}^{\,l} Q_l\rangle_{\mu=\mu_{gl}} \sim \left(m_l^{pole}\right)^2,$$
$$a_+(\mu=\mu_{gl}) = \left(\frac{\Lambda'_Q = m_h^{pole}}{\mu_{gl}}\right)^{\nu},$$
$$\langle \overline{Q}^{\,l} Q_l\rangle_{\mu=\mu_{gl}} = \langle \overline{Q}^{\,l} Q_l\rangle_{\mu=\Lambda_Q} \times \tag{6.4}$$
$$\times \left(\frac{\Lambda'_Q}{\Lambda_Q}\right)^{\gamma_Q = b_0/N_F} \left(\frac{\mu_{gl}}{\Lambda'_Q}\right)^{\gamma'_Q},$$
$$\langle \overline{Q}^{\,l} Q_l\rangle_{\mu=\Lambda_Q} = \frac{\Lambda_{YM}^3}{m_l}.$$

Therefore, as usually happens in this scenario No. 2 in the strong-coupling region, μ_{gl} and m_l^{pole} are parametrically the same, and this is "a point of tension". As before, we assume that $\mu_{gl} = m_l^{pole}/(\text{several})$, and hence the $Q_l, \overline{Q}^{\,l}$ quarks are not higgsed and the HQ$_l$ phase is self-consistent.

All quarks of the direct theory are in the HQ phase (and are therefore confined, but the string tension is small in comparison with quark masses, $\sqrt{\sigma} \sim \Lambda_{YM} \ll \ll m_l^{pole} \ll m_h^{pole}$). The mass spectrum then includes 1) a large number of heavy hh-flavored mesons with the mass scale $\sim m_h^{pole} \ll \Lambda_Q$; 2) a large number of hybrid hl-mesons and baryons $B_{hl}, \overline{B}_{hl}$ with the same mass scale $\sim m_h^{pole}$; 3) a large number of ll-flavored mesons and baryons $B_l, \overline{B}_l$ with the mass scale $\sim m_l^{pole} \ll m_h^{pole}$; and, finally, 4) gluonia, which are the lightest, with their mass scale $\sim \Lambda_{YM} \ll m_l^{pole}$.

7. DIRECT AND DUAL THEORIES. EQUAL QUARK MASSES, $N_c < N_F < 3N_c/2$, $N_l > N_c$

As regards the direct theory, this case is obtained from the one in the preceding section by a simple change of notation. The quark pole masses are

$$\frac{m_Q^{pole}}{\Lambda_Q} = \frac{m_Q}{\Lambda_Q}\left(\frac{\Lambda_Q}{m_Q = m_Q(\mu=\Lambda_Q)}\right)^{\gamma_Q} = = \left(\frac{m_Q}{\Lambda_Q}\right)^{1/(1+\gamma_Q)}, \tag{7.1}$$

while the gluon masses due to a possible higgsing of quarks are

$$\mu_{gl}^2 \sim a(\mu=\mu_{gl})\langle \overline{Q}Q\rangle_{\mu=\mu_{gl}}, \quad a(\mu=\mu_{gl}) = \left(\frac{\Lambda_Q}{\mu_{gl}}\right)^{\nu},$$
$$\nu = \frac{N_F\gamma_Q - \mathrm{b}_0}{N_c},$$
$$\langle \overline{Q}Q\rangle_{\mu=\mu_{gl}} = \langle \overline{Q}Q\rangle_{\mu=\Lambda_Q}\left(\frac{\mu_{gl}}{\Lambda_Q}\right)^{\gamma_Q}. \tag{7.2}$$

It follows from (7.2) that

$$\frac{\mu_{gl}}{\Lambda_Q} \sim \left(\frac{m_Q}{\Lambda_Q}\right)^{1/(1+\gamma_Q)} \sim \frac{m_Q^{pole}}{\Lambda_Q},$$
$$\frac{1}{1+\gamma_Q} = \frac{N_F - N_c}{N_c} \quad \text{for} \quad \gamma_Q = \frac{2N_c - N_F}{N_F - N_c} \tag{7.3}$$

and, as previously, we assume that $m_Q^{pole} = (\text{several})\mu_{gl}$ and the quarks are not higgsed but confined (see footnote 3). After all quarks are integrated out as heavy ones at $\mu = m_Q^{pole}$, we are left with the $SU(N_c)$ YM theory in the strong-coupling regime and with the scale factor $\Lambda_{YM} \ll m_Q^{pole}$ of its gauge coupling, and so on.

As regards the dual theory, its mass spectrum has been described in [6] (see Sec. 7), and we only recall it here briefly. The Lagrangian at $\mu = \Lambda_Q$ is taken in the form

$$\overline{L} = \left[\mathrm{Tr}\left(q^\dagger e^{\overline{V}} q + \overline{q}^\dagger e^{-\overline{V}}\overline{q}\right) + \frac{\mathrm{Tr}\,(M^\dagger M)}{\Lambda_Q^2}\right]_D +$$
$$+\left[-\frac{2\pi}{\overline{\alpha}(\mu=\Lambda_Q)}\overline{s} + \frac{\mathrm{Tr}\,(\overline{q}Mq)}{\Lambda_Q} + \mathrm{Tr}\,(m_Q M)\right]_F, \tag{7.4}$$
$$\overline{s} = \overline{w}_\alpha^2/32\pi^2, \quad \langle M(\mu=\Lambda_Q)\rangle = \mathcal{M}_{ch}^2,$$
$$\overline{a}(\mu=\Lambda_Q) = \overline{N}_c\overline{\alpha}(\mu=\Lambda_Q)/2\pi = O(1).$$

The dual quarks are in the HQ phase and are therefore confined, and their pole masses are (from now on in this section we neglect the logarithmic renormalization factors z_q and z_M for simplicity)

$$\frac{\mu_q^{pole}}{\Lambda_Q} = \frac{\langle M(\mu=\Lambda_Q)\rangle = \mathcal{M}_{ch}^2 = \langle \overline{Q}Q(\mu=\Lambda_Q)\rangle}{z_q(\Lambda_Q, \mu_q^{pole})\Lambda_Q^2} \sim \sim \left(\frac{m_Q}{\Lambda_Q}\right)^{(N_F-N_c)/N_c}. \tag{7.5}$$

After integrating out the dual quarks as heavy ones, we are left with the dual gauge theory with $\overline{N}_c$ colors and with the scale factor $\langle \Lambda_L(M)\rangle = \Lambda_{YM}$ of its gauge coupling, and with the frozen mion fields M. Finally, integrating the dual gluons by means of the VY procedure [10], we obtain the Lagrangian of mions

$$\overline{L} = \left[\frac{1}{\Lambda_Q^2}\mathrm{Tr}\,(M^\dagger M)\right]_D + \left[-\overline{N}_c\left(\frac{\det M}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + + m_Q \mathrm{Tr}\, M\right]_F, \quad \mu \ll \Lambda_{YM}. \tag{7.6}$$

It describes the mions M with the masses

$$\mu_M \sim m_Q \left(\frac{\Lambda_Q^2}{\mathcal{M}_{ch}^2}\right) \sim m_Q \left(\frac{\Lambda_Q}{m_Q}\right)^{(N_F-N_c)/N_c}, \quad m_Q \ll \mu_M \ll \Lambda_{YM}. \tag{7.7}$$

Clearly, there is no analog of these parametrically light particles in the direct theory.

