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ON MASS SPECTRUM IN SQCD. UNEQUAL QUARK MASSES

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The $\mathcal{N}=1$ SQCD with N_c colors and two types of light quarks, N_l flavors with the smaller mass m_l and $N_h = N_F - N_l$ flavors with the larger mass m_h, $N_c < N_F < 3N_c$, $0 < m_l \le m_h \ll \Lambda_Q$, is considered within the dynamical scenario in which quarks can form a coherent colorless diquark condensate $\langle \overline{Q}Q \rangle$. There are several phase states at different values of the parameters $r = m_l/m_h$, N_l, and N_F. Properties of these phases and their mass spectra are described.

1. INTRODUCTION

We generalize the results obtained in [1] for equal quark masses to the case of unequal masses. We do not consider the most general case of arbitrary quark masses here. Only one specific (but sufficiently representative) choice of unequal masses is considered: there are $N_l \neq N_c$ flavors with the smaller mass m_l and $N_h = N_F - N_l$ flavors with the larger mass $m_h \ge m_l > 0$, $N_c < N_F < 3N_c$. Some abbreviations used below are follows: DC is the diquark condensate, HQ is a heavy quark, the l-quarks are the quarks with the smaller mass m_l, and the h-quarks are those with the larger mass m_h. The masses $m_l = m_l(\mu = \Lambda_Q)$ and $m_h = m_h(\mu = \Lambda_Q)$ are the running current quark masses normalized at $\mu = \Lambda_Q$, and $\mathcal{M}^l_{ch}$ or $\mathcal{M}^h_{ch}$ are the chiral diquark condensates of the l- or h-quarks, also normalized at $\mu = \Lambda_Q$, $\langle \overline{Q}_{\bar l} Q^l(\mu = \Lambda_Q)\rangle = \delta^l_{\bar l}\mathcal{M}^l_{ch}$, $\langle \overline{Q}_{\bar h} Q^h(\mu = \Lambda_Q)\rangle = \delta^h_{\bar h}\mathcal{M}^h_{ch}$, and Λ_Q (independent of quark masses) is the scale parameter of the gauge coupling constant. All quark masses are small, but nonzero: $0 < m_l \le m_h \ll \Lambda_Q$.

The whole theory can therefore be regarded as being defined by the three numbers N_c, N_F, and N_l and three dimensional parameters Λ_Q, m_l, and m_h (i. e., all dimensional observables are expressed through these three).

It is shown below that within the dynamical scenario used, there are different phase states in this theory at different values of the parameters $r = m_l/m_h \le 1$, N_l, and N_F:

a) the DC$_l$–DC$_h$ phase appears for $m_h^{pole} \ll \mathcal{M}^h_{ch} < \mathcal{M}^l_{ch} \ll \Lambda_Q$ in both cases $N_l > N_c$ and $N_l < N_c$ (m_h^{pole} is the perturbative pole mass of the h-quarks);

b) the DC$_l$–HQ$_h$ phase appears for $\mathcal{M}^h_{ch} \ll \mathcal{M}^l_{ch} \ll m_h^{pole} \ll \Lambda_Q$ at $N_l > N_c$ only;

c) another regime of the DC$_l$–HQ$_h$ phase appears for $\mathcal{M}^h_{ch} \ll m_h^{pole} \ll \mathcal{M}^l_{ch} \ll \Lambda_Q$ in both cases $N_l > N_c$ and $N_l < N_c$;

d) the Higgs$_l$–DC$_h$ or Higgs$_l$–HQ$_h$ phases appear at $\mathcal{M}^l_{ch} \gg \Lambda_Q$ at $N_l < N_c$ only.

It is implied that the reader is familiar with the previous paper [1], because all the results in [1] are essentially used in this paper.

The paper is organized as follows. The properties of the DC$_l$–DC$_h$ phase are considered in Sec. 2. The DC$_l$–HQ$_h$ phase (in two regimes) is considered in Secs. 3 and 4. The Higgs$_l$–DC$_h$ and Higgs$_l$–HQ$_h$ phases with Higgsed l-quarks are considered in Sec. 5. Section 6 contains a short conclusion.

2. THE DC$_l$–DC$_h$ PHASE

We first recall the effective Lagrangian for equal-mass quarks just below the physical threshold at $\mu < \mu_H = \mathcal{M}_{ch}$, after the evolution of all quark degrees of freedom has terminated [1] ($b_0 = 3N_c - N_F$, $\overline{N}_c = N_F - N_c$, see also footnote 5 in [1]):

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$$L = \int d^2\theta\, d^2\overline{\theta} \times \times \left\{ \mathrm{Tr}\sqrt{\Pi^\dagger\Pi} + Z_Q \,\mathrm{Tr}\left(Q^\dagger e^V Q + \overline{Q}^\dagger e^{-V}\overline{Q}\right)\right\} + + \int d^2\theta \left\{ -\frac{2\pi}{\alpha(\mu)}\, S + W_Q \right\},$$
$$W_Q = \left(\frac{\det \Pi}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} \left\{ \mathrm{Tr}\left(\overline{Q}\,\Pi^{-1}\, Q\right) - N_F\right\} + + \mathrm{Tr}\left(m_Q \Pi\right), \tag{1}$$
$$Z_Q = \left(\frac{\mathcal{M}_{ch}}{\Lambda_Q}\right)^{b_0/\overline{N}_c} = \frac{\Lambda^3_{YM}}{\mathcal{M}^3_{ch}} = \frac{m_Q}{\mathcal{M}_{ch}}.$$

Here, $(m_Q)^{\overline{j}}_i \equiv m_Q(\mu = \Lambda_Q)^{\overline{j}}_i$, where $m_Q(\mu)^{\overline{j}}_i$ are the running quark masses, and $\langle \Pi^{i}_{\overline{j}}\rangle = \langle (\overline{Q}_{\overline{j}} Q^i)_{\mu=\Lambda_Q}\rangle \equiv \equiv (\mathcal{M}^2_{ch})^{i}_{\overline{j}}$. For equal quark masses, $(m_Q)^{\overline{j}}_i = = m_Q\,\delta^{\overline{j}}_i$, $(\mathcal{M}^2_{ch})^i_{\overline{j}} = \mathcal{M}^2_{ch}\,\delta^i_{\overline{j}}$, $\langle S\rangle = m_Q \mathcal{M}^2_{ch} = = (\Lambda_Q^{b_0} \det m_Q)^{1/N_c} \equiv \Lambda^3_{YM}$ (as regards the specific forms of the pion Kähler terms for DC phases here and everywhere below, see footnote 5 in [1]).

Well above the highest physical threshold, $\mu_H \ll \ll \mu \ll \Lambda_Q$, the quark fields $\overline{Q}$ and Q describe the original quarks with the small running current masses $m_Q(\mu)$, while below the threshold, they become the fields of heavy quarks with the large constituent masses $\mu_C = \mathcal{M}_{ch}$ [1)].

The fields Π are defined as "the light part of $\overline{Q}Q$". In other words, well above the threshold, when the large constituent mass of quarks is not yet formed, Π and $\overline{Q}Q$ are both the same living diquark operator of light quarks, and hence $\overline{Q}\Pi^{-1}Q$ is a unit c-number matrix, and the projector $\mathcal{P} = \mathrm{Tr}(\overline{Q}\,\Pi^{-1}\,Q\Big) - N_F = 0$. Moreover, the term $(\det\Pi/\Lambda_Q^{b_0})^{1/\overline{N}_c}$ is dominated by contributions of light quantum quark fields, and represents not a constant mass but a living interaction. But below the threshold, at $\mu < \mathcal{M}_{ch}$, after the appearance of a large constituent mass $\mathcal{M}_{ch}$, the light Π and heavy $\overline{Q}Q$ become quite different, such that $\mathcal{P}$ becomes a nontrivial nonzero term. Besides, below the threshold, all the N_F^2 fields Π become "frozen", in the sense that all of them contain the large c-number vacuum part $\mathcal{M}^2_{ch}$ and the light quantum pion fields π with the small masses m_Q, whose contributions to amplitudes are smaller, $|\pi| \leq \mu < \mathcal{M}_{ch}$. As a result, the entire term $(\det\Pi/\Lambda_Q^{b_0})^{1/\overline{N}_c}(Z_Q\Pi)^{-1}$ in W_Q is now dominated by the c-number vacuum part, which becomes the large constituent mass $\mathcal{M}_{ch}$ of the quark fields Q and $\overline{Q}$.

We start with $m_l = m_h$ and begin to make $m_h > > m_l$, so as to produce a gap between $\mathcal{M}^l_{ch} > \mathcal{M}^h_{ch}$ [2)]:

$$\left(\mathcal{M}^l_{ch}\right)^2 = \frac{1}{m_l}\left(\Lambda_Q^{b_0}\det m\right)^{1/N_c} = = \Lambda_Q^{b_0/N_c}\, m_l^{(N_l - N_c)/N_c} m_h^{(N_F - N_l)/N_c}, \tag{2}$$

$$\left(\mathcal{M}^h_{ch}\right)^2 = \frac{1}{m_h}\left(\Lambda_Q^{b_0}\det m\right)^{1/N_c} = \Lambda_Q^{b_0/N_c} \times \times m_l^{N_l/N_c} m_h^{(N_F - N_c - N_l)/N_c} = \frac{m_l}{m_h}\left(\mathcal{M}^l_{ch}\right)^2. \tag{3}$$

Clearly, at scales $\mu \gg \mathcal{M}^l_{ch}$, the large constituent masses $\mu^l_C = \mathcal{M}^l_{ch}$ and $\mu^h_C = \mathcal{M}^h_{ch}$ are not yet formed, and all quarks behave as perturbative massless particles. Therefore, the fields Π are not yet frozen, the factor $(\det\Pi)^{1/\overline{N}_c}$ in (1) is actually given by $(\det\overline{Q}Q)^{1/\overline{N}_c}$, and this is still a living interaction, not a mass. As a result, there is still no difference between the fields Π (light at lower scales) and the fields $\overline{Q}Q$ (heavy at lower scales). Therefore, the projector $\mathcal{P}$ in curly brackets in (1) is still zero:

$$\mathcal{P} = \mathrm{Tr}\,(\overline{Q}\Pi^{-1}Q) - N_F = 0, \quad \mu > \mathcal{M}^l_{ch} > \mathcal{M}^h_{ch}. \tag{4}$$

