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GRAVITATING GLOBAL MONOPOLES IN EXTRA DIMENSIONS AND THE BRANEWORLD CONCEPT

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Multidimensional configurations with a Minkowski external space–time and a spherically symmetric global monopole in extra dimensions are discussed in the context of the braneworld concept. The monopole is formed with a hedgehog-like set of scalar fields ϕ^i with a symmetry-breaking potential V depending on the magnitude $\phi^2 = \phi^i\phi^i$. All possible kinds of globally regular configurations are singled out without specifying the shape of $V(\phi)$. These variants are governed by the maximum value ϕ_m of the scalar field, characterizing the energy scale of symmetry breaking. If $\phi_m < \phi_{cr}$ (where ϕ_{cr} is a critical value of ϕ related to the multidimensional Planck scale), the monopole reaches infinite radii, whereas in the «strong field regime», when $\phi_m \geq \phi_{cr}$, the monopole may end with a finite-radius cylinder or have two regular centers. The warp factors of monopoles with both infinite and finite radii may either exponentially grow or tend to finite constant values far from the center. All such configurations are shown to be able to trap test scalar matter, in striking contrast to RS2 type five-dimensional models. The monopole structures obtained analytically are also found numerically for the Mexican hat potential with an additional parameter acting as a cosmological constant.

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1. INTRODUCTION

According to a presently popular idea, our observable Universe can be located on a 4-dimensional surface, called the brane, embedded in a higher-dimensional manifold, called the bulk. This «braneworld» concept, suggested in the 1980s [1], is broadly discussed nowadays, mainly in connection with the recent developments in supersymmetric string/M-theories [2]. The reason why we do not see any extra dimensions is that the observed matter is confined to the brane, and only gravity propagates in the bulk. There are numerous applications of the braneworld concept to particle physics, astrophysics, and cosmology, such as the hierarchy problem and the description of dark matter and dark energy [3].

Most of the studies are restricted to infinitely thin branes with delta-like localization of matter. A well-known example is Randall and Sundrum's second model (RS2) [4], in which a single Minkowski brane is embedded in a 5-dimensional anti-de Sitter (AdS) bulk.

Thin branes can, however, be only treated as a rough approximation because any fundamental underlying theory, be it quantum gravity, string or M-theory, must contain a fundamental length beyond which the classical space–time description is impossible. It is therefore necessary to justify the infinitely thin brane approximation as a well-defined limit of a smooth structure, a thick brane, obtainable as a solution of coupled gravitational and matter field equations. Such a configuration is then required to be globally regular, stable,

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and properly concentrated around a 3-dimensional surface that is meant to describe the observed spatial dimensions. Topological defects emerging in phase transitions with spontaneous symmetry breaking (SSB) are probably the best candidates for this role.

It should be mentioned that the evolution of the Universe, according to modern views, contained a sequence of phase transitions with SSB. A decisive step toward cosmological applications of the SSB concept was made in 1972 by Kirzhnits [5]. He assumed that, as in the case of solid substances, a symmetry of a field system, existing at sufficiently high temperatures, could be spontaneously broken as the temperature falls down. A necessary consequence of such phase transitions is the appearance of topological defects. The first quantitative analysis of the cosmological consequences of SSB was given by Zel'dovich, Kobzarev, and Okun' [6]. Later, the SSB phenomenon and various topological defects were widely used in inflationary Universe models and in attempts to explain the origin of the large-scale structure of the Universe, see, e.g., [7, 8].

The properties of global topological defects are generally described with the aid of a multiplet of scalar fields playing the role of an order parameter. If a defect is to be interpreted as a braneworld, its structure is determined by the self-gravity of the scalar field system and may be described by a set of Einstein and scalar equations.

In this paper, we analyze the gravitational properties of candidate (thick) braneworlds with the 4-dimensional Minkowski metric as global topological defects in extra dimensions. Our general formulation covers particular cases such as a brane (domain wall) in 5-dimensional space – time (one extra dimension), a global cosmic string with winding number $n = 1$ (two extra dimensions), and global monopoles (three or more extra dimensions). We restrict ourselves to Minkowski branes because most of the existing problems are clearly seen even in these comparatively simple systems; on the other hand, in the majority of physical situations, the inner curvature of the brane itself is much smaller than the curvature related to brane formation, and therefore the main qualitative features of Minkowski branes should survive in curved branes.

