ЖЭТФ, 2011, том 139, вып. 4, стр. 627–635

STRONG-COUPLING REGIME OF THE NONLINEAR LANDAU–ZENER PROBLEM FOR PHOTO- AND MAGNETOASSOCIATION OF COLD ATOMS

R. Sokhoyan [a,b], ***H. Azizbekyan*** [a], ***C. Leroy*** [b], ***A. Ishkhanyan*** [a*]

[a] *Institute for Physical Research, National Academy of Sciences of Armenia*
0203, Ashtarak-2, Armenia

[b] *Institut Carnot de Bourgogne, UMR 5209 CNRS, Université de Bourgogne*
BP 47870, 21078 Dijon, France

Received May 28, 2009
Revised version October 7, 2010

We discuss the strong-coupling regime of the nonlinear Landau–Zener problem occurring at coherent photo- and magneto-association of ultracold atoms. We apply a variational approach to an exact third-order nonlinear differential equation for the molecular state probability and construct an accurate approximation describing the time dynamics of the coupled atom–molecule system. The resultant solution improves the accuracy of the previous approximation [22]. The obtained results reveal a remarkable observation that in the strong-coupling limit, the resonance crossing is mostly governed by the nonlinearity, while the coherent atom–molecule oscillations occurring soon after crossing the resonance are principally of a linear nature. This observation is supposedly general for all nonlinear quantum systems having the same generic quadratic nonlinearity, due to the basic attributes of the resonance crossing processes in such systems. The constructed approximation turns out to have a larger applicability range than it was initially expected, covering the whole moderate-coupling regime for which the proposed solution accurately describes all the main characteristics of the system evolution except the amplitude of the coherent atom–molecule oscillation, which is rather overestimated.

1. INTRODUCTION

In contrast to atomic Bose condensates [1, 2], achieving molecular condensates via standard laser cooling techniques [3–5] is difficult since the laser cooling freezes only the centre-of-mass motion of a quantum object. In the case of atoms this is sufficient. But molecules have rotational and vibrational degrees of freedom. Hence, to create ultracold molecules, different approaches should be used. Currently, there are several approaches to this problem, among which the most widely used techniques are optical laser photoassociation [6, 7] and magnetic Feshbach resonance [8, 9].

For theoretical discussion of the specific field configurations applied within these techniques, the Landau–Zener model of linear resonance crossing ($\delta_t(t) = = 2\delta_0 t$, $\delta_0 = \text{const}$) at constant field amplitude ($U(t) = U_0 = \text{const}$) is particularly interesting [10, 11]. This is because to achieve a high conversion efficiency, a level-crossing field configuration must be used [12]. Then, as is well appreciated, the Landau–Zener model inevitably emerges as a natural starting point for studying such models. For this reason, this model has been a subject of intensive investigations over the last decades in different physical and mathematical contexts (see, e. g., [13–22]).

In this paper, we re-examine the strong-coupling regime of the resonance crossing in nonlinear systems involving quadratic nonlinearities generic for all boson field theories. We reveal the general property of such processes that the time dynamics of the transition process is effectively divided into two distinct regimes. We find that in the strong-coupling limit, the time dynamics of the atom–molecule conversion process consists in crossing the resonance in an essentially nonlinear manner, which is followed by atom–molecule coherent oscillations that are principally of a linear nature. This separation of the two processes is rather unexpected because of generic mixing of the corresponding terms

*E-mail: aishkhanyan@gmail.com

in the governing equations. The general nature of this effect is due to the basic attributes of the specific form of the quadratic nonlinearity involved.

Applying a variational approach to an exact third-order nonlinear differential equation obeyed by the molecular state probability, we develop an approximation that accurately describes the time dynamics of the coupled atom–molecule system in the strong-coupling limit. The formulas improve the accuracy of the previous result in [22] by providing the next approximation term. It turns out that the proposed approximation is also applicable for the intermediate regime of moderate coupling. In this regime, the solution accurately describes all the main characteristics of the system dynamics except the amplitude of coherent atom–molecule oscillations occurring at the end of the association process.