The main point is that the projector $\mathcal{P}$ begins to be nonzero only after the decreasing scale μ crosses the physical threshold at $\mu \sim \mathcal{M}^h_{ch} < \mathcal{M}^l_{ch}$ (and not before, at $\mu \sim \mathcal{M}^l_{ch} > \mathcal{M}^h_{ch}$), where a mass gap between the heavy constituent quarks $(\overline{Q}_{\overline{h}}, Q^h)^{(const)}$ with the masses $\mathcal{M}^h_{ch}$ and the light pions $\Pi^{h}_{\overline{h}} = (\overline{Q}_{\overline{h}}Q^h)^{(light)}$ with the masses of the order of m_h appears and "begins to work", such that the fields Q^h and $\overline{Q}_{\overline{h}}$ become frozen. Before this, at $\mu > \mathcal{M}^h_{ch}$, the constituent mass $\mathcal{M}^h_{ch}$ is not yet formed and the operator $\Pi^{h}_{\overline{h}}$ is not yet frozen and represents two still living light quarks $\overline{Q}_{\overline{h}}Q^h$, whose

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[1)] The Konishi anomaly [2] for the canonically normalized constituent quark fields $C = Q/Z_Q^{1/2}$ and $\overline{C} = \overline{Q}/Z_Q^{1/2}$ is given by $\langle\overline{C}C\rangle = \langle S\rangle/\mu_C$. But the form of its explicit realization is a matter of convention. One convention is that it is realized directly through the one-loop triangle diagram with the heavy constituent quarks forming a loop and emitting two gluinos. Another convention is that the one-loop constituent quark contributions to the vacuum polarization are transferred to the gluon kinetic term at a first stage, and then a term $S\ln\mu_C$ appears, while the quark term in W_Q in (1) has to be used for calculations with the valence constituent quarks only. The Konishi anomaly then originates from this vacuum polarization term and is given by $\langle\overline{C}C\rangle = \langle\partial/\partial\mu_C\,(S\ln\mu_C)\rangle = \langle S\rangle/\mu_C$.

[2)] But to remain in the same DC phase for all flavors, there must be a restriction on the values of m_l and m_h, such that $r = m_l/m_h$ cannot be too small. The explicit form of this restriction is presented below.

quantum part still dominates over its c-number vacuum part. Therefore, in the first term in W_Q, the common factor $(\det\Pi)^{1/\overline{N}_c}$ is not yet completely frozen, and still describes some interaction, not a mass. Hence, the constituent masses are not yet formed not only for the $\overline{Q}_{\bar h}$ and Q^h quarks but also for the $\overline{Q}_{\bar l}$ and Q^l quarks. This shows that the very presence of the still living perturbative light quarks $\overline{Q}_{\bar h}$ and Q^h at $\mathcal{M}^h_{ch} < \mu < \mathcal{M}^l_{ch}$ also prevents the quarks $\overline{Q}_{\bar l}$ and Q^l from acquiring the large constituent mass $\mathcal{M}^l_{ch}$. Hence, nothing happens yet at $\mu \sim \mathcal{M}^l_{ch}$ and the perturbative regime does not stop here, but continues down to $\mu \sim \mathcal{M}^h_{ch}$. This is the real physical threshold μ_H, and the nonzero nonperturbative contributions to the quark superpotential appear only after crossing this region, and they appear simultaneously for all flavors[3)].

Therefore, at $\mu < \mathcal{M}^h_{ch}$, instead of (1), the effective Lagrangian becomes

$$
\begin{gathered}
L = \int d^2\theta\, d^2\overline{\theta} \left\{ \mathrm{Tr}\sqrt{\Pi^\dagger\Pi} + Z_l\,\mathrm{Tr}_l\Big(Q^\dagger e^V Q\Big) + \right. \\
\left. + Z_h\,\mathrm{Tr}_h\Big(Q^\dagger e^V Q\Big) + \Big(Q \to \overline{Q}\Big)\right\} + \\
+ \int d^2\theta \left\{ -\frac{2\pi}{\alpha(\mu)}\,S + W_Q \right\}, \\
W_Q = \left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} \left\{ \mathrm{Tr}\Big(\overline{Q}\,\Pi^{-1}\,Q\Big) - N_F \right\} + \\
+ \mathrm{Tr}(m\,\Pi), \\
Z_l = \left(\frac{\Lambda_{YM}}{\mathcal{M}^l_{ch}}\right)^3 = \frac{m_l}{\mathcal{M}^l_{ch}}, \\
Z_h = \left(\frac{\Lambda_{YM}}{\mathcal{M}^h_{ch}}\right)^3 = \frac{m_h}{\mathcal{M}^h_{ch}}.
\end{gathered}
\tag{5}
$$

Here, Π is the total $N_F \times N_F$ matrix of all pions, and $\overline{Q}$ and Q with l or h flavors are the constituent quarks with the respective masses $\mathcal{M}^l_{ch}$ or $\mathcal{M}^h_{ch}$.

After integrating out all heavy constituent quarks (which leaves behind a large number of hadrons made of constituent quarks that are weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mathcal{M}^h_{ch} < \mathcal{M}^l_{ch}$) and proceeding the same as in [1], we obtain the same form as in [1][4)]:

$$
\begin{gathered}
L = \int d^2\theta\, d^2\overline{\theta} \left\{ \mathrm{Tr}\sqrt{\Pi^\dagger\Pi} \right\} + \\
+ \int d^2\theta \left\{ -\frac{2\pi}{\alpha_{YM}(\mu,\Lambda_L)} S - N_F \left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + \right. \\
\left. + \mathrm{Tr}\Big(m\Pi\Big) \right\}, \quad \Lambda_L^3 = \left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c}, \\
\langle\Lambda_L\rangle = \Lambda_{YM} \ll \mu \ll \mathcal{M}^h_{ch}.
\end{gathered}
\tag{6}
$$

Hence, the only difference from the case of equal quark masses is that the masses entering $\mathrm{Tr}(m\Pi)$ are no longer equal.

Proceeding further as in [1] and going through the Veneziano–Yankielowicz (VY) procedure for gluons [3], we obtain that there is a large number of gluonia with masses $M_{gl} \sim \Lambda_{YM} = (\Lambda_Q^{b_0}\det m)^{1/3N_c}$, and the lightest particles are the pions with the Lagrangian

$$
\begin{gathered}
L_\pi = \int d^2\theta\, d^2\overline{\theta} \left\{ \mathrm{Tr}\sqrt{\Pi^\dagger\Pi} \right\} + \\
+ \int d^2\theta \left\{ -\overline{N}_c \left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + \mathrm{Tr}\Big(m\,\Pi\Big) \right\}, \\
\mu \ll \Lambda_{YM}.
\end{gathered}
\tag{7}
$$

The pion masses are proportional to the sum of their two quark masses:

$$
M_\pi^{(ll)} = c_0 2m_l, \quad M_\pi^{(hh)} = c_0 2m_h,
$$
$$
M_\pi^{(lh)} = c_0(m_l + m_h),
$$

where c_0 is a constant $O(1)$.

Clearly, when the quark masses become equal, $m_h \to m_l$, Lagrangian (7) smoothly matches those in (1) (and *vice versa*). This is as it should be, until both types of quarks remain in the same DC phase.

On the whole, it is seen that starting with the case of equal quark masses and splitting them smoothly yields very similar results. The only essential restriction is that the model has to stay in the DC$_l$–DC$_h$ phase. And the only new nontrivial point is that there is only one common physical threshold μ_H where nonperturbative effects turn on and change the form of the

3) In a sense, the constituent quarks can be thought of as extended solitons. And this also shows that the characteristic size of the heavier constituent quarks $\overline{Q}_{\bar l}$, Q^l is not $R_l \sim 1/\mathcal{M}^l_{ch}$ but a larger value $R_l \sim 1/\mathcal{M}^h_{ch} \gg 1/\mathcal{M}^l_{ch}$; this is typical for a soft soliton, whose size is much larger than its Compton wavelength, $R_{sol}^{(soft)} \gg 1/M_{sol}^{(soft)}$. In other words, the size R_l is the same as the size R_h of the lighter constituent quarks $\overline{Q}_{\bar h}$ and Q^h: $R_h \sim 1/\mathcal{M}^h_{ch}$, which are, in this sense, hard solitons.

4) It is worth noting that because there is only one common threshold $\mu_H = \mathcal{M}^h_{ch}$ for all flavors, the renormalization factors Z_π of all the N_F^2 pions are the same: $Z_\pi = z_Q^{-1}(\Lambda_Q, \mathcal{M}^h_{ch}) \equiv \equiv z_Q^{-1}$, where $z_Q \ll 1$ is the perturbative renormalization factor of the massless quark (see [1]).

Lagrangian, and this threshold is determined by the smallest diquark condensate $\mu_H = \mathcal{M}^h_{ch} < \mathcal{M}^l_{ch}$ (see (2) and (3)).

We finally write the conditions for the theory to be in the DC$_l$–DC$_h$ phase. In going down from $\mu \sim \Lambda_Q$, the massless perturbative evolution is to be stopped either at $\mu_H = \mathcal{M}^h_{ch}$ if $\mathcal{M}^h_{ch} > m_h^{pole}$ or at $\mu_H = m_h^{pole}$ if $m_h^{pole} > \mathcal{M}^h_{ch}$. Hence, the h-quarks are in the DC$_h$ phase at $\mathcal{M}^h_{ch} > m_h^{pole}$ and in the HQ$_h$ phase at $\mathcal{M}^h_{ch} < m_h^{pole}$. The phase transition occurs at $\mathcal{M}^h_{ch} \sim m_h^{pole}$. Using (3), we obtain that the DC$_h$ phase persists from $r = m_l/m_h = 1$ down to $r > r_1$:

$$\mathcal{M}^h_{ch} = m_h^{pole} = m_h\Big(\frac{\Lambda_Q}{m_h^{pole}}\Big)^{\gamma_+} \to r = \frac{m_l}{m_h} = = r_1 \equiv \Big(\frac{m_h}{\Lambda_Q}\Big)^{\sigma} \ll 1,$$
$$\sigma = \frac{1}{N_l}\Big[\frac{2N_c}{1+\gamma_+} - (N_F - N_c)\Big],$$
$$\frac{3N_c}{2} < N_F < 3N_c: \quad \gamma_+ = \frac{b_0}{N_F} \to \sigma = \frac{b_0}{3N_l},$$
$$N_c < N_F < \frac{3N_c}{2}: \quad \gamma_+ = \frac{2N_c - N_F}{N_F - N_c} \to \to \sigma = \frac{N_F - N_c}{N_l}, \tag{8}$$

where γ_+ is the quark anomalous dimension. It is known in the conformal window; to have definite answers, the value $\gamma_+ = (2N_c - N_F)/(N_F - N_c)$ used in [1] for $N_c < N_F < 3N_c/2$ is also used here and below in the text.