8. DIRECT AND DUAL THEORIES. UNEQUAL QUARK MASSES, $N_c < N_F < 3N_c/2$, $N_l > N_c$

As regards the direct theory, the mass spectrum in this case with $r = m_l/m_h < 1$ is not much different from the one in the preceding section. All quarks are in the HQ phase, and the highest physical mass is the pole mass of h-quarks (see footnote 9, $\gamma_+ = (2N_c - N_F)/(N_F - N_c)$):

$$m_h^{pole} = m_h \left(\frac{\Lambda_Q}{m_h^{pole}}\right)^{\gamma_+} \to \frac{m_h^{pole}}{\Lambda_Q} = \left(\frac{m_h}{\Lambda_Q}\right)^{(N_F-N_c)/N_c}. \tag{8.1}$$

After integrating the $Q_h, \overline{Q}^h$ quarks as heavy ones at $\mu = m_h^{pole}$, the lower-energy theory remains with N_c colors and $N_c < N_l < 3N_c/2$ l-quarks. The next independent physical scale is the pole mass of the l-quarks ($\gamma_- = (2N_c - N_l)/(N_l - N_c)$):

$$m_l^{pole} = \Bigl(m_l(\mu = m_h^{pole}) = r\, m_h^{pole}\Bigr) \left(\frac{m_h^{pole}}{m_l^{pole}}\right)^{\gamma_-} \to \left(\frac{m_l^{pole}}{m_h^{pole}}\right) = r^{(N_l-N_c)/N_c} \ll 1. \tag{8.2}$$

Integrating the l-quarks as heavy ones at $\mu = m_l^{pole}$ leaves us with the $SU(N_c)$ YM theory in the strong-coupling regime. The scale factor Λ'_{YM} of its gauge coupling is determined from

$$\left(\frac{m_l^{pole}}{\Lambda'_{YM}}\right)^3 = \left(\frac{\Lambda_Q}{m_h^{pole}}\right)^{\nu_+} \left(\frac{m_h^{pole}}{m_l^{pole}}\right)^{\nu_-} \to \Lambda'_{YM} = \Lambda_{YM} = \left(\Lambda_Q^{b_0} m_l^{N_l} m_h^{N_h}\right)^{1/3N_c}, \tag{8.3}$$

where

$$\nu_+ = \frac{3N_c - 2N_F}{N_F - N_c}, \quad \nu_- = \frac{3N_c - 2N_l}{N_l - N_c} > \nu_+.$$

To verify the self-consistency, we also estimate the gluon masses due to the possible higgsing of the Q_h and/or Q_l quarks. For the Q_h quarks,

$$\frac{\mu_{gl,h}^2}{\Lambda_Q^2} \sim \left[a_+(\mu = \mu_{gl,h}) = \left(\frac{\Lambda_Q}{\mu_{gl,h}}\right)^{\nu_+}\right] \frac{\langle \overline{Q}^h Q_h\rangle}{\Lambda_Q^2} \times \left(\frac{\mu_{gl,h}}{\Lambda_Q}\right)^{\gamma_+} \to \frac{\mu_{gl,h}}{m_h^{pole}} \sim r^{N_l/N_c} \ll 1, \tag{8.4}$$

and hence there is no problem, but for the Q_l quarks, we now have

$$\frac{\mu_{gl,l}^2}{\Lambda_Q^2} \sim \left[a_-(\mu = \mu_{gl,l}) = \left(\frac{\Lambda_Q}{m_h^{pole}}\right)^{\nu_+} \left(\frac{m_h^{pole}}{\mu_{gl,l}}\right)^{\nu_-}\right] \times \frac{\langle \overline{Q}^l Q_l\rangle}{\Lambda_Q^2} \left(\frac{m_h^{pole}}{\Lambda_Q}\right)^{\gamma_+} \left(\frac{\mu_{gl,l}}{m_h^{pole}}\right)^{\gamma_-} \to \tag{8.5}$$

$$\to \frac{\mu_{gl,l}}{\Lambda_Q} \sim \frac{\langle \overline{Q}^l Q_l\rangle}{\Lambda_Q^2} \sim \frac{m_l^{pole}}{\Lambda_Q} = r^{(N_l-N_c)/N_c} \frac{m_h^{pole}}{\Lambda_Q}, \tag{8.6}$$

and this is also "a point of tension". As before, we assume that it is in favor of m_l^{pole}, i.e., $m_l^{pole} = (\text{several})\, \mu_{gl,l}$.

In the direct theory, all quarks are in the HQ phase and the mass spectrum consists of a) a large number of hh- and hybrid hl-mesons and baryons with the mass scale $\sim m_h^{pole} \sim \Lambda_Q (m_h/\Lambda_Q)^{(N_F-N_c)/N_c} \ll \Lambda_Q$, b) a large number of ll-mesons and baryons with the mass scale $\sim m_l^{pole} \sim m_h^{pole} r^{(N_l-N_c)/N_c} \ll m_h^{pole}$; all quarks are weakly confined, i.e., the string tension $\sqrt{\sigma} \sim \Lambda_{YM}$ is much smaller than their masses; and c) gluonia, which are the lightest, with the masses $\sim \Lambda_{YM} \ll m_l^{pole}$.