2. MATHEMATICAL TREATMENT

In the mean-field two-mode approximation, both photoassociation and Feshbach resonance are described by a basic semiclassical time-dependent nonlinear two-state model [12, 23, 24]:

$$\begin{aligned} i\frac{da_1}{dt} &= U(t)e^{-i\delta(t)}a_1^* a_2, \\ i\frac{da_2}{dt} &= \frac{U(t)}{2}e^{i\delta(t)}a_1 a_1, \end{aligned} \tag{1}$$

where a_1 and a_2 are the respective probability amplitudes of atomic and molecular states, and «$*$» denotes complex conjugation. Real functions $U(t)$ and $\delta(t)$ are the characteristics of the applied field. When the photoassociation terminology is used, $U(t)$ is referred to as the Rabi frequency of the laser field, and $\delta(t)$ as the frequency detuning modulation function. The time derivative of this function, $\delta_t(t)$, is the detuning of the laser field frequency from that of transition from the atomic state to the molecular one. Hereafter, an alphabetical subscript denotes differentiation with respect to the corresponding variable. We note that system (1) describes a lossless process, and hence the total number of particles is conserved:

$$|a_1|^2 + 2|a_2|^2 = \text{const} = 1.$$

We also note that $|a_1|^2 \in [0,1]$, whereas $|a_2|^2 \in [0,1/2]$.

Our previous experience (see, e. g., Refs. [21, 22, 25]) has revealed that the exact equation obeyed by the quantity $p = |a_2|^2$ (conventionally referred to as the molecular state probability) is quite helpful in treating the nonlinear two-state problem in (1). Because the above equation plays a decisive role in the subsequent development, we briefly outline its derivation. First, it can be verified by direct differentiation that the function $p = |a_2|^2$ satisfies the relations

$$\begin{aligned} p_t = a_{2t}^* a_2 + a_2^* a_{2t} = \\ = \frac{U}{2i}\left(a_1^2 a_2^* e^{i\delta(t)} - a_1^{*2} a_2 e^{-i\delta(t)}\right), \end{aligned} \tag{2}$$

$$\begin{aligned} p_{tt} = \frac{U_t}{U}\,p_t + \frac{U^2}{2}(1 - 8p + 12p^2) + \\ + \frac{U}{2}\delta_t\left(a_1^2 a_2^* e^{i\delta(t)} + a_1^{*2} a_2 e^{-i\delta(t)}\right). \end{aligned} \tag{3}$$

Next, straightforward differentiation shows that the function

$$Z = a_1^2 a_2^* e^{i\delta(t)} + a_1^{*2} a_2 e^{-i\delta(t)} \tag{4}$$

satisfies the relation

$$Z_t = -\delta_t \frac{2p_t}{U}\,. \tag{5}$$

Finally, differentiation of Eq. (5) followed by some algebraic transformations yields a nonlinear third-order ordinary differential equation for the molecular state probability in the form [25]

$$\begin{aligned} p_{ttt} - \left(\frac{\delta_{tt}}{\delta_t} + 2\frac{U_t}{U}\right)p_{tt} + \\ + \left[\delta_t^2 + 4U^2(1-3p) - \left(\frac{U_t}{U}\right)_t + \frac{U_t}{U}\left(\frac{\delta_{tt}}{\delta_t} + \frac{U_t}{U}\right)\right]p_t + \\ + \frac{U^2}{2}\left(\frac{\delta_{tt}}{\delta_t} - \frac{U_t}{U}\right)(1 - 8p + 12p^2) = 0. \end{aligned} \tag{6}$$

It is worth stressing that the normalization condition is incorporated in this equation.

The exact equation (6) for the molecular state probability is considerably simplified in the case of the Landau–Zener model: in this case, Eq. (6) is written in the factored form

$$\left(\frac{d}{dt} - \frac{1}{t}\right)\left[p_{tt} - \frac{\lambda}{2}(1 - 8p + 12p^2)\right] + 4t^2 p_t = 0. \tag{7}$$

Here, we have passed to dimensionless time by the scaling transformation $t \to t/\sqrt{\delta_0}$ and have introduced the conventional Landau–Zener parameter $\lambda = U_0^2/\delta_0$. Because the initial set of equations (1) and, consequently, the exact equation for the molecular state probability (7) contain only one combined parameter λ to characterize the external field, we conclude that the application of high laser field intensities U_0^2 along with large sweep rates $2\delta_0$ or, alternatively, small laser field intensities U_0^2 together with small sweep rates $2\delta_0$ results in

the same final molecular population as $t \to \infty$ if the ratio $\lambda = U_0^2/\delta_0$ remains unchanged.