However, (8) is not the only condition, because if $r = m_l/m_h$ is too small at $N_l < N_c$, then $\mathcal{M}^l_{ch}$ becomes larger than Λ_Q and the l-quarks are Higgsed. This happens (see (2)) at

$$N_l < N_c: \mathcal{M}^l_{ch} = \Lambda_Q \to r = \frac{m_l}{m_h} = = r_2 \equiv \Big(\frac{m_h}{\Lambda_Q}\Big)^{(N_F-N_c)/(N_c-N_l)} \ll 1. \tag{9}$$

On the whole, the theory is in the DC$_l$–DC$_h$ phase under the following conditions:

$$a) \quad \frac{m_l}{m_h} > r_1 \quad \text{for} \quad N_l > N_c;$$
$$b) \quad \frac{m_l}{m_h} > \max(r_1, r_2) \quad \text{for} \quad N_l < N_c;$$
$$r_2 > r_1 \quad \text{at} \quad N_l < N_0,$$
$$N_0 = \begin{cases} N_c\, b_0/2N_F & \text{for } 3N_c/2 < N_F < 3N_c, \\ N_c/2 & \text{for } N_c < N_F < 3N_c/2. \end{cases} \tag{10}$$

,

3. THE DC$_l$–HQ$_h$ PHASE: $\mathcal{M}^l_{ch} \ll m_h^{pole}$ AND $N_l > N_c$

3.1. $3N_c/2 < N_F < 3N_c$, $3N_c/2 < N_l < N_F$

This is a separate phase when the lighter l-quarks Q^l and $\overline{Q}_{\bar l}$ are in the DC phase, while the heavier h-quarks Q^h and $\overline{Q}_{\bar h}$ are in the HQ phase.

For definiteness, we agree to use the following procedure below. The theory is defined at $\mu = \Lambda_Q$ by the quark mass values $m_l \equiv m_l(\mu = \Lambda_Q) \le m_h \equiv m_h(\mu = \Lambda_Q) \ll \Lambda_Q$. Starting with $m_l = m_h$, unequal quark masses are obtained with m_h staying intact, while m_l decreases, $m_l \ll m_h \ll \Lambda_Q$.

At $r \equiv m_l/m_h = 1$, the theory is in the DC$_l$–DC$_h$ phase, with the highest physical scale μ_H given by $\mu_H = \mathcal{M}^h_{ch} = \mathcal{M}^l_{ch} \ll \Lambda_Q$. As explained in Sec. 2, the constituent masses of Q^l and $\overline{Q}_{\bar l}$ quarks cannot be formed alone, and are formed only after all flavors are frozen. Hence, as r begins to decrease, the highest physical scale μ_H is determined by a competition between $\mathcal{M}^h_{ch} < \mathcal{M}^l_{ch}$ and the pole mass of Q^h and $\overline{Q}_{\bar h}$ quarks, $m_h^{pole} = m_h(\mu = m_h^{pole})$.

The DC$_l$–DC$_h$ phase persists until $\mathcal{M}^h_{ch} > m_h^{pole}$, while the coherent condensate of Q^h, $\overline{Q}_{\bar h}$ quarks can no longer be maintained at $\mathcal{M}^h_{ch} < m_h^{pole}$, and therefore a phase transition occurs from the DC$_l$–DC$_h$ phase to the DC$_l$–HQ$_h$ one. This happens at $r \sim r_1 \ll 1$ (see (8)).

Although the theory is in the DC$_l$–HQ$_h$ phase at $r < r_1$, there are two different regimes (see Sec. 4 below), depending on whether $r < r_1' \ll r_1$ or $r_1' < r < r_1$, with r_1' determined by

$$\mathcal{M}^l_{ch} = m_h^{pole} \to r = r_1' = \left(\frac{m_h}{\Lambda_Q}\right)^{\rho} \ll r_1 \ll 1,$$
$$\rho = \frac{1}{N_l - N_c}\Big[\frac{2N_c}{1+\gamma_+} - (N_F - N_c)\Big]. \tag{11}$$

The regime at $r < r_1' \ll r_1$ is much simpler and is considered first in this section. We therefore take $r \ll r_1'$ and consider the properties of this DC$_l$–HQ$_h$ phase. The highest physical scale μ_H is then given by the pole mass $m_h^{pole} = \Lambda_Q(m_h/\Lambda_Q)^{1/(1+\gamma_+)} \ll \Lambda_Q$ of the heavier quarks Q^h and $\overline{Q}_{\bar h}$ (see (8)).

The condition $r \ll r_1'$, i.e., $\mathcal{M}^l_{ch} \ll m_h^{pole}$ (see (11)) means that even if the Q^l and $\overline{Q}_{\bar l}$ quarks were trying to freeze in the threshold region around $\mu \sim m_h^{pole}$ by forming the largest possible constituent mass $\mu_C^l = \mathcal{M}^l_{ch}$, this is impossible if $\mathcal{M}^l_{ch} \ll m_h^{pole}$, because even this mass is too small for freezing. Therefore, no nonperturbative effects turn on at $\mu \sim m_h^{pole}$ in this case, and the region $\mu \sim m_h^{pole}$ is crossed in the purely perturbative regime.

At $\mu < m_h^{pole}$, the heavy h-quarks decouple from the lower-energy theory and can be integrated out. What remains is the lower-energy theory with N_c colors and $N_l > 3N_c/2$ light flavors Q^l and $\overline{Q}_{\bar{l}}$, which are also in the conformal regime at $\mu'_H < \mu \ll \mu_H = m_h^{pole}$ (μ'_H is the new highest physical scale of this lower-energy theory). We let $\hat{\Lambda}_Q$ denote the scale parameter of the new gauge coupling. Its value can be found from the following considerations. At $m_h^{pole} < \mu \ll \Lambda_Q$, the original coupling $\alpha(\mu)$ is already frozen at the value $\alpha_1^* = O(1)$. At $\mu \ll m_h^{pole}$, the new coupling is also frozen at a new value $\alpha_2^* = O(1)$, $\alpha_2^* > \alpha_1^*$. Hence, in passing from $\mu \ll m_h^{pole}$ to $\mu \sim m_h^{pole}$, the coupling of the lower-energy theory becomes living in the interval $\delta\mu \sim m_h^{pole}$ around $\mu = m_h^{pole}$, where it decreases significantly from α_2^* to α_1^*. This is only possible if the scale factor $\hat{\Lambda}_Q$ of the lower-energy theory is $\hat{\Lambda}_Q \sim m_h^{pole}$.

Therefore, at $\mu < \hat{\Lambda}_Q$, we remain with N_c colors, $N_l > 3N_c/2$ light flavors with the small current mass

$$\hat{m}_l \equiv m_l(\mu = \hat{\Lambda}_Q) = z_Q^{-1}(\Lambda_Q, m_h^{pole})\, m_l \ll \hat{\Lambda}_Q,$$

and the coupling $\alpha(\mu)$ with the scale parameter $\hat{\Lambda}_Q = m_h^{pole}$. Moreover, the value of the diquark condensate of l-flavors is (see (3) and (8))

$$\begin{gathered}\langle(\overline{Q}_{\bar{l}}\, Q^l)_{\mu=\hat{\Lambda}_Q}\rangle \equiv \delta_{\bar{l}}^l \left(\hat{\mathcal{M}}_{ch}^l\right)^2,\\ \hat{\mathcal{M}}_{ch}^l = z_Q^{1/2}(\Lambda_Q,\, m_h^{pole})\, \mathcal{M}_{ch}^l \ll \mathcal{M}_{ch}^l \ll \ll \hat{\Lambda}_Q = m_h^{pole},\end{gathered} \tag{12}$$

$$z_Q(\Lambda_Q,\, m_h^{pole}) = \left(\frac{m_h^{pole}}{\Lambda_Q}\right)^{\gamma_+} = \left(\frac{m_h}{\Lambda_Q}\right)^{\gamma_+/(1+\gamma_+)} \ll 1\,,$$

$$\gamma_+^{conf} = b_0/N_F.$$

The properties of this theory have been described in [1], and it is in the DC_l phase. Its highest physical scale is $\mu'_H = \hat{\mathcal{M}}_{ch}^l \ll \hat{\Lambda}_Q$, and hence it is in the conformal regime at $\hat{\mathcal{M}}_{ch}^l < \mu \ll \hat{\Lambda}_Q$, while below the threshold at $\mu \sim \hat{\mathcal{M}}_{ch}^l$, the quarks Q^l and $\overline{Q}_{\bar{l}}$ acquire the constituent masses $\mu_C^l = \hat{\mathcal{M}}_{ch}^l$ and N_l^2 light pions appear. The low-energy Lagrangian of these pions at $\mu \ll \Lambda_{YM} = (\hat{\Lambda}_Q^{3N_c-N_l} \det \hat{m}_l)^{1/3N_c} = (\Lambda_Q^{3N_c-N_F} \det m)^{1/3N_c}$ is

$$L_\pi = \int d^2\overline{\theta}\, d^2\theta\, \sqrt{\mathrm{Tr}\, \hat{\Pi}_l^\dagger \hat{\Pi}_l} + \int d^2\theta \Biggl\{ -(N_l - N_c) \times \times \left(\frac{\det \hat{\Pi}_l}{\hat{\Lambda}_Q^{3N_c-N_l}}\right)^{1/(N_l-N_c)} + \hat{m}_l\, \mathrm{Tr}\, \hat{\Pi}_l \Biggr\}. \tag{13}$$

The normalization of the pion fields

$$\hat{\Pi}_l \equiv (\overline{Q}_{\bar{l}}\, Q^l)_{\mu=\hat{\Lambda}_Q}, \quad \langle \hat{\Pi}_l \rangle = (\hat{\Lambda}_Q^{3N_c-N_l} \hat{m}_l^{N_l-N_c})^{1/N_c},$$

is the most natural one from the standpoint of the lower-energy theory. But it is also useful to rewrite (13) with the "old normalization" of fields at $\mu = \Lambda_Q$:

$$\Pi_l \equiv (\overline{Q}_{\bar{l}} Q^l)_{\mu=\Lambda_Q},$$

$$\langle \Pi_l \rangle = \frac{\langle S \rangle}{m_l} \equiv \Lambda_{YM}^3/m_l = \frac{1}{m_l}(\Lambda_Q^{b_0} \det m)^{1/N_c}.$$