In the dual theory with Lagrangian (7.4), there are several regimes depending on the value of $r = m_l/m_h < 1$.

i) at $r_1 = (m_h/\Lambda_Q)^{(3N_c-2N_F)/2N_l} < r < 1$, the hierarchy of masses is given by $\mu_{q,l} > \mu_{q,h} > \mu_{gl}^h$, where $\mu_{q,l}$ and $\mu_{q,h}$ are the masses of dual quarks and μ_{gl}^h is the gluon mass due to the possible higgsing of q_h, $\overline{q}_h$ quarks (all logarithmic renormalization effects are neglected here and below in this section for simplicity):

$$\frac{\mu_{q,l}^{pole}}{\Lambda_Q} \sim \frac{\mu_{q,l} = \mu_{q,l}(\mu = \Lambda_Q)}{\Lambda_Q} = \frac{\langle \overline{Q}_l Q_l(\mu = \Lambda_Q)\rangle}{\Lambda_Q^2} = r^{(N_l-N_c)/N_c} \left(\frac{m_h}{\Lambda_Q}\right)^{(N_F-N_c)/N_c},$$

$$\frac{\mu^{pole}_{q,h}}{\Lambda_Q} \sim \frac{\mu_{q,h} = \mu_{q,h}(\mu = \Lambda_Q)}{\Lambda_Q} = \\ = \frac{\langle \overline{Q}_h Q_h(\mu = \Lambda_Q)\rangle}{\Lambda_Q^2} = \\ = r^{N_l/N_c} \left(\frac{m_h}{\Lambda_Q}\right)^{(N_F-N_c)/N_c}, \quad (8.7)$$

$$\mu^h_{gl} \sim \sqrt{|\langle \overline{q}_h q_h\rangle|} = \sqrt{m_h \Lambda_Q}.$$

This hierarchy shows that all dual quarks are in the HQ phase and not higgsed.

ii) at $r_2 = (m_h/\Lambda_Q)^{(3N_c-2N_F)/2(N_l-N_c)} < r < r_1 = (m_h/\Lambda_Q)^{(3N_c-2N_F)/2N_l}$, the hierarchy of masses is given by $\mu_{q,l} > \mu^h_{gl} > \mu_{q,h}$. This shows that $\overline{q}_l$, q_l quarks are in the HQ phase and are the heaviest ones, while $\overline{q}_h$, q_h quarks are higgsed. The phase transition of $\overline{q}_h$, q_h quarks from the HQ_h phase to the Higgs_h phase occurs in the region $r \sim r_1$.

iii) at $r < r_2 = (m_h/\Lambda_Q)^{(3N_c-2N_F)/2(N_l-N_c)}$, the hierarchy of masses is given by $\mu^h_{gl} > \mu_{q,l} > \mu_{q,h}$. Here, $\overline{q}_h$, q_h quarks are higgsed and μ^h_{gl} is the highest mass, while the $\overline{q}_l$, q_l quarks are lighter and are in the HQ phase.

We now give some details.

i) After the heaviest dual l-quarks are integrated out at $\mu < \mu_{q,l}$, a theory remains with $\overline{N}_c$ colors and $N_h < \overline{N}_c$ quarks $\overline{q}_h$, q_h (and mions M), and with the scale factor of its gauge coupling

$$(\Lambda'_q)^{\overline{b}'_0} = \Lambda_Q^{\overline{b}_0} \mu_{q,l}^{N_l}, \quad \overline{b}'_0 = (3\overline{N}_c - N_h) > 0, \\ \left(\frac{\Lambda'_q}{\mu_{q,h}}\right)^{\overline{b}'_0/\overline{N}_c} = \left(\frac{r_1}{r}\right)^{2N_l/N_c} < 1. \quad (8.8)$$

Hence, after integrating $\overline{q}_h$, q_h quarks as heavy ones with masses $\mu_{q,h} > \Lambda'_q$ and gluons through the VY procedure, we obtain the Lagrangian of mions

$$\overline{L} = \left[\frac{1}{\Lambda_Q^2}\,\mathrm{Tr}\,\left(M^\dagger M\right)\right]_D + \\ + \left[-\overline{N}_c\left(\frac{\det M}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + \mathrm{Tr}\,(m_Q M)\right]_F. \quad (8.9)$$

It describes mions M with the masses

$$\mu(M^i_j) \sim \frac{m_i m_j \Lambda_Q^2}{\Lambda_{YM}^3} < \Lambda_{YM}, \quad i,j = l,h. \quad (8.10)$$

ii) The first step of integrating out the heaviest l-quarks with masses $\mu_{q,l}$ is the same. But now, at $r_2 < r < r_1$, the next physical mass is $\overline{\mu}_{ql,h} > \Lambda'_q$ due to higgsing of dual h-quarks, with $\overline{N}_c \to \overline{N}_c - N_h$ and formation of N_h^2 nions N_{hh}. After integrating out the heavy higgsed gluons and their superpartners with masses $\overline{\mu}_{ql,h} \sim \sqrt{m_h \Lambda_Q}$, and unhiggsed gluons through the VY procedure, the Lagrangian of the remaining degrees of freedom takes the form

$$L = \left[\frac{\mathrm{Tr}\,\left(M^\dagger M\right)}{\Lambda_Q^2} + 2\,\mathrm{Tr}\sqrt{N^\dagger_{hh} N_{hh}}\right]_D + \\ + \left[-(N_l - N_c)\left(\frac{\det M_{ll}}{\Lambda_Q^{b_0}\det(-N_{hh}/\Lambda_Q)}\right)^{1/(N_l-N_c)} + \right. \\ + \frac{1}{\Lambda_Q}\mathrm{Tr}\,N_{hh}\left(M_{hh} - M_{hl}M_{ll}^{-1}M_{lh}\right) + \\ \left. + \mathrm{Tr}\,(m_l M_{ll} + m_h M_{hh})\right]_F. \quad (8.11)$$

From this, the masses of mions M_{hh}, M_{ll}, hybrids M_{lh}, M_{hl}, and nions N_{hh} are

$$\mu(M_{hh}) \sim \mu(N_{hh}) \sim \sqrt{m_h\Lambda_Q} \sim \overline{\mu}_{ql,h}, \\ \mu(M_{lh}) \sim \mu(M_{hl}) \sim \frac{m_l m_h \Lambda_Q^2}{\Lambda_{YM}^3}, \\ \mu(M_{ll}) \sim \frac{m_l^2\Lambda_Q^2}{\Lambda_{YM}^3}. \quad (8.12)$$

iii) The heaviest particles in this region $r < r_2$ are higgsed gluons and their superpartners, with masses $\overline{\mu}_{ql,h} \sim \sqrt{m_h \Lambda_Q}$. After these particles are integrated out, $\overline{N}'_c = N_l - N_c$ dual colors and N_l flavors with active unhiggsed colors remain and the regime at $\mu < \overline{\mu}_{ql,h}$ is IR free logarithmic, $\overline{b}'_0 = 3\overline{N}'_c - N_l = 2N_l - 3N_c < 0$. In this theory, the next independent physical scale is given by the mass $\mu_{q,l}$ of the N_l active l-quarks with unhiggsed colors.