We assume that the system starts from the all-atom state, such that the initial conditions for system (1) are $a_1(-\infty) = 1$ and $a_2(-\infty) = 0$, and hence $p(-\infty) = 0$. Then, to find the remaining initial conditions for Eq. (7), we use Eqs. (2) and (3) that define the derivatives p_t and p_{tt} in terms of atomic and molecular state probability amplitudes. Equation (2) immediately implies that $p_t(-\infty) = 0$. As regards the second derivative of p, we note that the factor δ_t in the last term in the right-hand side of Eq. (3) diverges as $t \to -\infty$. Hence, to define the correct limit of p_{tt} as $t \to -\infty$, we should take the asymptotic behavior of the functions $a_1(t)$ and $a_2(t)$ into account. This behavior can be found from the governing set of equations (1) using Picard's successive approximations and taking the linear Landau–Zener solution as the appropriate physical limit at vanishing nonlinearity. As a result, we obtain $p_{tt}(-\infty) = 0$. Thus the initial conditions for Eq. (7) become

$$p(-\infty) = 0, \quad p_t(-\infty) = 0, \quad p_{tt}(-\infty) = 0. \tag{8}$$

Because we consider the strong-coupling regime, we suppose that the Landau–Zener parameter is large (equivalently, the field intensity U_0^2 is large enough or the detuning sweep across the resonance is sufficiently slow, i. e., the sweep rate $2\delta_0$ is small). Hence, the second term in the square brackets in Eq. (7) takes large values in general. Because for large t the last term of the equation also takes a large value, we suppose that the leading terms in Eq. (7) are the last two, and we can therefore temporarily neglect the term p_{tt}, thus arriving at a limit nonlinear equation of the first order. This equation admits two trivial stationary solutions, $p = 1/2$ and $p = 1/6$, and a nontrivial one.

Unfortunately, for the initial condition $p(-\infty) = 0$, the nontrivial solution diverges as $t \to \infty$ [22], and hence cannot be directly applied as a proper initial approximation. In Ref. [22], an appropriate initial approximation was constructed via combination of the nontrivial solution with the trivial one $p = 1/2$. With the constructed function as the zeroth-order approximation, the nonadiabatic transition probability has been calculated, and it turned out that the final transition probability is expressed as a power of the Landau–Zener parameter [17, 22], in contrast to the familiar exponential prediction of the linear theory [10, 11]. However, this approach is rather complicated and does not provide a clear treatment of the time dynamics of the association process.

Here, we make a step forward by proposing a much simpler treatment of the problem that gives comprehensive understanding of the whole time evolution of the system. For this, we use an augmented limit equation that differs from that used in Ref. [22] by a term of the form A/t, where A is a constant that is supposed to be small compared with other involved terms in order not to change the leading asymptotes. Due to this modification of the limit equation, we manage to construct a simple two-term approximation that accurately describes the whole time dynamics of the system. Importantly, the constructed solution reveals the main characteristics of the process in a simple and natural manner.

The augmented limit equation, involving a fitting constant A, is written as

$$\left(\frac{d}{dt} - \frac{1}{t}\right)\left[-\frac{\lambda}{2}(1 - 8p + 12p^2) + A\right] + 4t^2 p_t = 0. \tag{9}$$

This equation is integrated via transformation of the independent variable followed by interchange of the dependent and independent variables. This results in a fourth-degree polynomial equation for the limit solution $p_0(t)$:

$$\frac{\lambda}{4t^2} = \frac{C_0 + p_0(p_0 - \beta_1)(p_0 - \beta_2)}{9(p_0 - \alpha_1)^2(p_0 - \alpha_2)^2}, \tag{10}$$

where C_0 is the integration constant and

$$\alpha_{1,2} = \frac{1}{3} \mp \frac{1}{6}\sqrt{1 + \frac{6A}{\lambda}}, \quad \beta_{1,2} = \frac{1}{2} \mp \sqrt{\frac{A}{2\lambda}}. \tag{11}$$

For the initial condition $p_0(-\infty) = 0$, the integration constant $C_0 = 0$. We note that at $A = 0$, quartic equation (10) reduces to a quadratic one because three of the four parameters $\alpha_{1,2}$, $\beta_{1,2}$ then become equal, $\alpha_2 = \beta_1 = \beta_2 = 1/2$. The solution of this quadratic equation diverges as $t \to \infty$. However, for positive A, the solution of quartic equation (10) defines a bounded, monotonically increasing function that tends to a finite value less than $1/2$ as $t \to \infty$ (Fig. 1). This solution has all the needed features to be used as an appropriate initial approximation for solving the problem. We thus see that the introduction of the parameter A is, indeed, an essential point.