It then becomes

$$\begin{gathered} L_\pi = \int d^2\overline{\theta}\, d^2\theta \Biggl\{ z_Q(\Lambda_Q,\, m_h^{pole})\, \sqrt{\mathrm{Tr}\, \Pi_l^\dagger \Pi_l} \Biggr\} + \\ + \int d^2\theta \Biggl\{ -(N_l - N_c) \times \\ \times \left(\frac{\det \Pi_l}{\Lambda_Q^{b_0}\, \det m_h}\right)^{1/(N_l-N_c)} + m_l\, \mathrm{Tr}\, \Pi_l \Biggr\}, \\ z_Q(\Lambda_Q, m_h^{pole}) = \left(\frac{m_h}{\Lambda_Q}\right)^{b_0/3N_c} \ll 1. \end{gathered} \tag{14}$$

In this case, on the whole, the mass spectrum includes a) a large number of heaviest hh-hadrons with their mass scale of the order of m_h^{pole}, b) a large number of ll-mesons with masses of the order of $\hat{\mathcal{M}}_{ch}^{(l)}$ made of nonrelativistic quarks Q^l and $\overline{Q}_{\bar{l}}$ with the constituent masses $\hat{\mathcal{M}}_{ch}^{(l)} \ll m_h^{pole}$, c) a large number of hybrid hl-mesons made of the above constituents (all quarks are weakly confined and the string tension is $\sqrt{\sigma} \sim \Lambda_{YM} \ll \hat{\mathcal{M}}_{ch}^l \ll m_h^{pole}$), d) a large number of gluonia with masses $\sim \Lambda_{YM} \equiv (\Lambda_Q^{3N_c-N_F} \det m)^{1/3N_c} \ll \hat{\mathcal{M}}_{ch}^{(l)}$, $\det m \equiv m_l^{N_l} m_h^{N_F-N_l}$, and e) N_l^2 lightest l-pions $\hat{\Pi}_l$ with masses $M_\pi^l \sim \hat{m}_l = z_Q^{-1}(\Lambda_Q,\, m_h^{pole})\, m_l \ll \Lambda_{YM}$.

3.2. $3N_c/2 < N_F < 3N_c$, $N_c < N_l < 3N_c/2$

The difference from the case 3.1 above is that at $\mu < \mu_H = m_h^{pole}$, after the heaviest quarks Q^h, $\overline{Q}_{\bar{h}}$ are integrated out, the lower-energy theory is not in the conformal regime at $\mu'_H < \mu < m_h^{pole}$ but in the strong-coupling regime (see [1]). In other words, its new coupling increases in a power-like fashion at $\mu \ll \mu_H = m_h^{pole}$. This allows determining its new scale parameter Λ' from matching of the couplings at $\mu = \mu_H = m_h^{pole}$, where both are $O(1)$. This is only possible with $\Lambda' = \mu_H = m_h^{pole} \equiv \hat{\Lambda}_Q$. Therefore, at $\mu < \hat{\Lambda}_Q$, we remain with N_c colors, $N_c < N_l < 3N_c/2$ light flavors with the current masses $\hat{m}_l \equiv m_l(\mu = \hat{\Lambda}_Q) \ll \hat{\Lambda}_Q$, and the coupling with the scale parameter $\hat{\Lambda}_Q$. All this is exactly as it was in the case 3.1 above, only the value of N_l is now smaller.

As was explained in [1], only the perturbative behavior in the interval of scales $\mu'_H = \hat{\mathcal{M}}^l_{ch} < \mu < \hat{\Lambda}_Q$ differs in this case from the conformal behavior in the case 3.1 above, while at $\mu < \hat{\mathcal{M}}^l_{ch}$, all properties and mass spectra are the same. In particular, the lowest-energy pion Lagrangian is the same as in (13) and (14).

3.3. $N_c < N_F < 3N_c/2$, $N_c < N_l < N_F$

In this case, the original theory (at $\mu_H = m_h^{pole} < \mu < \Lambda_Q$) and the lower-energy theory (at $\mu < \mu_H$) are both in the strong-coupling regime. Their couplings $\alpha_\pm(\mu)$ are to be matched at $\mu = \mu_H = m_h^{pole}$, where m_h^{pole} is the pole mass of the Q^h and $\overline{Q}_{\bar{h}}$ quarks: $m_h^{pole} = m_h(\mu = m_h^{pole})$. Because $m_h^{pole} \ll \Lambda_Q$, the upper (i. e., original) coupling $\alpha_+(\mu = m_h^{pole})$ is parametrically large, and so is $\alpha_-(\mu = m_h^{pole})$. It is therefore clear that its scale parameter $\Lambda' \gg m_h^{pole}$.

To obtain definite expressions, we make a (sufficiently weak) assumption that at $N_c < N_F < 3N_c/2$, the quark perturbative anomalous dimension γ_Q is constant in the infrared region. Then the quark renormalization factor $z_Q^+(\Lambda, \mu)$ and the coupling $a_+ \equiv N_c\alpha_+/2\pi$ of the higher-energy theory, as well as $z_Q^-(\Lambda', \mu)$ and $a_- \equiv N_c\alpha_-/2\pi$ of the lower-energy one, behave as [1]

$$\begin{gathered}
z_Q^+(\Lambda_Q, \mu) = \left(\frac{\mu}{\Lambda_Q}\right)^{\gamma_+} \ll 1, \\
a_+(\mu) = \left(\frac{\Lambda_Q}{\mu}\right)^{\nu_+} \gg 1, \\
\nu_+ = \frac{N_F\gamma_+ - b_0}{N_c} > 0, \quad \mu \ll \Lambda_Q, \\
z_Q^-(\Lambda', \mu) = \left(\frac{\mu}{\Lambda'}\right)^{\gamma_-} \ll 1, \\
a_-(\mu) = \left(\frac{\Lambda'}{\mu}\right)^{\nu_-} \gg 1, \\
\nu_- = \frac{N_l\gamma_- - b'_0}{N_c} > \nu_+, \quad \mu \ll \Lambda', \\
b_0 = 3N_c - N_F, \quad b'_0 = 3N_c - N_l,
\end{gathered} \tag{15}$$

and the matching of the couplings at $\mu = \mu_H = m_h^{pole}$ takes the form

$$\begin{gathered}
\left(\frac{\Lambda_Q}{m_h^{pole}}\right)^{\nu_+} = \left(\frac{\Lambda'}{m_h^{pole}}\right)^{\nu_-}, \\
m_h^{pole} = \frac{m_h}{z_Q^+(\Lambda_Q, m_h^{pole})} = \\
= m_h \left(\frac{\Lambda_Q}{m_h}\right)^{\gamma_+/(1+\gamma_+)} \gg m_h.
\end{gathered} \tag{16}$$

Because $\nu_- > \nu_+ > 0$, it follows from (16) that $m_h^{pole} \ll \Lambda' \ll \Lambda_Q$ [5)].

Therefore, after the heaviest quarks Q^h and $\overline{Q}_{\bar{h}}$ are integrated out at $\mu < m_h^{pole}$, we have N_c colors, $N_c < N_l < 3N_c/2$ flavors, and the gauge coupling with the scale parameter Λ', $m_h^{pole} \ll \Lambda' \ll \Lambda_Q$, determined from (16). The value of the current mass m'_l of the Q^l and $\overline{Q}_{\bar{l}}$ quarks and the pion fields $(\Pi'_l)^{\bar{l}} \equiv (\overline{Q}_{\bar{l}} Q^l)^{(light)}_{\mu=\Lambda'}$ normalized at $\mu = \Lambda'$ are given by

$$\begin{gathered}
m'_l \equiv m_l(\mu = \Lambda') = z_Q^-(\Lambda', m_h^{pole})\, \hat{m}_l, \\
z_Q^-(\Lambda', m_h^{pole}) = \left(\frac{m_h^{pole}}{\Lambda'}\right)^{\gamma_-} \ll 1, \\
\hat{m}_l = m_l \left(\frac{\Lambda_Q}{m_h}\right)^{\gamma_+/(1+\gamma_+)}, \quad \langle(\Pi'_l)^i_{\bar{j}}\rangle \equiv \delta^i_{\bar{j}}\Pi'_l, \\
\Pi'_l = \frac{1}{z_Q^-(\Lambda', m_h^{pole})}\, \hat{\Pi}_l, \\
\hat{\Pi}_l = \Pi_l\, z_Q^+(\Lambda_Q, m_h^{pole}) = \Pi_l \left(\frac{m_h}{\Lambda_Q}\right)^{\gamma_+/(1+\gamma_+)}.
\end{gathered} \tag{17}$$

Therefore [1], the low-energy pion Lagrangian has form (13) with the replacements $\hat{\Pi}_l \to \Pi'_l$, $\hat{m}_l \to m'_l$, and $\hat{\Lambda} \to \Lambda'$. The pion mass is now m'_l. Being expressed through the pion fields Π_l normalized at $\mu = \Lambda_Q$, the superpotential has the universal form (14), and only the Z^l_π factor multiplying the Kähler term of l-pions is different, being given by

$$\begin{gathered}
Z^l_\pi = \left(\frac{m_h}{\Lambda_Q}\right)^{\Delta}, \quad \Delta = \frac{\delta}{1+\gamma_+}, \\
\delta = \left[\gamma_+ - \gamma_- \left(\frac{\nu_+}{\nu_-}\right)\right], \quad m'_l = \frac{m_l}{Z^l_\pi}.
\end{gathered} \tag{18}$$

Hence, in the case considered, the mass spectrum contains a) the heaviest h-hadrons with the mass scale of the order of m_h^{pole} given by (16); b) the ll-mesons made of the nonrelativistic quarks Q^l and $\overline{Q}_{\bar{l}}$ with the constituent masses $\mu'_C = \langle\Pi'_l\rangle^{1/2}$ (17); c) the hybrid hl-mesons made of the above constituents; d) the gluonia with the universal mass scale Λ_{YM}; and d) N_l^2 lightest l-pions with the masses $m'_l \ll \Lambda_{YM}$ (see (18)).