The masses of $2N_l N_h$ hybrid mions $M_{hl} + M_{lh}$ and $2N_l N_h$ nions $N_{lh} + N_{hl}$ (these are the dual l-quarks that have higgsed colors) are determined mainly by their common mass term in the superpotential:

$$L_{hybr} \approx \left[\mathrm{Tr}\,\left(\frac{M^\dagger_{hl} M_{hl}}{\Lambda_Q^2}\right) + \mathrm{Tr}\,\left(N^\dagger_{lh} N_{lh}\right)\right]_D + \\ + \left[(m_h \Lambda_Q)^{1/2}\,\mathrm{Tr}\,\left(\frac{M_{hl} N_{lh}}{\Lambda_Q}\right)\right]_F, \quad (8.13)$$

and hence their masses are

$$\mu(M_{hl}) \sim \mu(M_{lh}) \sim \mu(N_{hl}) \sim \mu(N_{lh}) \sim \overline{\mu}_{ql,h}. \quad (8.14)$$

Passing to lower scales and integrating out first the active l-quarks as heavy ones with masses $\mu_{q,l}$ and then the unhiggsed gluons through the VY procedure, we obtain

$$L=\left[\frac{\mathrm{Tr}\left(M^{\dagger}_{ll}M_{ll}+M^{\dagger}_{hh}M_{hh}\right)}{\Lambda_Q^2}+2\mathrm{Tr}\sqrt{N^{\dagger}_{hh}N_{hh}}\right]_D+ \\ +\left[-(N_l-N_c)\left(\frac{\det M_{ll}}{\Lambda_Q^{b_0}\det\left(-N_{hh}/\Lambda_Q\right)}\right)^{1/(N_l-N_c)}+\right. \\ \left.+\frac{\mathrm{Tr}\left(N_{hh}M_{hh}\right)}{\Lambda_Q}+\mathrm{Tr}\left(m_lM_{ll}+m_hM_{hh}\right)\right]_F. \quad (8.15)$$

Finally, the masses are given by

$$\mu(N_{hh})\sim\mu(M_{hh})\sim\overline{\mu}_{ql,h}\sim\sqrt{m_h\Lambda_Q}, \\ \mu(M_{ll})\sim m_l^2\Lambda_Q^2/\Lambda_{YM}^3. \quad (8.16)$$

9. DIRECT THEORY. UNEQUAL QUARK MASSES, $N_c<N_F<3N_c/2$, $N_l<N_c-1$

In this regime, at $r=m_l/m_h\ll 1$, the highest physical scale μ_H is determined by the gluon masses $\mu_{gl,l}$ that arise due to higgsing of the $Q_l,\overline{Q}^l$ quarks[10]):

$$\frac{\mu^2_{gl,l}}{\Lambda_Q^2}\sim\left[a_+(\mu=\mu_{gl,l})=\left(\frac{\Lambda_Q}{\mu_{gl,l}}\right)^{\nu_+}\right]\times \\ \times\frac{\langle\overline{Q}^lQ_l\rangle}{\Lambda_Q^2}\left(\frac{\mu_{gl,l}}{\Lambda_Q}\right)^{\gamma_+}\to\frac{\mu_{gl,l}}{\Lambda_Q}\sim\frac{\langle\overline{Q}^lQ_l\rangle}{\Lambda_Q^2}= \\ =\frac{(\mathcal{M}^l_{ch})^2}{\Lambda_Q^2}=\frac{\Lambda^3_{YM}}{m_l\Lambda_Q^2}\sim\left(\frac{1}{r}\right)^{(N_c-N_l)/N_c}\times \\ \times\left(\frac{m_h}{\Lambda_Q}\right)^{(N_F-N_c)/N_c}\ll 1. \quad (9.1)$$

The lower-energy theory includes $\hat{N}_c=N_c-N_l$ unbroken colors, $2N_lN_h$ scalar hybrids $\Pi_{hl}+\Pi_{lh}$ (these are the Q_h and $\overline{Q}^h$ quarks with higgsed colors), N_h flavors of active $Q_h,\overline{Q}^h$ quarks with unbroken colors, and, finally, N_l^2 pions Π_{ll}, $\langle\Pi_{ll}\rangle=\langle\overline{Q}^lQ_l\rangle=(\mathcal{M}^l_{ch})^2$. Their Lagrangian at $\mu<\mu_{gl,l}$ takes the form (with all fields normalized at $\mu=\Lambda_Q$)

$$L=\left[z_Q\mathrm{Tr}\sqrt{\Pi^{\dagger}_{ll}\Pi_{ll}}+z_Q\mathrm{Tr}_h\left(Q^{\dagger}e^{\hat{V}}Q+\overline{Q}^{\dagger}e^{-\hat{V}}\overline{Q}\right)+\right. \\ \left.+z_Q\mathrm{Tr}\left(\Pi^{\dagger}_{hl}\Pi_{hl}+\Pi^{\dagger}_{lh}\Pi_{lh}\right)+\ldots\right]_D+ \\ +\left[-\frac{2\pi}{\hat{\alpha}(\mu)}\hat{S}+m_l\,\mathrm{Tr}\,\Pi_{ll}+\right. \\ \left.+m_h\mathrm{Tr}_h\,\overline{Q}Q+m_h\mathrm{Tr}\,\Pi_{hl}\Pi_{lh}\right]_F, \quad (9.2)$$

where $\hat{V}$ are the $SU(\hat{N}_c)$ gluons and $\hat{\alpha}(\mu)$ is their gauge coupling (with the pions Π_{ll} sitting down inside). The dots in D-terms denote residual interactions. It is assumed that they play no essential role for what follows. The quark renormalization factor due to the perturbative evolution in the interval of scales $\mu_{gl,l}<\mu<\Lambda_Q$ is

$$z_Q=z_Q(\Lambda_Q,\mu_{gl,l})=\left(\frac{\mu_{gl,l}}{\Lambda_Q}\right)^{\gamma_+}\ll 1, \\ \gamma_+=(2N_c-N_F)/(N_F-N_c).$$

All pion fields, Π_{hl}, Π_{lh} and Π_{ll}, are frozen and do not evolve any more at $\mu<\mu_{gl,l}$. The numbers of colors and flavors have already changed in the threshold region $\mu_{gl,l}/(\text{several})<\mu<(\text{several})\mu_{gl,l}$, $N_F\to\hat{N}_F=N_F-N_l=N_h$, $N_c\to\hat{N}_c=N_c-N_l$, while the coupling $\hat{\alpha}(\mu)$ does not change essentially and remains $\sim\alpha(\mu=\mu_{gl,l})\gg 1$. Therefore, the new quark anomalous dimension $\gamma_-(\hat{N}_c,\hat{N}_F=N_h,\hat{a}\gg 1)$ and the new $\hat{\beta}$-function have the form

$$\frac{d\hat{a}(\mu)}{d\ln\mu}=-\nu_-\hat{a}(\mu), \\ \nu_-=\frac{\hat{N}_F\gamma_--\hat{b}_0}{\hat{N}_c}=\frac{3\hat{N}_c-2\hat{N}_F}{\hat{N}_F-\hat{N}_c}= \\ =\nu_+-\frac{N_l}{N_F-N_c}, \\ \hat{b}_0=3\hat{N}_c-\hat{N}_F=b_0-2N_l, \\ \gamma_-=\frac{2\hat{N}_c-\hat{N}_F}{\hat{N}_F-\hat{N}_c}=\gamma_+-\frac{N_l}{N_F-N_c}. \quad (9.3)$$

Depending on the value of $\hat{N}_F/\hat{N}_c$, the lower-energy theory is in different regimes. We consider only two cases below.

i) $1<\hat{N}_F/\hat{N}_c<3/2$. In this case, $\nu_->0$, and hence the coupling $\hat{a}(\mu)$ continues to increase with decreasing μ, but more slowly than before.