We consider the properties of the limit solution $p_0(t)$ defined by Eq. (10) with $C_0 = 0$. The final value $p_0(\infty)$ is easily found by noting that the left-hand side of Eq. (10) tends to zero as $t \to \infty$. It is then seen that either $p_0(\infty) = 0$ or $p_0(\infty) = \beta_1$, or $p_0(\infty) = \beta_2$ should hold. Because $p_0(t)$ is a monotonically increasing function with $p_0(-\infty) = 0$ and because $\beta_2 > 1/2$,

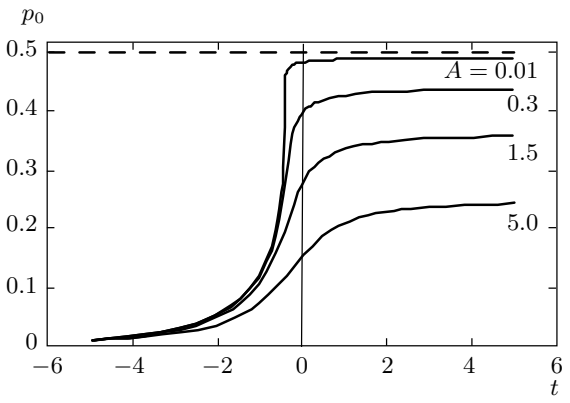

Fig. 1. The limit solution $p_0(t)$ for positive $A > 0$ and a fixed λ

we deduce that $p_0(\infty) = \beta_1$. In a similar way, we find that $p_0(0) = \alpha_1$. Thus,

$$p_0(0) = \frac{1}{3} - \frac{1}{6}\sqrt{1+\frac{6A}{\lambda}}\,, \quad p_0(\infty) = \frac{1}{2} - \sqrt{\frac{A}{2\lambda}}\,. \quad (12)$$

To determine the appropriate value of the parameter A, we substitute $p_0(t, A)$ in the exact equation for the molecular state probability (7) and examine the remainder

$$R = \left(\frac{d}{dt} - \frac{1}{t}\right)[p_{0tt} - A]. \quad (13)$$

Obviously, the better the approximation p_0 is, the smaller the remainder. We next note that if $p_{0tt} - A \neq$ $\neq 0$, the remainder diverges at the resonance crossing point $t = 0$, while it is finite at all other points of time. We can therefore eliminate this divergence by requiring A to obey the equation

$$p_{0tt}(0) - A = 0. \quad (14)$$

After some algebra, this equation is rewritten as

$$A = \frac{2}{9\lambda}\left(1 + \frac{1 - 18A/\lambda}{(1 + 6A/\lambda)^{3/2}}\right). \quad (15)$$

An approximate solution of the derived equation can be constructed by Newton's successive approximations starting, e. g., from $A = 0$. It turns out that the first approximation is already good enough. We therefore set $A = 0$ in the right-hand side of the equation and obtain

$$A = \frac{4}{9\lambda}\,. \quad (16)$$

This value of A leads to a good zeroth-order approximation $p_0(t)$. Numerical simulations show that for large λ, this function accurately describes the time evolution of the system in the interval covering the prehistory (up to the resonance point) and an interval after the resonance has been crossed. After that, however, p_0 misses several essential features of the process. For instance, the coherent oscillations between atomic and molecular populations that occur at a certain time point after the resonance has been passed are not captured by this solution. Furthermore, the final transition probability as $t \to \infty$ predicted by p_0 is always lower than what is shown by numerical solution of the exact equation.

It is understood that the shortcomings of the suggested limit solution are due to the singularity of the procedure used to obtain it. Indeed, we have constructed p_0 by neglecting the term p_{tt} in the square brackets in Eq. (7), i. e., the two highest-order derivative terms of the equation. Certainly, when determining the appropriate value of A by imposing Eq. (14), we have taken these terms into account (in fact, to some extent). Yet, this was an indirect procedure and we have convinced ourselves that it does not suffice.

Therefore, to improve the result, we need a correction that accounts for the second and third-order derivatives of p. This is not a simple task because the equation obeyed by the correction term $u \equiv p - p_0$ is still essentially nonlinear. Moreover, attempting to linearize the exact Eq. (7) using p_0 as the zeroth-order approximation and supposing the correction u to be small compared with p_0 ($u \ll p_0$), we arrive at a complicated equation with variable coefficients (depending on p_0), whose solution is not known.