On the whole, the hierarchy of scales in the mass spectrum is always the same for this regime of the DC_l–HQ_h phase with $N_l > N_c$:

a) the largest masses are the pole masses $m_h^{pole} \ll \Lambda_Q$ of the Q^h and $\overline{Q}_{\bar{h}}$ quarks;

5) As a specific example, we can use the values from [1]: $\gamma_+ = (2N_c - N_F)/(N_F - N_c)$, $\gamma_- = (2N_c - N_l)/(N_l - N_c)$, $\nu_+ = (3N_c - 2N_F)/(N_F - N_c)$, and $\nu_- = (3N_c - 2N_l)/(N_l - N_c)$. The value of Δ in (18) is $0 < \Delta = (N_F - N_l)/(3N_c - 2N_l) < 1/2$ in this case.

b) the next ones are the constituent masses μ^l_C of the Q^l and $\overline{Q}_{\bar l}$ quarks, which are always much smaller than m^{pole}_h, although their concrete values depend on the case considered;

c) the next one is the universal mass scale of gauge particles, which is always given by $\Lambda_{YM} = (\Lambda_Q^{b_0}\det m)^{1/3N_c}$;

d) the lightest are the N_l^2 l-pions, whose low-energy Lagrangian has the universal form (14), but the value of the Z^l_π factor in front of the Kähler term (and hence their mass M^l_π) depends on the case considered.

4. THE DC$_l$–HQ$_h$ PHASE: $\mathcal{M}^h_{ch} \ll m^{pole}_h \ll \mathcal{M}^l_{ch}$

We now consider the most difficult regime with $r_1' \ll r \ll r_1$, i. e., $\mathcal{M}^h_{ch} \ll m^{pole}_h \ll \mathcal{M}^l_{ch}$.

We trace the RG flow when the running scale μ starts at $\mu = \Lambda_Q$ and decreases. As was argued in Sec. 2, even a large value of the running coherent condensate $\mathcal{M}^l_{ch}(\mu)$ does not necessarily mean that the large constituent mass $\mathcal{M}^l_{ch}(\mu)$ of the Q^l and $\overline{Q}_{\bar l}$ quarks is already formed, because the projector $\mathcal{P}$ in (4) becomes nonzero only after the decreasing μ reaches a value μ_2 such that both flavors, l and h, entering $\det(\overline{Q}Q)$ acquire masses larger than μ_2 and become frozen. Therefore, the first point where this can happen in the DC$_l$–HQ$_h$ phase with $\mathcal{M}^h_{ch} \ll m^{pole}_h \ll \mathcal{M}^l_{ch}$ is the pole mass m^{pole}_h. Hence, there is a narrow threshold region $\mu_2 = m^{pole}_h/(\text{several}) < \mu < \mu_1 = (\text{several})\, m^{pole}_h$ around m^{pole}_h where the nonperturbative effects turn on at μ_1 and saturate at μ_2. In a sense, what is occurring in this transition region is qualitatively similar to what was described in Sec. 2 for the DC$_l$–DC$_h$ phase, but with the role of the coherent condensate $\mathcal{M}^h_{ch}$ of the Q^h and $\overline{Q}_{\bar h}$ quarks now played by their perturbative pole mass m^{pole}_h. Therefore, all flavors become frozen in the threshold region $\mu_2 < \mu < \mu_1$ around m^{pole}_h. For the Q^h and $\overline{Q}_{\bar h}$ quarks, this is because their evolution is stopped by their pole mass m^{pole}_h, and for the Q^l and $\overline{Q}_{\bar l}$ quarks, because their large constituent mass $\mathcal{M}^l_{ch} \gg m^{pole}_h$ is formed in this threshold region.

What form does the superpotential take at $\mu < m^{pole}_h$, after the nonperturbative RG flow terminates and all quark masses become frozen? (The heaviest are the constituent Q^l and $\overline{Q}_{\bar l}$ quarks with the mass $\mathcal{M}^l_{ch}$, the next ones are the Q^h and $\overline{Q}_{\bar h}$ quarks with the mass m^{pole}_h, and the lightest are the pions Π_l with the mass m_l, plus all gluons, which are still massless). We consider the superpotential in (1) or (5). Because there are only Π_l-pions, while the h-quarks are in the HQ phase and there is no difference between $(\overline{Q}_{\bar h}Q^h)$ and $\Pi^{h}_{\bar h}$, the h-quark contributions cancel in the projector $\mathcal{P} = \mathrm{Tr}\,(\overline{Q}\Pi^{-1}Q) - N_F$, and it takes the form $\mathcal{P} = \mathrm{Tr}\,(\overline{Q}_{\bar l}\Pi_l^{-1}Q^l) - N_l$. Now, what form can $\det\Pi$ in (1) or (5) take at $\mu < m^{pole}_h$, after the evolution of all quark degrees of freedom terminates and the Q^h and $\overline{Q}_{\bar h}$ quarks are integrated out? In other words, what their fields $\Pi^{h}_{\bar h} = (\overline{Q}_{\bar h}Q^h)$ are substituted by in $\det\Pi$? The only possible form is[6)]

$$\Pi^{h}_{\bar h} = \left(\overline{Q}_{\bar h}Q^h\right) \to \to \left(m^{-1}\right)^{h}_{\bar h}\left(\frac{\det\Pi_l}{\Lambda_Q^{b_0}\det m_h}\right)^{1/(N_l-N_c)}, \quad \left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/(N_F-N_c)} \to \to \left(\frac{\det\Pi_l}{\Lambda_Q^{b_0}\det m_h}\right)^{1/(N_l-N_c)}. \tag{19}$$

Hence, instead of (5), the effective Lagrangian at $\mu < m^{pole}_h$ takes the form (we recall that all fields entering (1), (5), and (20) are normalized at $\mu = \Lambda_Q$)

$$L = \int d^2\theta\, d^2\overline{\theta}\left\{\mathrm{Tr}\sqrt{\Pi_l^\dagger\Pi_l} + Z_l\,\mathrm{Tr}_l\left(Q^\dagger e^V Q\right) + + Z_h\mathrm{Tr}_h\left(Q^\dagger e^V Q\right) + \left(Q \to \overline{Q}\right)\right\} + + \int d^2\theta\left\{-\frac{2\pi}{\alpha(\mu)}\,S + W_Q\right\}, \quad Z_l = \frac{m_l}{\mathcal{M}^l_{ch}}, \quad Z_h = \frac{m_h}{m^{pole}_h}, \tag{20}$$

$$m^{pole}_h = \frac{m_h}{z^+_Q(\Lambda_Q, m^{pole}_h)} = m_h\left(\frac{\Lambda_Q}{m_h}\right)^{\gamma_+/(1+\gamma_+)} \gg m_h,$$

[6)] The form given in (19) is determined uniquely by the symmetries a) the flavor symmetry $SU(N_l)_L\times SU(N_l)_R\times SU(N_h)_L\times \times\, SU(N_h)_R$; b) the R-charges of the higher-energy theory, $R(Q^l) = R(\overline{Q}_{\bar l}) = R(Q^h) = R(\overline{Q}_{\bar h}) = R(\Pi_l)/2 = (N_F - N_c)/N_F$ and $R(m_l) = R(m_h) = 2N_c/N_F$; c) the R'-charges of the lower-energy theory, $R'(Q^l) = R'(\overline{Q}_{\bar l}) = R'(\Pi_l)/2 = (N_l - N_c)/N_l$, $R'(Q^h) = R'(\overline{Q}_{\bar h}) = 1$, $R'(m_l) = 2N_c/N_l$, and $R'(m_h) = 0$. The overall normalization in (19) is determined by the Konishi anomaly $m_h\langle\overline{Q}^hQ_h\rangle = \langle S\rangle$ (see also (2)).

$$W_Q = \left(\frac{\det \Pi_l}{\Lambda_Q^{b_0} \det m_h}\right)^{1/(N_l-N_c)} \times \\ \times \left\{ \mathrm{Tr}_l \left(\overline{Q}\, \Pi_l^{-1}\, Q\right) - N_l \right\} + \\ + m_h \mathrm{Tr}_h \left(\overline{Q}Q\right) + m_l \mathrm{Tr}\left(\Pi_l\right).$$

Equation (20) has the same meaning as Eqs. (1) or (5). All terms with the quark fields are retained only to keep track of the values of their masses, and it is implied in addition that they can be used, for instance, for some calculations where these quarks appear as valence ones. If one is not interested in all this at $\mu < m_h^{pole}$, all quark terms in (20) can be omitted.

We now write the explicit form of the inverse Wilsonian coupling $2\pi/\alpha_W(\mu)$ [4] in (20). It is simplest to write the result of the overall RG flow from $\mu = \Lambda_Q$ down to $\mu_2 = m_h^{pole}/(\text{several})$ (because the RG is a group). We thus obtain

$$\frac{2\pi}{\alpha_W(\mu_2)} = N_c \ln\Big(\frac{\mu_2^3}{\Lambda_Q^3}\Big) - \ln\Big(\frac{\det \mu_C^l}{\Lambda_Q^{N_l}}\Big) - \\ - N_h \ln\Big(\frac{m_h^{pole}}{\Lambda_Q}\Big) + N_l \ln\Big(\frac{1}{Z_l}\Big) + N_h \ln\Big(\frac{1}{Z_h}\Big). \quad (21)$$

In (21), the specific properties of the case considered are as follows: a) the Z_h factor of the Q^h and $\overline{Q}_{\bar{h}}$ quarks is $Z_h = m_h/m_h^{pole}$ because their mass $m_h(\mu)$ started with the value m_h at $\mu = \Lambda_Q$ and finished with the value m_h^{pole} at $\mu = \mu_2$; b) the constituent mass μ_C^l of the Q^l and $\overline{Q}_{\bar{l}}$ quarks in (21) has the form (see (20))

$$\left(\mu_C^l\right)_i^{\bar{j}} = \frac{1}{Z_l}\left(\frac{\det \Pi_l}{\Lambda_Q^{b_0} \det m_h}\right)^{1/(N_l-N_c)} \left(\Pi_l^{-1}\right)_i^{\bar{j}}. \quad (22)$$