[10]) Here and below, the value of r is taken to be not too small, such that $\mu_{gl,l}\ll\Lambda_Q$. At r so small that $\mu_{gl,l}\gg\Lambda_Q$, the quarks $Q_l,\overline{Q}^l$ are higgsed in the logarithmic weak-coupling region (as in scenario No. 1 in [7]), and the form of the RG flow is different, but the regime is qualitatively the same for $\mu_{gl,l}\ll\Lambda_Q$ or $\mu_{gl,l}\gg\Lambda_Q$, and nothing happens as $\mu_{gl,l}$ overshoots Λ_Q in scenario No. 2 considered here. Everywhere below, we neglect the additional dependence of Kähler terms on the quantum pion fields $\pi^{\overline{l}}_l/\mathcal{M}^l_{ch}$ (originating from the dependence on $\pi/\mathcal{M}^l_{ch}$ of the quark renormalization factor $z_Q(\Pi_{ll}{}^{\dagger},\Pi_{ll})$), because this would affect the particle mass values with nonparametric factors $O(1)$ only.

The next physical scale is given by the pole mass of the active $Q_h, \overline{Q}^h$ quarks:

$$\frac{m_h^{pole}}{\Lambda_Q} = \left[\frac{m_h(\mu=\mu_{gl,l})}{\Lambda_Q} = \frac{m_h}{\Lambda_Q}\left(\frac{\Lambda_Q}{\mu_{gl,l}}\right)^{\gamma_+}\right] \times \times \left(\frac{\mu_{gl,l}}{m_h^{pole}}\right)^{\gamma_-} \to \frac{m_h^{pole}}{\mu_{gl,l}} = r \ll 1, \quad (9.4)$$

$$\frac{m_h^{pole}}{\Lambda_Q} = r^{N_l/N_c}\left(\frac{m_h}{\Lambda_Q}\right)^{(N_F-N_c)/N_c} \gg \frac{\Lambda_{YM}}{\Lambda_Q}.$$

After integrating out these $Q_h, \overline{Q}^h$ quarks as heavy ones at $\mu = m_h^{pole}$, we are left with the $SU(\hat{N}_c)$ YM theory in the strong-coupling regime $a_{YM}(\mu = m_h^{pole}) \gg 1$ (and pions). The scale of its gauge coupling $\hat{\Lambda}_{YM} = \langle \Lambda_L(\Pi_{ll})\rangle$ is determined from

$$a_{YM}(\mu=m_h^{pole}) = \left(\frac{m_h^{pole}}{\hat{\Lambda}_{YM}}\right)^3 = \hat{a}(\mu=m_h^{pole}) = = \left(\frac{\Lambda_Q}{\mu_{gl,l}}\right)^{\nu_+}\left(\frac{\mu_{gl,l}}{m_h^{pole}}\right)^{\nu_-} \to \hat{\Lambda}_{YM} = \Lambda_{YM}. \quad (9.5)$$

Finally, tracing the fate of pions Π_{ll} and applying the VY procedure at $\mu < \Lambda_{YM}$, we obtain the Lagrangian

$$L = \left[z_Q \operatorname{Tr}\sqrt{\Pi_{ll}^\dagger\Pi_{ll}} + z_Q \operatorname{Tr}\left(\Pi_{hl}^\dagger\Pi_{hl} + \Pi_{lh}^\dagger\Pi_{lh}\right)\right]_D + \left[(N_c-N_l)\left(\frac{\Lambda_Q^{b_0}\det m_h}{\det \Pi_{ll}}\right)^{1/(N_c-N_l)} + + m_l \operatorname{Tr}\Pi_{ll} + m_h \operatorname{Tr}\Pi_{hl}\Pi_{lh}\right]_F. \quad (9.6)$$

From this, the masses of Π_{hl}, Π_{lh}, and Π_{ll} are

$$\mu(\Pi_{hl}) = \mu(\Pi_{lh}) \sim r^{\gamma_-}\, m_h^{pole}, \quad \mu(\Pi_{ll}) \sim m_l/z_Q \sim r\mu(\Pi_{hl}). \quad (9.7)$$

To check the self-consistency, i.e., that the active $Q_h, \overline{Q}^h$ quarks are indeed in the HQ phase and are not higgsed, we estimate the possible value of the gluon mass $\mu_{gl,h}$:

$$\mu_{gl,h}^2 \sim \left[\hat{a}(\mu=\mu_{gl,h}) = \left(\frac{\Lambda_Q}{\mu_{gl,l}}\right)^{\nu_+}\left(\frac{\mu_{gl,l}}{\mu_{gl,h}}\right)^{\nu_-}\right] \times \times \langle \overline{Q}^h Q_h\rangle_{\mu=\mu_{gl,h}}, \quad (9.8)$$

$$\frac{\langle \overline{Q}^h Q_h\rangle_{\mu=\mu_{gl,h}}}{\Lambda_Q^2} = r\frac{\mu_{gl,l}}{\Lambda_Q}\left(\frac{\mu_{gl,l}}{\Lambda_Q}\right)^{\gamma_+}\left(\frac{\mu_{gl,h}}{\mu_{gl,l}}\right)^{\gamma_-} \to \to \mu_{gl,h} \sim m_h^{pole}, \quad (9.9)$$

as could be expected. This is "the standard point of tension" in scenario No. 2.

ii) $3/2 < \hat{N}_F/\hat{N}_c < 3$. In this case, $\nu_- < 0$, and hence the RG flow is reversed and the coupling $\hat{a}(\mu)$ starts to decrease with decreasing μ at $\mu < \mu_{gl,l}$, approaching its fixed-point value $\hat{a}_* < 1$ from above (unless it stops before at $\mu = m_h^{pole}$). Until $\hat{a}(\mu) \gg 1$, it behaves as

$$\hat{a}(\mu < \mu_{gl,l}) = a_+(\mu=\mu_{gl,l})\left(\frac{\mu}{\mu_{gl,l}}\right)^{-\nu_-} = = \left(\frac{\Lambda_Q}{\mu_{gl,l}}\right)^{\nu_+}\left(\frac{\mu_{gl,l}}{\mu}\right)^{\nu_-}, \quad \nu_- < 0. \quad (9.10)$$