We now introduce an approach that allows overcoming these difficulties. Importantly, the resultant solution not only correctly accounts for the higher-order derivate terms in the equation for the correction term u but also takes the nonlinear terms into account to a very good extent. The constructed solution displays much more improved results. It both treats the oscillations accurately and fits the final transition probability well. For the most part of the variation range of the Landau–Zener parameter $\lambda \gg 1$, the resultant graphs are practically indistinguishable from the numerical solution.

We consider a correction u defined as

$$p = p_0 + u. \quad (17)$$

This function obeys the exact equation

$$\left(\frac{d}{dt} - \frac{1}{t}\right)\left[u_{tt} + 4\lambda(1 - 3p_0)u + p_{0tt} - \right.$$
$$\left. - A - 6\lambda u^2\right] + 4t^2 u_t = 0. \quad (18)$$

Taking the initial conditions discussed here into account, we impose the conditions

$$u(-\infty) = 0, \quad u_t(-\infty) = 0, \quad u_{tt}(-\infty) = 0. \qquad (19)$$

Because the limit solution $p_0(t)$ is supposed to be a good approximation, the correction u is expected to be small. We therefore temporarily neglect the nonlinear term $-6\lambda u^2$ in Eq. (18), thus arriving at a linear equation. Although we now have a linear equation, there is only little progress since the solution of the derived equation is not known in the general case of a variable $p_0(t)$. However, we note that in the case of a constant p_0, we can construct the solution using the scaling transformation

$$u = \frac{A}{2\lambda(1-3p_0)}\,\nu. \qquad (20)$$

As a result, we then obtain a linear Landau–Zener problem for ν with the effective Landau–Zener parameter $\lambda_1 = \lambda(1-3p_0)$.

This observation suggests the conjecture that the exact solution of Eq. (18) can be approximated as

$$u = C_1 \frac{p_{LZ}(\lambda_1, t)}{p_{LZ}(\lambda_1, \infty)}, \qquad (21)$$

where $p_{LZ}(\lambda_1, t)$ is the solution of the linear Landau–Zener equation with an effective Landau–Zener parameter λ_1,

$$\left(\frac{d}{dt} - \frac{1}{t}\right)(p_{LZ_{tt}} + 4\lambda_1 p_{LZ} - 2\lambda_1) + 4t^2 p_{LZ_t} = 0, \qquad (22)$$

satisfying initial conditions (8). This solution is conveniently written in terms of the Kummer hypergeometric functions [26] (see, e. g., [21]).

This proves to be a good conjecture. Numerical simulations show that C_1, λ_1, and A can always be found such that approximate solution (21) accurately fits the numerical solution of Eq. (18).

Now, to derive analytic formulas for the fitting parameters C_1 and λ_1, we substitute expression (21) in the exact Eq. (18) and try to minimize the remainder

$$R = \left(\frac{d}{dt} - \frac{1}{t}\right)\left\{4\left[\lambda(1-3p_0) - \lambda_1\right]\frac{p_{LZ}(\lambda_1, t)}{p_{LZ}(\lambda_1, \infty)} + \right.$$
$$+ \frac{2\lambda_1}{p_{LZ}(\lambda_1, \infty)} + \frac{1}{C_1}(p_{0tt} - A) -$$
$$\left. - 6\lambda C_1 \frac{p_{LZ}^2(\lambda_1, t)}{p_{LZ}^2(\lambda_1, \infty)}\right\} \qquad (23)$$

by appropriately choosing these parameters.

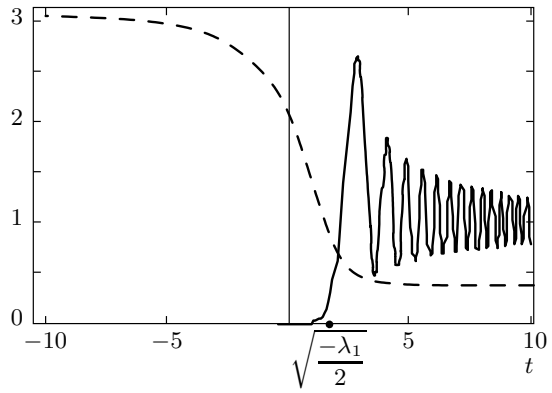

Fig. 2. Behavior of the functions $4[\lambda(1-3p_0) - \lambda_1]$ (dashed line) and $p_{LZ}(\lambda_1, t)/p_{LZ}(\lambda_1, \infty)$ (solid line)