Therefore, the coupling in (20) at $\Lambda_{YM} \ll \mu < \mu_2 = m_h^{pole}/(\text{several})$ is weak and is given by (see also Sec. 2 in [1])

$$\frac{2\pi}{\alpha_W\left(\mu, \Lambda_L\right)} = \frac{2\pi}{\alpha\left(\mu, \Lambda_L\right)} - \\ - N_c \ln \frac{1}{g^2(\mu, \langle\Lambda_L\rangle)} = 3N_c \ln\Big(\frac{\mu}{\Lambda_L}\Big), \\ \Lambda_L^3 = \left(\frac{\det \Pi_l}{\Lambda_Q^{b_0} \det m_h}\right)^{1/(N_l-N_c)}, \\ \langle \Lambda_L\rangle = \left(\Lambda_Q^{b_0} \det m\right)^{1/3N_c} = \Lambda_{YM}, \quad (23)$$

and the Lagrangian at $\mu < \mu_2$ is

$$L = \int d^2\theta\, d^2\overline{\theta} \left\{ \mathrm{Tr} \sqrt{\Pi_l^{\dagger} \Pi_l} \right\} + \int d^2\theta\, \times \\ \times \left\{ -\frac{2\pi}{\alpha(\mu, \Lambda_L)}\, S - N_l \left(\frac{\det \Pi_l}{\Lambda_Q^{b_0} \det m_h}\right)^{1/(N_l-N_c)} + \\ + \mathrm{Tr}\left(m_l \Pi_l\right)\right\}. \quad (24)$$

It describes gluonia with the universal mass scale $M_{gl} \sim \Lambda_{YM}$ (coupled to the pions Π_l), and after integrating them out via the VY procedure [3], we finally obtain the lowest-energy Lagrangian of pions

$$L = \int d^2\theta\, d^2\overline{\theta} \left\{ \mathrm{Tr} \sqrt{\Pi_l^{\dagger} \Pi_l} \right\} + \int d^2\theta\, \times \\ \times \left\{ -(N_l - N_c)\left(\frac{\det \Pi_l}{\Lambda_Q^{b_0} \det m_h}\right)^{1/(N_l-N_c)} + \\ + \mathrm{Tr}\left(m_l \Pi_l\right)\right\}. \quad (25)$$

It describes N_l^2 l-pions Π_l with the masses of the order of m_l, and their superpotential has the standard universal form for the DC_l–HQ_h phase (see (14)).

In this case, on the whole, the mass spectrum includes a) the ll-hadrons made of the Q^l and $\overline{Q}_{\bar{l}}$ quarks with the constituent mass $\mu_C^l = \mathcal{M}_{ch}^l \ll \Lambda_Q$, b) the hh-hadrons made of the nonrelativistic Q^h and $\overline{Q}_{\bar{h}}$ quarks with the pole mass $m_h^{pole} \ll \mathcal{M}_{ch}^l$, c) the hybrid hl-hadrons made of the above constituents (all quarks are weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll m_h^{pole} \ll \mathcal{M}_{ch}^l$), d) the gluonia with their universal mass scale $M_{gl} \sim \Lambda_{YM} \ll m_h^{pole}$, and e) N_l^2 lightest l-pions with the mass of the order of $m_l \ll \Lambda_{YM}$ and superpotential (25), which is universal for the DC_l–HQ_h phase. In a sense, this mass spectrum is similar to those described in Sec. 3, the main difference being that the hierarchy $m_h^{pole} \gg \mu_C^l$ in Sec. 3 is reversed here.

We finally consider how the mass spectrum changes on both sides of the phase transition at $r \sim r_1$, with $\mathcal{M}_{ch}^h \sim m_h^{pole} \ll \mathcal{M}_{ch}^l$ (see (8) and Sec. 2).

a. The h-flavors. In the DC_l–DC_h phase at $r > r_1$, there are many heavy h-hadrons made of the nonrelativistic Q^h and $\overline{Q}_{\bar{h}}$ quarks with the constituent mass $\mathcal{M}_{ch}^h$ (see (3)), and N_h^2 light h-pions with the mass of the order of m_h. As r crosses r_1, the coherent condensate of the h-flavors breaks down and the theory enters the DC_l–HQ_h phase. The above N_h^2 light h-pions with the mass of the order of m_h disappear from the mass spectrum. At the same time, because

$\mathcal{M}^h_{ch} = m^{pole}_h$, the constituent masses $\mathcal{M}^h_{ch}$ of Q^h and $\overline{Q}_{\bar h}$ quarks are substituted smoothly by their perturbative pole masses m^{pole}_h, such that the mass spectrum of the heavy h-hadrons, now made of the nonrelativistic current quarks Q^h and $\overline{Q}_{\bar h}$, changes smoothly.

b. The l-flavors. In the DC_l–DC_h phase at $r > r_1$, there are many heavy l-hadrons made of the Q^l and $\overline{Q}_{\bar l}$ quarks with the constituent mass $\mathcal{M}^l_{ch} \gg \mathcal{M}^h_{ch}$ (see (2)), and N_l^2 lightest l-pions with the mass of the order of $m_l \ll m_h$. In the DC_l–HQ_h phase with $\mathcal{M}^l_{ch} \gg m^{pole}_h$ at $r < r_1$, all these l-hadrons and the N_l^2 l-pions are still present in the spectrum and their masses remain the same.

c. The hybrid hl-flavors. In the DC_l–DC_h phase at $r > r_1$, there are many heavy hl-mesons with the mass $\mathcal{M}^h_{ch} + \mathcal{M}^l_{ch}$ and the corresponding hl-pions with the small mass $m_h + m_l$. In the DC_l–HQ_h phase at $r < r_1$, these light hybrid pions are absent. As regards the heavy hybrid mesons, their masses change smoothly from $\mathcal{M}^h_{ch} + \mathcal{M}^l_{ch}$ to $m^{pole}_h + \mathcal{M}^l_{ch}$.

d. Finally, all gluons remain massless down to the scale $\mu \sim \Lambda_{YM}$, and there is a large number of gluonia with the same mass $M_{gl} \sim \Lambda_{YM}$ in both phases.

5. THE HIGGS$_l$–DC$_h$ AND HIGGS$_l$–HQ$_h$ PHASES. $\mathcal{M}^l_{ch} > \Lambda_Q$, $N_l < N_c - 1$

There are only two different phases at $N_l > N_c$ (because the condition $\mathcal{M}^l_{ch} \ll \Lambda_Q$ is always satisfied and the lighter quarks are never Higgsed), DC_l–DC_h and DC_l–HQ_h.

At $N_l < N_c$, in addition to the above two phases, two new phases appear at $\mathcal{M}^l_{ch} > \Lambda_Q$, when the lighter l-quarks are Higgsed, $\langle Q^l\rangle = \langle \overline{Q}_{\bar l}\rangle \neq 0$, while the heavier quarks are either in the DC phase or in the HQ phase.

We therefore take $r \ll r_2$ (see (9); the value of r must not be too small, see below) and find the mass spectrum in this phase. We can proceed in a close analogy with the case of the Higgs phase for $N_F < N_c - 1$ in [1], the only difference being that not all flavors are now Higgsed (only Q^l and $\overline{Q}_{\bar l}$ are).

We hence begin with the scale of the large gluon mass,

$$\mu = \mu_{gl} = g_H \hat{\mathcal{M}}^l_{ch} \gg \Lambda_Q, \quad g_H^2 = 4\pi\alpha(\mu = \mu_{gl}) \ll 1,$$

$$\langle Q^l_a\rangle_{\mu=\mu_{gl}} = \delta^l_a\, \hat{\mathcal{M}}^l_{ch}, \ \ \langle \overline{Q}^a_{\bar l}\rangle_{\mu=\mu_{gl}} = \delta^a_{\bar l}\, \hat{\mathcal{M}}^l_{ch}.$$

The gauge symmetry $SU(N_c)$ is broken down to $SU(N_c - N_l)$ at this high scale $\mu_H = \mu_{gl}$ and $2N_lN_c - N_l^2$ gluons become massive. The same number of the degrees of freedom of the Q^l and $\overline{Q}_{\bar l}$ quarks acquire the same mass and become superpartners of these massive gluons (in a sense, they can be considered the heavy "constituent quarks"), and there remain N_l^2 light complex pion fields $\hat\Pi_l = (\overline{Q}_{\bar l}Q^l)_{\mu=\mu_{gl}}$, $\langle\hat\Pi_l\rangle = (\hat{\mathcal{M}}^l_{ch})^2$. The value of $m_l(\mu)$ at this scale is $\hat m_l \equiv m_l(\mu = \mu_{gl})$ (this is to become the l-pion mass), and similarly, the mass of h-quarks at this scale is $\hat m_h \equiv m_h(\mu = \mu_{gl})$. Besides, we let $\hat\Pi_{hl}$ and $\hat\Pi_{lh}$ denote the hybrids (in essense, these are the h-quark fields Q^h_a and $\overline{Q}^a_{\bar h}$ with broken colors $a = 1, \ldots, N_l$), while Q^h and $\overline{Q}_{\bar h}$ are still the active h-quark fields with unbroken colors.

We first consider the case $N_l < b_0/2$, i.e., $b_0' = b_0 - 2N_l > 0$. After integrating out all heaviest particles with masses of the order of μ_{gl} and proceeding in the same way as in [1], we obtain the lower-energy Lagrangian at the scale $\mu \lesssim \mu_{gl}$:

$$L = \int d^2\theta\, d^2\overline{\theta} \left\{ 2\,\mathrm{Tr}\sqrt{\hat\Pi_l^\dagger \hat\Pi_l} + \right.$$
$$+ \mathrm{Tr}_h\left(\hat Q^\dagger e^{\hat V}\hat Q + \left(\hat Q \to \hat{\overline{Q}}\right)\right) +$$
$$\left. + \mathrm{Tr}\left(\hat\Pi^\dagger_{hl}\hat\Pi_{hl} + \hat\Pi^\dagger_{lh}\hat\Pi_{lh}\right) + \ldots \right\} +$$
$$+ \int d^2\theta \left\{ -\frac{2\pi}{\alpha(\mu,\hat\Lambda)}\hat S + \hat m_l \mathrm{Tr}\,\hat\Pi_l + \right.$$
$$\left. + \hat m_h \mathrm{Tr}_h\left(\hat{\overline{Q}}\hat Q\right) + \hat m_h \mathrm{Tr}\left(\hat\Pi_{lh}\hat\Pi_{hl}\right)\right\},$$

$$\hat\Lambda^{b_0'} = \frac{\Lambda_Q^{b_0}}{z_Q^{N_l}\det\hat\Pi_l}\left(\frac{z_Q'}{z_Q}\right)^{N_F - N_l}, \quad b_0' = b_0 - 2N_l > 0, \tag{26}$$

$$z_Q = z_Q(\mu_{gl}, \Lambda_Q \,|\, N_c, N_F) = \frac{\hat m_l}{m_l}, \quad z_Q' = z_Q(\mu_{gl}, \langle\hat\Lambda\rangle \,|\, N_c - N_l, N_F - N_l) = \frac{\hat m_h}{m_h'}. \tag{27}$$

Here, $\hat S = \hat W_\alpha^2/32\pi^2$, $\hat W_\alpha$ are the gauge field strengths of the remaining $(N_c - N_l)^2 - 1$ massless gluon fields, $\alpha(\mu,\hat\Lambda)$ is the gauge coupling of this lower-energy theory and $\hat\Lambda$ is its scale parameter, $z_Q \ll 1$ is the massless quark renormalization factor from $\mu = \mu_{gl}$ down to $\mu = \Lambda_Q$ in the original theory with N_c colors and N_F flavors, $z_Q' \ll 1$ is the analogous renormalization factor from $\mu = \mu_{gl}$ down to $\mu = \langle\hat\Lambda\rangle$ in the lower-energy theory with $N_c - N_l$ colors and N_h remaining active h-flavors Q^h and $\overline{Q}_{\bar h}$, and $m_h' \ll \langle\hat\Lambda\rangle$ is the current mass of the Q^h and $\overline{Q}_{\bar h}$ quarks in this lower-energy theory at $\mu = \langle\hat\Lambda\rangle$[7]. All fields in (26) are normalized at $\mu = \mu_{gl}$. Finally, the dots in (26) denote residual D-term interactions, which are supposed to play no significant role in the case considered in this section and are neglected in what follows.