It therefore decreases to ~ 1 at $\mu \sim \Lambda_0$,

$$\hat{a}(\Lambda_0) \sim 1 \to \frac{\Lambda_0}{\Lambda_Q} \sim \left(\frac{\mu_{gl,l}}{\Lambda_Q}\right)^{\tilde{\Delta}}, \quad \tilde{\Delta} = \frac{N_l}{2\hat{N}_F - 3\hat{N}_c} > 1. \quad (9.11)$$

We first consider the case $m_h^{pole} \gg \Lambda_0$. The value of m_h^{pole} is then given by (9.4), and this requires

$$m_h^{pole} = r\,\mu_{gl,l} \gg \Lambda_0 \to \to \left(\frac{\mu_{gl,l}}{\Lambda_Q}\right)^{\tilde{\Delta}-1>0} \ll r \ll 1. \quad (9.12)$$

Then the running of $\hat{a}(\mu)$ stops at $\hat{a}(m_h^{pole}) \gg 1$, and hence this coupling does not enter the conformal regime. The situation is here similar to those described above in item **i**. The active quarks $Q_h, \overline{Q}^h$ decouple at $\mu < m_h^{pole}$, and the $SU(\hat{N}_c)$ YM theory (and pions) remains in the strong coupling regime, $a_{YM}(\mu) = (\mu/\Lambda_{YM})^3 \gg 1$ at $\Lambda_{YM} \ll \mu < m_h^{pole}$, etc. It can be dealt with as before in item **i**.

The new regime is realized for the parameter values such that $m_h^{pole} \ll \Lambda_0$, but still $\mu_{gl,l} \ll \Lambda_Q$. In this case, as the scale μ decreases below $\mu_{gl,l} \ll \Lambda_Q$, the large but decreasing coupling $\hat{a}(\mu \gg \Lambda_0) \gg 1$ crosses unity at $\mu = \Lambda_0$ and becomes $\hat{a}(\mu < \Lambda_0) < 1$, and the theory enters the conformal regime, but with $\hat{a}(\mu)$ approaching its fixed-point value $\hat{a}_* < 1$ from above. The self-consistency of this regime then requires very specific behavior of the quark anomalous dimension $\hat{\gamma}(\mu) = \gamma_-(\hat{N}_F, \hat{N}_c, \hat{a}(\mu))$ in the region $\mu \sim \Lambda_0$, when the decreasing $\hat{a}(\mu)$ undershoots unity. Qualitatively, the behavior has to be as follows: a) $\hat{\gamma}(\mu)$ stays nearly intact at its value $(2\hat{N}_c - \hat{N}_F)/(\hat{N}_F - \hat{N}_c) < \hat{b}_0/\hat{N}_F$, as far as the coupling remains large, $\hat{a}(\mu \gg \Lambda_0) \gg 1$; b) $\hat{\gamma}(\mu)$ changes rapidly in the threshold region $\Lambda_0/(\text{several}) < \mu < (\text{several})\Lambda_0$. It begins to increase at $\mu = (\text{several})\Lambda_0$ and crosses the value $\hat{b}_0/\hat{N}_F$ just at the point $\mu = \Lambda_0$, where $\hat{a}(\mu)$ crosses unity, such that the β-function remains smooth and nonzero and does not change sign; c) $\hat{\gamma}(\mu)$ continues to increase at $\mu < \Lambda_0$ and reaches its maximal positive value at $\mu = \Lambda_0/(\text{several})$, and then begins to decrease with further decreasing μ, approaching its limit value (equal to $\hat{b}_0/\hat{N}_F$ at $\hat{b}_0 > 0$ or zero at $\hat{b}_0 < 0$) from above at $\mu \ll \Lambda_0$. It would be useful to confirm this very specific behavior of $\hat{\gamma}(\mu)$ independently. But once this is accepted, we can trace the lower-energy behavior proceeding similarly to what we did for the conformal regime (but additionally taking the presence of pions which are remnants of the l-quarks higgsed previously at the higher scale $\mu_{gl,l}$ into account).

10. CONCLUSIONS

The mass spectra of the $\mathcal{N} = 1$ SQCD with $SU(N_c)$ colors and N_F flavors of light quarks $Q, \overline{Q}$ (with masses $0 < m_i \ll \Lambda_Q$) have been described above, within dynamical scenario No. 2. This scenario implies that quarks can be in two different phases only: the HQ phase where they are confined, or the Higgs phase. Besides, we have compared this (direct) theory with its Seiberg dual variant [1, 4], which contains $SU(N_F - N_c)$ dual colors, N_F dual quarks $q, \overline{q}$, and N_F^2 additional mesons M (mions).

As was shown above, in those regions of the parameter space where an additional small parameter is available (this is $0 < b_0/N_F = (3N_c - N_F)/N_F \ll 1$ at the right end of the conformal window or its dual analog $0 < \overline{b}_0/N_F = (2N_F - 3N_c)/N_F \ll 1$ at the left end), there are parametrical differences in the mass spectra of the direct and dual theories, and therefore they are clearly not equivalent. In fact, this implies that even when both $b_0/N_F \sim \overline{b}_0/N_F \sim 1$, there are no reasons for these two theories to become exactly the same[11)].

Besides, as was shown in Sec. 5, unavoidable internal inconsistencies of the Seiberg duality can be traced in variant "b", i. e., "confinement without flavor chiral symmetry breaking" (this implies that at $N_c < N_F < 3N_c/2$, the direct quarks and gluons form a large number of massive hadrons with masses $\sim \Lambda_Q$, while new light composite particles with masses $\mu_i \ll \Lambda_Q$ appear, described by the dual theory)[12)]. This agrees with some general arguments presented previously in Sec. 7 in [6] that the duality in the variant "b" cannot be realized (it is also worth recalling that those arguments were not related with the use of scenario No. 1 with the diquark condensate).

As regards the mass spectra of the direct theory in this scenario No. 2, their main features are as follows (for $b_0/N_F = O(1)$; see Sec. 2 for $b_0/N_F \ll 1$).

1) In all the cases considered, there is a large number of gluonia with masses $\sim \Lambda_{YM} = (\Lambda_Q^{b_0} \det m_Q)^{1/3N_c}$.