The first term in the curly brackets is a product of two functions. The function $p_{LZ}(\lambda_1, t)/p_{LZ}(\lambda_1, \infty)$ is an increasing (although oscillating) function that starts from zero at $t = -\infty$ and noticeably differs from zero only for $t > 0$. On the other hand, the function $4[\lambda(1-3p_0) - \lambda_1]$ is a monotonically decreasing function that tends to a large final value as $t \to \infty$ since λ is a large parameter (Fig. 2). It is then understood that this term is highly suppressed if we choose

$$\lambda_1 = \lambda\left[1 - 3p_0(\infty)\right]. \qquad (24)$$

For $\lambda \gg 1$, this gives

$$\lambda_1 \approx -\lambda/2 \qquad (25)$$

and hence for large λ, the value of λ_1 becomes a large negative parameter. Interestingly, this choice of λ_1 leads to other relevant observations. First, it is known that

$$\lim_{t\to\infty} p_{LZ}(\lambda_1, t) = 1 - \exp(-\pi\lambda_1), \qquad (26)$$

and therefore, in the case of negative λ_1, the function $p_{LZ}(\lambda_1, \infty)$ increases exponentially with $|\lambda_1|$. Consequently, with this choice of λ_1, the second term in the curly brackets in Eq. (23) is also essentially suppressed. Second, in contrast to positive λ_1, for negative λ_1, the Landau–Zener function $p_{LZ}(\lambda_1, t)$ starts to noticeably differ from zero not merely for nonnegative times $t \geq 0$ but exclusively for those of the order of or larger than $\sqrt{-\lambda_1/2}$ (see Fig. 2). Hence, the first term in the curly brackets in Eq. (23) is even smaller than it was initially expected. Thus, the choice made in (24) essentially suppresses the first two terms in Eq. (23).

Regarding the two last terms in Eq. (23), we should minimize them with respect to the parameter C_1. This implies the condition

$$\frac{\partial R}{\partial C_1} = \left(\frac{d}{dt} - \frac{1}{t}\right) \times$$
$$\times \left[-\frac{1}{C_1^2}(p_{0tt} - A) - 6\lambda \frac{p_{LZ}^2(\lambda_1, t)}{p_{LZ}^2(\lambda_1, \infty)}\right] = 0. \quad (27)$$

Because the last term of this equation is proportional to (large) λ and $p_{LZ}(\lambda_1, t)/p_{LZ}(\lambda_1, \infty)$ is an increasing function of time, it is clear that the "worst" point is $t = \infty$. Hence, we consider the minimization at $t = \infty$. This immediately leads to the following value for C_1:

$$C_1 = \sqrt{A/6\lambda}\,. \quad (28)$$

This result, together with relation (24), is of considerable general importance. Indeed, we see that although we use the solution $p_{LZ}(\lambda_1, t)$ of a linear equation, the parameters of this solution, λ_1 and C_1, are essentially changed due to the nonlinear terms involved.

The obtained formulas (24) and (28) provide a rather good approximation. As can be checked numerically, solution (17), $p = p_0 + u$, where p_0 is the exact solution of the limit equation (9) and u is a linear Landau–Zener function, describes the process qualitatively well. This solution can then be used as an initial approximation for linearization of the initial equation (7).

However, more elaborate approaches can be suggested. For example, an immediate observation is that if we try approximation (17), (21) without imposing the initial restriction that the introduced parameter A be already determined by Eq. (14), we can modify this last equation such that the resultant value of A would take the correction term u into account. Following this approach leads to the formulas

$$\lambda_1 = -\frac{\lambda}{2} + \lambda \ln\left(1 + \frac{1}{\lambda}\right), \quad (29)$$

$$C_1 = \frac{1}{4\lambda} + \frac{1}{27\lambda^3}. \quad (30)$$

These formulas define a fairly good approximation. Indeed, starting already from $\lambda = 3$, the produced graphs (Fig. 3) are practically indistinguishable from the numerical solution of the exact Eqs. (1). The derived approximation notably improves the accuracy of the previous approximation in Ref. [22]. However, importantly, it is applicable far beyond the strong-coupling limit and provides a sufficiently good description also for the intermediate regime of moderate field intensities (or sweeping rates) down to $\lambda = 1$ and even slightly less ($0.95 < \lambda < 1$) (Fig. 4). Although the amplitude of oscillations predicted in this regime differs from that displayed by the numerical solution, it is seen