Therefore, the hybrids $\hat\Pi_{hl}$ and $\hat\Pi_{lh}$ appear in the spectrum as (weakly interacting) particles with the mass $\hat m_h$. These hybrids are not written explicitly (but are understood) below.

The lower-energy theory with $N_c' = N_c - N_l$ colors, $N_F' = N_h$ flavors of the active Q^h and $\overline{Q}_{\bar h}$ quarks, with $b_0' > 0$ and $m_h' \ll \langle\hat\Lambda\rangle$, is in the DC$_h$ phase [1]. The constituent mass $\mu_C^h = (z_Q')^{1/2}\hat{\mathcal M}_{ch}^h \ll \langle\hat\Lambda\rangle$ is formed in the threshold region $\mu \sim \mu_C^h$, and N_h^2 h-pions Π_h', $\langle(\Pi_h')^{i}_{\bar j}\rangle = \delta^i_{\bar j}(\mu_C^h)^2$, appear with masses m_h'. After integrating out these constituent h-quarks, we are left with the Yang–Mills theory with $N_c' = N_c - N_l$ colors and the new scale factor Λ_L of the gauge coupling, and with N_h^2 h-pions [1]:

$$L = \int d^2\theta\, d^2\overline{\theta}\left\{2\mathrm{Tr}\sqrt{\hat\Pi_l^\dagger\hat\Pi_l} + \mathrm{Tr}\sqrt{(\Pi_h')^\dagger\Pi_h'}\right\} + \int d^2\theta\left\{-\frac{2\pi}{\alpha(\mu,\Lambda_L)}\hat S - N_F'\left(\frac{\det\Pi_h'}{\hat\Lambda^{b_0'}}\right)^{1/(N_F'-N_c')} + \hat m_l\mathrm{Tr}\,\hat\Pi_l + m_h'\mathrm{Tr}\,\Pi_h'\right\}, \tag{28}$$

$$\Lambda_L^3 = \left(\frac{\det\Pi_h'}{\hat\Lambda^{b_0'}}\right)^{1/(N_F'-N_c')},$$
$$\langle\Lambda_L\rangle = \Lambda_{YM} = \left(\Lambda_Q^{b_0}\det m\right)^{1/3N_c},$$
$$\det m = m_l^{N_l}\, m_h^{N_F-N_l}.$$

Follwing the VY procedure, we obtain the lowest-energy Lagrangian of pions

$$L = \int d^2\theta\, d^2\overline{\theta}\left\{2\,\mathrm{Tr}\sqrt{\hat\Pi_l^\dagger\hat\Pi_l} + \mathrm{Tr}\sqrt{(\Pi_h')^\dagger\Pi_h'}\right\} + \int d^2\theta\left\{-(N_F'-N_c')\left(\frac{\det\Pi_h'}{\hat\Lambda^{b_0'}}\right)^{1/(N_F'-N_c')} + \hat m_l\mathrm{Tr}\,\hat\Pi_l + m_h'\mathrm{Tr}\,\Pi_h'\right\}. \tag{29}$$

[7] Both z_Q and z_Q' are only logarithmic in the case considered.

With $\hat\Lambda$ in (26), this becomes

$$L = \int d^2\theta\, d^2\overline{\theta}\left\{2\,\mathrm{Tr}\sqrt{\hat\Pi_l^\dagger\hat\Pi_l} + \mathrm{Tr}\sqrt{(\Pi_h')^\dagger\Pi_h'}\right\} + \int d^2\theta\left\{-(N_F-N_c)\left(\frac{z_Q^{N_F}\det\hat\Pi_l\det\Pi_h'}{\Lambda_Q^{b_0}\,(z_Q')^{N_F-N_l}}\right)^{1/(N_F-N_c)} + \hat m_l\mathrm{Tr}\,\hat\Pi_l + m_h'\mathrm{Tr}\,\Pi_h'\right\}. \tag{30}$$

The Lagrangian (30) (with the hybrid pions $\hat\Pi_{hl}$ and $\hat\Pi_{lh}$ reinstalled), being expressed in terms of the fields Π_l, Π_h, Π_{hl}, and Π_{lh} and the masses m_l and m_h normalized at the "old scale" $\mu = \Lambda_Q$, takes the form

$$L = \int d^2\theta\, d^2\overline{\theta}\left\{\frac{2}{z_Q}\mathrm{Tr}\sqrt{\Pi_l^\dagger\Pi_l} + \frac{z_Q'}{z_Q}\mathrm{Tr}\sqrt{\Pi_h^\dagger\Pi_h} + \frac{1}{z_Q}\mathrm{Tr}\left(\Pi_{hl}^\dagger\Pi_{hl} + \Pi_{lh}^\dagger\Pi_{lh}\right) + \ldots\right\} + \int d^2\theta \times \left\{-(N_F-N_c)\left(\frac{\det\Pi_l\det\Pi_h}{\Lambda_Q^{b_0}}\right)^{1/(N_F-N_c)} + m_l\,\mathrm{Tr}\,\Pi_l + m_h\,\mathrm{Tr}\left(\Pi_h + \Pi_{hl}\Pi_{lh}\right)\right\}, \tag{31}$$

$$\Pi_h' = \frac{z_Q'}{z_Q}\Pi_h,\quad \hat\Pi_l = \frac{1}{z_Q}\Pi_l,\quad m_l = \frac{\hat m_l}{z_Q},$$
$$m_h' = \frac{\hat m_h}{z_Q'} = \frac{z_Q}{z_Q'}m_h.$$

On the whole in this case, when theory is deeply in the Higgs$_l$–DC$_h$ phase (i.e., when $\hat{\mathcal M}_{ch}^l \gg \Lambda_Q$), the mass spectrum is as follows. There are

a) $2N_lN_c - N_l^2$ massive gluons and the same number of their superpartners (the "constituent l-quarks" with heaviest masses $\mu_{gl} \gg \Lambda_Q$);

b) a large number of hadrons made of nonrelativistic constituent Q^h and $\overline{Q}_{\bar h}$ quarks with masses of the order of $\mu_C^h \ll \langle\hat\Lambda\rangle \ll \Lambda_Q \ll \mu_{gl}$;

c) a large number of strongly coupled gluonia with the mass scale $M_{gl} \sim \Lambda_{YM} \ll \mu_C^h$;

d) N_h^2 h-pions with masses of the order of $m_h' \ll \Lambda_{YM}$;

e) the hybrid pions Π_{hl} and Π_{lh} (which are Q^h and $\overline{Q}_{\bar h}$ quarks with Higgsed colors) with masses $\hat m_h \ll m_h'$;

f) N_l^2 lightest l-pions with masses $\hat m_l \ll \hat m_h$.

At $N_l < N_c - 1$, starting with $r \equiv m_l/m_h = 1$, when all quarks are in the DC phase, a number of phase transitions occurs as r decreases. The DC_l–DC_h phase is maintained until (10) is fulfilled.

We take $N_l < N_0$ (see (10)). Then, as r approaches r_2 from above, $\mathcal{M}^l_{ch}$ approaches Λ_Q from below, with all quarks being in the DC_l–DC_h phase. When $\mathcal{M}^l_{ch}$ exceeds Λ_Q, a phase transition occurs as the l-quarks become Higgsed. The crucial parameter here (i.e., at $\mathcal{M}^l_{ch} > \Lambda_Q$, but not too large, see below) is $b'_0 = 3N'_c - N'_F = b_0 - 2N_l$. The Q^h and $\overline{Q}_{\bar h}$ quarks are in the DC_h phase at $b'_0 > 0$, and in the HQ_h phase at $b'_0 < 0$. If $N_l < N_0$, then also $N_l < b_0/2$, and hence as $\mathcal{M}^l_{ch}$ exceeds Λ_Q and the l-quarks are Higgsed, the DC_h phase of h-quarks is maintained.

We trace how the mass spectrum changes on both sides of this phase transition between the DC_l–DC_h and Higgs_l–DC_h phases at $r \sim r_2 \ll 1$ (see Sec. 2).

a. The gluons. In the DC_l–DC_h phase at $r < r_2$, all the $N_c^2 - 1$ gluons can be thought of as having the small mass $M_{gl} \sim \Lambda_{YM}$. In the Higgs_l–DC_h phase at $r > r_2$, the $SU(N_c)$ gauge symmetry is broken down to the non-Abelian $SU(N_c - N_l)$ one, with $N_l < N_c - 1$. Hence, $2N_lN_c - N_l^2$ gluons acquire the large mass $M_{gl} \sim \Lambda_Q \gg \Lambda_{YM}$, while $(N_c - N_l)^2 - 1$ gluons remain with the same small masses of the order of Λ_{YM}.

b. The *ll*-flavors. In the DC_l–DC_h phase at $r < r_2$, the confined Q^l and $\overline{Q}_{\bar l}$ quarks have large constituent masses $\mu^l_C = \mathcal{M}^l_{ch} \sim \Lambda_Q$, and there are N_l^2 light ll-pions with small masses $M^l_\pi \sim m_l$. In the Higgs_l–DC_h phase at $r > r_2$, there are $2N_lN_c - N_l^2$ massive quarks that are superpartners of the massive gluons and hence have the same masses of the order of Λ_Q. In a sense, these quarks can be considered remnants of the previous constituent l-quarks, and their masses match smoothly across $r \sim r_2$. As regards the ll-pions, their number and masses also match smoothly across the phase transition.

c. The *hh*-flavors. Nothing happens to the confined constituent Q^h and $\overline{Q}_{\bar h}$ quarks (i.e., those with unbroken colors) with the masses $\mu^h_C = \mathcal{M}^h_{ch} \ll \Lambda_Q$, and to the N_h^2 hh-pions with masses $m_h \gg m_l$. But in the Higgs_l–DC_h phase at $r > r_2$, the Q^h and $\overline{Q}_{\overline{h}}$ quarks with broken colors appear now individually in the spectrum as light particles with the masses m_h. They can be considered remnants of the previous hybrid Π_{hl} and Π_{lh} pions with the masses $m_h + m_l \sim m_h$, which were present in the spectrum in the DC_l–DC_h phase at $r < r_2$.