2) When all quark masses are equal, they are (most probably) in the HQ phase (i. e., not higgsed but confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM}$), for the whole interval $N_c < N_F < 3N_c$, and hence form a large number of various hadrons with the mass scales a) $\sim m_Q^{pole} \sim \Lambda_{YM}$ at $3N_c/2 < N_F < 3N_c$ and b) $\sim m_Q^{pole} \sim \Lambda_Q (m_Q/\Lambda_Q)^{(N_F - N_c)/N_c} \gg \Lambda_{YM}$ at $N_c < N_F < 3N_c/2$. There are no additional lighter pions $\pi_i^{\overline{j}}$ with masses $\mu_\pi \ll m_Q^{pole}$, for all $N_c < N_F < 3N_c$.

3) The case with N_l flavors of smaller masses m_l and $N_h = N_F - N_l$ flavors with larger masses m_h, $0 < m_l < m_h \ll \Lambda_Q$, was also considered. When $N_l > N_c$, all quarks are also in the HQ phase for all $N_c < N_F < 3N_c$, and form a large number of hadrons whose masses depend on their flavor content, but there are no additional lighter pions.

4) Only when $N_l < N_c$, the l-flavored quarks $Q_l, \overline{Q}^{\overline{l}}$ are higgsed, $SU(N_c) \to SU(N_c - N_l)$, and there N_l^2 lighter pions $\pi_l^{\overline{l}}$ appear, while the heavier h-flavored quarks $Q_h, \overline{Q}^{\overline{h}}$ always remain in the HQ phase. In this case, the mass spectra and some new regimes with unusual properties of the RG flow were presented in Secs. 7–9.

We have considered not all possible regimes in this paper, but only those that reveal some qualitatively new features. We hope that if needed, the reader can deal with other regimes using the methods in [6, 7] and in this paper.

On the whole, the mass spectra have been obtained in both dynamical scenarios for $\mathcal{N} = 1$ SQCD (No. 1 considered in [6, 7] and No. 2 considered in this paper). Both scenarios are apparently possible, i. e., no unavoidable internal inconsistences are seen. Therefore,

11) But to see the possible differences more clearly, it is insufficient in this case to make rough estimates of particle masses as has been done in this paper. One has either to resolve the mass spectra in more detail or to calculate some Green's functions in both theories and to compare them. At present, it is unclear how to do this.

12) Similar problems with this variant "b" can also be traced in scenario No. 1 (but in this scenario, the differences between the direct and dual theories are much more pronounced; see also footnote 6).

with our present very limited abilities to trace the dynamics in more detail, it remains unclear which one of the scenarios (if any) is correct. Time will show. But in any case, the direct and dual theories are not equivalent in both scenarios.

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APPENDIX

The main purpose of this appendix is to show that the choice $\Lambda_q \sim \Lambda_Q$ used in the text is, in essence, most favorable for the dual theory. But we first present a few useful formulas.

The RG flow of the coupling in the region $\mu_H \le \mu \le \Lambda_Q$, where $\mu_H \ll \Lambda_Q$ is the highest physical mass, is given by

$$\begin{gathered}\frac{da}{d\ln\mu} = \beta(a) = -\frac{N_F}{N_c}\frac{a^2}{1-a}\left(\Delta_0 - \gamma_Q(a)\right),\\ \ln\frac{\Lambda_Q}{\mu} = \frac{N_c}{N_F}\int\limits_{a_\Lambda}^{a_\mu}\frac{da(1-a)}{a^2(\Delta_0-\gamma_Q)},\\ 0<\Delta_0 = \frac{b_0}{N_F} \ll 1,\\ a_\mu = a(\mu) = \frac{N_c\alpha(\mu)}{2\pi}, \quad a_* = \Delta_0 + O(\Delta^2),\\ a_\Lambda \equiv a(\mu=\Lambda_Q) = a_*(1-\delta),\\ 0<\delta\ll 1.\end{gathered} \tag{A.1}$$

Hence, using $\gamma_Q(a) \approx a$, we obtain from (A.1) that for sufficiently small $\mu/\Lambda_Q \ll 1$,

$$\begin{gathered}a(\mu) = a_*\left(1-\delta\epsilon_\mu\right),\\ \epsilon_\mu \approx \left(\frac{\mu}{\Lambda_Q}\right)^{3a_*^2} \approx \left(\frac{\mu}{\Lambda_Q}\right)^{b_0^2/3N_c^2},\\ \mu_H < \mu \ll \Lambda_Q.\end{gathered} \tag{A.2}$$

Expression (A.2) can even be used as a reasonable interpolation in the whole interval $\mu_H < \mu < \Lambda_Q$. It follows from (A.2) that $a(\mu)$ approaches its fixed-point value a_* very slowly.

The RG flow of the quark Kähler term renormalization factor from $z_Q(\mu=\Lambda_Q) = 1$ down to $z_Q(\Lambda_Q, \mu \ll \Lambda_Q) \ll 1$ is given by

$$\begin{gathered}\gamma_Q(a) = \frac{dz_Q}{d\ln\mu},\\ \gamma_Q(a) = \Delta_0 - (a_* - a) + O\left((a_*-a)^2\right),\end{gathered} \tag{A.3}$$

$$\ln\frac{1}{z_Q(\Lambda_Q,\mu)} = \frac{N_c}{N_F}\int\limits_{a_\Lambda}^{a_\mu}\frac{da(1-a)}{a^2}\,\frac{[\gamma_Q(a)-\Delta_0]+\Delta_0}{\Delta_0-\gamma_Q}, \tag{A.4}$$

$$\begin{gathered}z_Q(\Lambda_Q, \mu\ll\Lambda_Q) = \left(\frac{\mu}{\Lambda_Q}\right)^{\Delta_0}\rho \ll 1,\\ \rho = \left(\frac{a_\Lambda}{a_\mu}\right)^{N_c/N_F}\exp\left[\frac{N_c}{N_F}\left(\frac{a_\mu-a_\Lambda}{a_\mu a_\Lambda}\right)\right] \approx \\ \approx \exp\left(\frac{\delta}{3\Delta_0}\right).\end{gathered} \tag{A.5}$$

Clearly, the terms of the order of $\rho \sim \exp[(\delta \ll 1)/\Delta_0]$ are nonleading in comparison with the terms $\sim \exp[(c_0 \sim 1)/\Delta_0]$, $\Delta_0 \ll 1$, which are traced explicitly in the text (and we neglect such corrections in the main text).