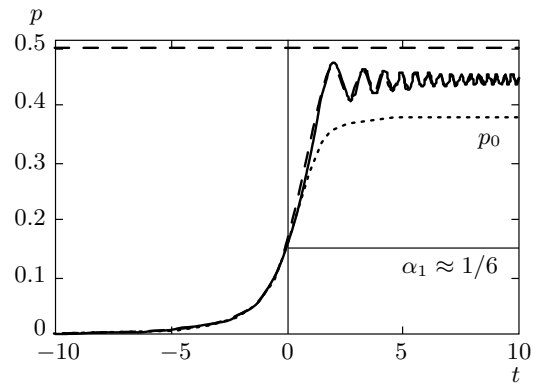

Fig. 3. Molecular state probability versus time at $\lambda = 4$ (the dashed line is the approximate solution with parameters (16), (29), and (30), and the dotted line is the limit solution). It is seen that in the strong-coupling limit $\lambda \gg 1$, the prehistory of the system and the resonance crossing are basically defined by the limit solution p_0, while the atom–molecule oscillations are described by the correction u

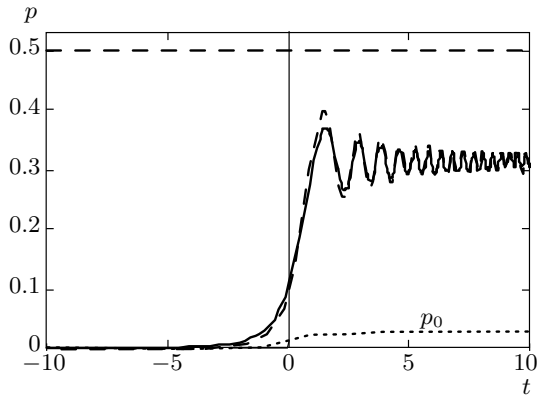

Fig. 4. The same as in Fig. 3 at $\lambda = 1$. The limit solution p_0 is small, and hence the "correction" u basically defines the time evolution of the system in the regime of moderate couplings $\lambda \geq 1$

from Fig. 4 that the approximation correctly describes many properties of the system time evolution including the effective transition time, the final transition probability, and the period of atom–molecule oscillations. This is, indeed, a rather unexpected result, especially if we note that at moderate couplings $\lambda \approx 1 \ldots 1.5$, the function $p_0(t)$ is very far from the exact solution, as can be seen from Fig. 4. An immediate conclusion following from this result is that the time evolution of the system in this regime is basically determined not by the limit solution p_0 but by the "correction" u, which during our

calculation was assumed to be small compared with the limit solution.

We note in conclusion that the obtained formulas show that the final probability of the molecular state is given by the simple formula

$$p(\infty) = \beta_1 + C_1. \tag{31}$$

Hence, the formula derived in Ref. [22] for the strong-coupling limit $\lambda \gg 1$ is modified to also include the intermediate regime of moderate couplings $\lambda \geq 1$ as follows:

$$p(\infty) = \frac{1}{2} - \left(\frac{\sqrt{2}}{3} - \frac{1}{4}\right)\frac{1}{\lambda} + \frac{1}{27\lambda^3} \approx \approx \frac{1}{2} - \frac{0.2214}{\lambda} + \frac{1}{27\lambda^3}. \tag{32}$$

Therefore, for the quadratic nonlinear interaction we have discussed here, the final probability for the system to stay in its initial all-atom state, $|a_1(\infty)|^2 = 1-2p(\infty)$, is not given by an exponential as predicted by the linear Landau–Zener theory [10, 11]. Instead, in the strong-coupling limit, it is a linear function of the sweep rate $2\delta_0 \sim 1/\lambda$ if the leading order of the approximation is discussed [17, 22]. This linear dependence of the nontransition probability on the sweep rate is also confirmed by many-body calculations [18–20]. We finally note that formula (32) suggests the next approximation term as $1/27\lambda^3$.

To conclude this section, we discuss the relation of the Landau–Zener model to contemporary physical experiments. The ramping of an external magnetic field across a Feshbach resonance is the most commonly adopted scheme to form Feshbach molecules. A typical example is the ^{85}Rb experiment by Hodby and coworkers at JILA [27], where a coherent formation of Rb_2 molecules via sweeping the magnetic field across the Feshbach resonance is realized. The magnetic field is changed at a given linear sweep rate $\dot{B}$, and the molecule conversion efficiency is measured as a function of the inverse sweep rate. The external field configuration used in this experiment therefore corresponds to the Landau–Zener model.