We now take $N_l > N_0$. The theory is in the DC_l–DC_h phase at $r = 1$. As r decreases, a phase transition to the DC_l–HQ_h phase first occurs at $r \sim r_1 \gg r_2$, which persists until r approaches r_2 from above. If $N_l > b_0/2$, as $\mathcal{M}^l_{ch}$ exceeds Λ_Q and the l-quarks are Higgsed, the HQ_h phase of the Q^h and $\overline{Q}_{\bar h}$ quarks is maintained.

But there are values of $N_c < N_F < 3N_c$ and $N_l < N_c - 1$ such that $N_0 < N_l < b_0/2$. In this case, the theory stays in the DC_l–HQ_h phase as $\mathcal{M}^l_{ch}$ approaches Λ_Q from below, while as $\mathcal{M}^l_{ch}$ exceeds Λ_Q, the h-quarks condense and Π_h pions appear, and theory is in the Higgs_l–DC_h phase. Hence, not only the l-quarks but also the Q^h and $\overline{Q}_{\bar h}$ quarks change their phase. The reason for this is as follows. At $\mathcal{M}^l_{ch}$ slightly above Λ_Q, when the l-quarks are already Higgsed, the remaining lower-energy theory has $\langle\hat\Lambda\rangle \sim \Lambda_Q$, with $N'_c = N_c - N_l$ colors, $N'_F = N_F - N_l$ flavors, and $m_h \ll \Lambda_Q$ and $\mathcal{M}^h_{ch}$ staying intact because $\langle\hat\Lambda\rangle \sim \Lambda_Q$. But the pole mass $\overline{m}^{pole}_h$ of the Q^h and $\overline{Q}_{\bar h}$ quarks is smaller in this new theory than it was before Higgsing, $\overline{m}^{pole}_h \ll m^{pole}_h$, because the quark anomalous dimension decreased. Therefore, while the hierarchy was $m^{pole}_h \gg \mathcal{M}^h_{ch}$ before Higgsing, it is reversed after Higgsing, $\overline{m}^{pole}_h \ll \mathcal{M}^h_{ch}$, and the Q^h and $\overline{Q}_{\bar h}$ quarks also change their phase simultaneously with the Q^l and $\overline{Q}_{\bar l}$ ones.

This is not the end of the story with $b'_0 > 0$, however, because to stay in the Higgs_l–DC_h phase, the condition $m'_h = m_h(\mu = \langle\hat\Lambda\rangle) \ll \langle\hat\Lambda\rangle$ is necessary, and hence $r = m_l/m_h$ must not be too small. As r decreases at $N_l < N_c - 1$ and $b'_0 > 0$, $\langle\hat\Lambda\rangle$ in (26) decreases in a power-like fashion because $\mathcal{M}^l_{ch}$ increases like $(1/r)^\omega$, $\omega = (N_c - N_l)/2N_c$, see (2) ($\hat{\mathcal{M}}^l_{ch} \sim \mathcal{M}^l_{ch}$ up to a logarithmic factor), while m'_h changes only logarithmically. Therefore, as r decreases and crosses the smaller value $r_3 \ll r_2$ where the decreasing $\langle\hat\Lambda\rangle$ becomes $\langle\hat\Lambda\rangle < m'_h$, the phase transition from the Higgs_l–DC_h phase to the Higgs_l–HQ_h one occurs. The coherent condensate of the Q^h and $\overline{Q}_{\bar h}$ quarks breaks down, the Π_h pions disappear, and the heavy Q^h and $\overline{Q}_{\bar h}$ quarks with unbroken colors are in the perturbative weak-coupling regime at $r \ll r_3$, like the h-quarks with Higgsed colors (but the Q^h and $\overline{Q}_{\bar h}$ quarks with unbroken colors are weakly confined, the string tension being small, $\sqrt{\sigma} \sim \Lambda_{YM}$). In other words, the lower-energy theory at $\mu \ll \mu_{gl}$ contains the unbroken non-Abelian gauge group $SU(N'_c)$ with $N'_c = N_c - N_l$ and with the scale factor $\hat\Lambda$ of its gauge coupling in (26), and $N'_F = N_F - N_l$ ($N'_c < N'_F < 3N'_c$) flavors of the heavy nonrelativistic quarks Q^h and $\overline{Q}_{\bar h}$ with their pole masses $m^{pole}_h \gg \langle\hat\Lambda\rangle$, plus the l-pions entering $\hat\Lambda$ (see (26)) and the hybrid pions Π_{hl} and Π_{lh} (these are the light Q^h and $\overline{Q}_{\bar h}$ quarks with Higgsed colors, weakly interacting through residual D-term interactions). We do not give further details here because this is a simple regime and it is evident how to deal with this case. The mass spectrum in this Higgs$_l$–HQ$_h$ phase at $r \ll r_3$ is as follows. There are a) $2N_l N_c - N_l^2$ massive gluons and the same number of their superpartners (the "constituent l-quarks" with the heaviest masses $\mu_{gl} \gg \Lambda_Q$); b) a large number of hadrons made of nonrelativistic Q^h and $\overline{Q}_{\bar h}$ quarks with the perturbative pole masses m_h^{pole} (the hierarchy here is $\langle\hat\Lambda\rangle \ll \Lambda_{YM} \ll m_h^{pole} \ll \Lambda_Q$, with $\hat\Lambda$ from (26), while Λ_{YM} is the gauge coupling scale arising after the Q^h and $\overline{Q}_{\bar h}$ quarks are integrated out); c) the hybrids Π_{hl} and Π_{lh} (which are Q^h and $\overline{Q}_{\bar h}$ quarks with Higgsed colors) with the masses $\Lambda_{YM} \ll \hat m_h = m_h(\mu = \mu_{gl}) \ll m_h^{pole}$; d) a large number of strongly coupled gluonia with the mass scale $M_{gl} \sim \Lambda_{YM}$; and e) N_l^2 lightest l-pions with the masses $2\hat m_l = 2m_l(\mu = \mu_{gl}) = 2z_Q\, m_l$. The lowest-energy Lagrangian of these l-pions has the same Kähler term as in (31), and the same universal superpotential as in (14).

We finally consider the case where $\mathcal{M}_{ch}^l > \Lambda_Q$, $N_l < N_c - 1$, and $b_0' = b_0 - 2N_l < 0$. As was noted above, as $\mathcal{M}_{ch}^l$ approaches Λ_Q from below, the theory is already in the DC$_l$–HQ$_h$ phase, and as $\mathcal{M}_{ch}^l$ exceeds Λ_Q and the l-quarks become Higgsed, the confined Q^h and $\overline{Q}_{\bar h}$ quarks remain in the HQ$_h$ phase. Therefore, this is the Higgs$_l$–HQ$_h$ phase on the whole. But now, with $b_0' = b_0 - 2N_l < 0$ and $\mathcal{M}_{ch}^l \gg \Lambda_Q$, $\langle\hat\Lambda\rangle \gg \mathcal{M}_{ch}^l$ (see (26)), and in the interval of scales $\Lambda_{YM} \ll \mu \ll \mu_{gl}$, the remaining non-Abelian $SU(N_c - N_l)$ gauge theory with $N_F - N_l$ of confined Q^h and $\overline{Q}_{\bar h}$ quarks is in the weak coupling logarithmic regime. On the whole, this is also a very simple case (see Secs. 2 and 8 in [1]), and it is clear what its mass spectrum is. Qualitatively, it is similar to the spectra described in the preceding paragraph with $b_0' > 0$ and in the same Higgs$_l$–HQ$_h$ phase at $r < r_3$ (and the lowest-energy Lagrangian of the lightest l-pions is then the same), and we do not therefore consider this case in more detail.

6. CONCLUSIONS

As was described above within the dynamical scenario considered in this paper, the $\mathcal{N} = 1$ SQCD with N_c colors (with the scale factor Λ_Q of their gauge coupling), $N_c < N_F < 3N_c$ flavors of light quarks, N_l lighter flavors with masses m_l, and $N_h = N_F - N_l$ heavier flavors with masses m_h, $0 < m_l < m_h \ll \Lambda_Q$, can be in different phase states depending on the values of the above parameters. In addition, the mass spectra are also highly sensitive to the values of these parameters.

The lighter Q^l and $\overline{Q}_{\bar l}$ quarks may be in two different phases: either in the DC (diquark condensate) phase at $\mathcal{M}_{ch}^l \ll \Lambda_Q$ (at both $N_l < N_c$ and $N_l > N_c$) or in the Higgs phase at $\mathcal{M}_{ch}^l \gg \Lambda_Q$ (at $N_l < N_c$ only). The heavier Q^h and $\overline{Q}_{\bar h}$ quarks may also be in two different phases: either in the DC phase at $\mathcal{M}_{ch}^h \gg m_h^{pole}$ or in the HQ (heavy quark) phase at $\mathcal{M}_{ch}^h \ll m_h^{pole}$. On the whole, four different phases are therefore realized in this theory. For each of them, we described the mass spectra and the corresponding interaction Lagrangians[8)].

We did not consider the Seiberg dual theories [5, 6] with unequal quark masses in this paper. As was argued in [1], the direct and dual theories are not equivalent even in the simpler case of equal quark masses. There are no chances that the situation will be better for unequal quark masses.

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8) The N_c dependence of various quantities (e.g., $\langle S\rangle \sim N_c^0$) used everywhere in the text is the same as in the main text in [1]. As in [1], the correct N_c dependence ($\langle S\rangle \sim N_c$, etc.) can be easily restored (see Sec. 9 in [1]).