We now consider the region $0 < \Delta_0 = b_0/N_F \ll 1$. Qualitatively, the value of the scale factor Λ_q shows the characteristic scale where the logarithmic behavior of the dual gauge coupling $\overline{a}(\mu) = \overline{N}_c\overline{\alpha}(\mu)/2\pi$ of the UV free dual theory at $\mu \gg \Lambda_q$ changes for the conformal freezing. At scales $\mu \sim \Lambda_q$, the coupling is $\overline{a}(\mu \sim \Lambda_q) = O(1)$, the dual β-function is also $O(1)$, and hence the dual theory enters quickly into the conformal regime (unlike the weakly coupled direct theory). We first consider the case $\Lambda_q \gg \Lambda_Q$ (the case $\Lambda_q \sim \Lambda_Q$ is considered in the main text). Then we can start to deal with the dual theory at the lower reference scale $\sim \Lambda_Q$, where it is already deep in the conformal regime, to match $\langle M(\mu=\Lambda_Q)\rangle = \langle \overline{Q}Q(\mu=\Lambda_Q)\rangle$, $\overline{m}_Q(\mu=\Lambda_Q) = m_Q(\mu=\Lambda_Q)$ etc., and to proceed further exactly as in the text.

We now consider the case $\Lambda_q \ll \Lambda_Q$ (for example, $\Lambda_q/\Lambda_Q \sim \exp(-1/\Delta_0^2)$; we recall that $\Delta_0 \ll 1$, although small, does not compete in any way with the main small parameter m_Q/Λ_Q; see footnote 1), such that at $\mu \sim \Lambda_Q$, the dual theory is still deep in the logarithmic regime. In this case, in the interval of scales $\Lambda_q \ll \mu < \Lambda_Q$, the direct and dual theories are clearly different. By the definition of Λ_Q, the direct theory already entered sufficiently deep into the conformal regime, $[a_* - a(\mu)]/a_* < \delta \ll 1$ at $\mu < \Lambda_Q$, while the dual theory is still deep in the logarithmic regime. We therefore consider lower energies $\mu \sim \Lambda_q$, where, by definition, even the dual theory entered sufficiently deep into the conformal regime. At this scale, some parameters of the direct theory are given by

$$\widetilde{\mathcal{M}}^2_{ch} = \langle \overline{Q} Q(\mu = \Lambda_q) \rangle \approx \left(\frac{\Lambda_q}{\Lambda_Q}\right)^{b_0/N_F} \times \times \left(\mathcal{M}^2_{ch} = \langle \overline{Q} Q(\mu = \Lambda_Q)\rangle\right), \quad \text{(A.6)}$$

$$\widetilde{m}_Q = m_Q(\mu = \Lambda_q) \approx \left(\frac{\Lambda_Q}{\Lambda_q}\right)^{b_0/N_F} m_Q(\mu = \Lambda_Q),$$

$$m_Q(\mu = \Lambda_Q)\mathcal{M}^2_{ch} = \widetilde{m}_Q \widetilde{\mathcal{M}}^2_{ch} = \Lambda^3_{YM},$$

and therefore (3.1) at the scale $\mu \sim \Lambda_q \ll \Lambda_Q$ has the same form, with the normalizations: $M(\mu = \Lambda_q) = \widetilde{\mathcal{M}}^2_{ch}$ and $\overline{m}_Q(\mu = \Lambda_q) = \widetilde{m}_Q$, while μ_1 in (3.1) is now rewritten as $\mu_1 \equiv \widetilde{Z}_q \Lambda_q$. After this, all calculations are the same as in Sec. 3, with only notational changes. The only point that deserves additional comment is the explicit form of Λ_{YM} (see (2.2) and (2.1)), which was used in (3.3) and (3.4) for finding Z_q. But it can be seen from (A.6) that Λ_{YM} stays intact. The same can be seen from an expression in [13] (see also review [3] and the references therein) for the gluino condensate of the direct theory in terms of the running scale μ,

$$\langle S \rangle = \mu^{b_0/N_c} m_Q^{N_F/N_c}(\mu) \frac{1}{a(\mu)} \exp\left(-\frac{1}{a(\mu)}\right), \quad \langle S \rangle = \frac{\langle \lambda\lambda \rangle}{32\pi^2} = \Lambda^3_{YM}, \quad \text{(A.7)}$$

which is valid from sufficiently large μ down to $\mu = \mu_H \ll \Lambda_Q$, where μ_H is the largest physical mass. Now, taking $\mu = \Lambda_Q$ in (A.7), we can write (with our exponential accuracy and neglecting $\delta \ll 1$ in comparison with unity in $a(\mu = \Lambda_Q) = a_*(1 - \delta) \approx \Delta_0 \approx b_0/3N_c$)

$$\Lambda^3_{YM} \approx \Lambda_Q^{b_0/N_c} \left(m_Q = m_Q(\mu = \Lambda_Q)\right)^{N_F/N_c} \times \times \left[\exp\left(-\frac{1}{a(\mu = \Lambda_Q)}\right) \approx \exp\left(-\frac{3N_c}{b_0}\right)\right]. \quad \text{(A.8)}$$

On the other hand, taking $\mu = \Lambda_q$ in (A.7), we obtain

$$\Lambda^3_{YM} \approx \Lambda_q^{b_0/N_c} \left(\widetilde{m}_Q = m_Q(\mu = \Lambda_q)\right)^{N_F/N_c} \times \times \left[\exp\left(-\frac{1}{a(\mu = \Lambda_q)}\right) \approx \exp\left(-\frac{3N_c}{b_0}\right)\right]. \quad \text{(A.9)}$$

Hence, instead of Λ_Q and $m_Q = m_Q(\mu = \Lambda_Q)$ in (A.8), Λ_{YM} can equivalently be expressed through Λ_q and $\widetilde{m}_Q = m_Q(\mu = \Lambda_q)$ in (A.9).

Therefore, we obtain the same result: $\widetilde{Z}_q = Z_q = \exp(-N_c/b_0)$, and at $\mu < \Lambda_q$, all results for observable masses in Sec. 3 will stay intact (with only notational changes). Nevertheless, in a sense, this variant with $\Lambda_q \ll \Lambda_Q$ is worse for the dual theory in comparison with $\Lambda_q \sim \Lambda_Q$ because both theories are additionally not equivalent in the interval of energies $\Lambda_q < \mu < \Lambda_Q$.

We now consider the region $0 < \overline{b}_0/N_F \ll 1$ where weakly coupled is the dual theory. Here, we can simply repeat all the above arguments to see that all the results in Sec. 4 remain valid, except for a change of notation as in (A.6) and the omission of the exponential factors in (A.8) and (A.9) in accordance with the direct theory being strongly coupled here, $a_* = O(1)$. The same applies to the results for observable masses in Sec. 5, because all these can be expressed in terms of Λ_{YM} and the appropriate powers of $r = m_l/m_h$ and $\widetilde{Z}_q = Z_q = \exp(-\overline{N}_c/7\overline{b}_0)$.

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