Another experiment that can be described by the Landau–Zener model was performed by Xu and coworkers at MIT [28]. In this experiment, a quantum-degenerate gas of 10^5 cold sodium molecules has been created. This was achieved with a fast magnetic field sweep through a Feshbach resonance, followed by quick removal of the remnant atoms with resonant light. This purification was necessary to avoid heating and decay of the molecules via inelastic collision processes.

3. CONCLUSION

We have presented an analysis of a quadratically nonlinear version of the Landau–Zener problem that occurs in various physical situations, e. g., in photoassociation of an atomic Bose–Einstein condensate, in controlling the scattering length of an atomic condensate by means of a Feshbach resonance, in second-harmonic generation, and generally in nonlinear field theories involving a Hamiltonian with a $2:1$ resonance. Using an exact third-order nonlinear differential equation for the molecular state probability, we have developed an effective variational method for constructing the approximate solution of the problem in the strong-coupling limit corresponding to large values of the Landau–Zener parameter, $\lambda \gg 1$. In the case of photoassociation, this implies that the intensity of the applied laser field is large enough or, equivalently, the sweep rate across the resonance is sufficiently slow.

We have shown that the approximation describing time evolution of the molecular state probability can be written as a sum of two distinct terms. In the strong-coupling limit, the first term, being a solution of a limit first-order nonlinear differential equation, effectively describes the process of molecule formation, while the second one, being the scaled solution of the linear Landau–Zener problem (but with a negative effective Landau–Zener parameter as long as the strong-coupling limit of high field intensities or, equivalently, slow sweeping rates are considered), describes the oscillation that occurs some time after the system has passed through the resonance. From this, we can conclude that in the strong-coupling limit, the time dynamics of the atom–molecule conversion consists of the essentially nonlinear process of resonance crossing followed by atom–molecule coherent oscillations that are principally linear in nature. The possibility of such a decomposition is quite surprising because the Hamiltonian of the system is essentially nonlinear.

The constructed approximation describes the molecule formation process with high accuracy. For $\lambda > 3$, the produced graphs are practically indistinguishable from the exact numerical solution (see Fig. 3). Interestingly, the approximation also works rather well in the regime of moderate couplings down to $\lambda = 1$ (see Fig. 4) and slightly less, $0.95 < \lambda < 1$. It correctly describes many properties of the system time evolution, including the effective transition

time, the final transition probability, and the period of atom–molecule oscillations. The only noticeable discrepancy is that the approximate solution overestimates the amplitude of the oscillations (the largest deviation is observed at the points of maxima and minima of the probability within the time interval covering first several periods of oscillations). The applicability of the proposed approximation to the intermediate regime of moderate coupling is, indeed, a rather unexpected result because at $\lambda \approx 1 \ldots 1.5$, the limit solution $p_0(t)$ is very far from the exact solution; hence, it is not the limit solution that mostly defines the evolution of the system in this regime. Using the constructed approximation, we can easily find the main characteristics of the association process such as the tunneling time, the frequency of the oscillations of the transition probability that start soon after crossing the resonance, as well as the final transition probability to the molecular state. In particular, we have confirmed that the nontransition probability in the leading approximation order is a linear function of the sweep rate. In addition, we have found that the next approximation term is $1/27\lambda^3$.

We finally note that the presented approach is not restricted to the Landau–Zener model only. It can also be generalized to other time-dependent level-crossing models [29, 30]. Also, it can be used in exploring other nonlinear regimes beyond by the Landau–Zener model [31]. Importantly, the developed approach allows treating the extended version of the nonlinear two-state state problem, when higher-order nonlinearities involving functions of the transition probability are added to the basic system (1). For example, one can analyze the role of the inter-particle elastic scattering described by Kerr-type cubic nonlinear terms [32]. Hence, the developed method may serve as a general strategy for attacking analogous nonlinear two-state problems involving the generic quadratic nonlinearity as discussed here.

This work was supported by the Armenian National Science and Education Fund (ANSEF Grant Nos. PS-1692 and PS-2591) and the International Science and Technology Center (ISTC Grant No. A-1241). R. Sokhoyan and H. Azizbekyan acknowledge the French Embassy in Armenia for the Grants Nos. 2006-4638 and 2007-3849 (Boursiers du Gouvernement Français). A. Ishkhanyan acknowledges Institut Carnot de l'Université de Bourgogne for the invited professorship in 2007 and 2009.

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