Brane worlds as thick domain walls in a 5-dimensional bulk have been discussed in many papers (see, e.g., [9] and the references therein). Such systems were analyzed in a general form in Refs. [10, 11], without specifying the symmetry-breaking potential; it was shown, in particular, that all regular configurations should have an AdS asymptotic form. Therefore, all possible thick branes are merely regularized versions of the RS2 model, with all concomitant difficulties in matter field confinement. Thus, it has been demonstrated [11] that a test scalar field has a divergent stress–energy tensor infinitely far from the brane, at the AdS horizon. A reason for that is the repulsive gravity of the RS2 and similar models: gravity repels matter from the brane and pushes it towards the AdS horizon. To overcome this difficuty, it is natural to try considering a greater number of extra dimensions. This was one of the reasons for us to consider higher-dimensional bulks.

We study the simplest possible realization of this idea, assuming a static, spherically symmetric configuration of the extra dimensions and a thick Minkowski brane as a concentration of the scalar field stress–energy tensor near the center. The possible trapping properties of gravity for test matter are then determined by the behavior of the so-called warp factor (the metric coefficient acting as a gravitational potential) far from the center, and we indeed find classes of regular solutions where gravity is attracting.

Some of our results repeat those obtained in Refs. [12, 13], which have discussed global and gauge ('t Hooft – Polyakov-type) monopoles in extra dimensions; a more detailed comparison is given in Sec. 7.

The paper is organized as follows. In Sec. 2, we formulate the problem, introduce space – times with global topological defects in the extra dimensions, write the equations and boundary conditions, and demonstrate a connection between the possibility of SSB and the properties of the potential at a regular center. In Sec. 3, we briefly discuss the trapping problem for RS2-type domain-wall models and show that they always have repulsive gravity and are unable to trap matter in the form of a test scalar field. Section 4 is devoted to a search for regular global monopole solutions in higher dimensions by analyzing their asymptotic properties far from the center. All regular configurations are classified by the behavior of the spherical radius r and by the properties of the potential. This leads to separation of the «weak gravity» and «strong gravity» regimes, related to maximum values of the scalar field magnitude.

In the weak gravity regime, the spherical radius r tends to infinity along with the distance from the center. Such moderately curved configurations exist without any restrictions of fine-tuning type. If the scalar field magnitude exceeds some critical value, the radius r either tends to a finite value far from the center or returns to zero at a finite distance from the center, thus forming one more center, which should also be regular. Some cases require fine tuning of the parameters of the potential, and hence one may believe that static configurations can only exist if the scalar and gravitational forces are somewhat mutually balanced.

In Sec. 5, we show that in contrast to domain walls, global monopoles in different regimes do provide scalar field trapping on the brane. Section 6 is a brief description of numerical experiments with the Mexican hat potential admitting shifts up and down, equivalent to introducing a bulk cosmological constant. Their results confirm and illustrate the conclusions in Sec. 4. Section 7 summarizes the results.

2. PROBLEM SETTING

2.1. Geometry

We consider a $(D = d_0 + d_1 + 1)$-dimensional space–time with the structure $\mathbb{M}^{d_0} \times \mathbb{R}_u \times \mathbb{S}^{d_1}$ and the metric

$$ds^2 = e^{2\gamma(u)}\eta_{\mu\nu}dx^\mu dx^\nu - \left(e^{2\alpha(u)}du^2 + e^{2\beta(u)}d\Omega^2\right). \quad (1)$$

Here,

$$\eta_{\mu\nu}dx^\mu dx^\nu = dt^2 - (d\mathbf{x})^2$$

is the Minkowski metric in the subspace $\mathbb{M}^{d_0}$,

$$\eta_{\mu\nu} = \mathrm{diag}(1,\ -1, \ldots,\ -1);$$

$d\Omega$ is a linear element on a d_1-dimensional unit sphere $\mathbb{S}^{d_1}$; α, β, and γ are functions of the radial coordinate u with the definition domain $\mathbb{R}_u \subseteq \mathbb{R}$, to be specified later. The Riemann tensor has a diagonal form, and its nonzero components are

$$\begin{aligned}
&R^{\mu\nu}{}_{\rho\sigma} = -e^{-2\alpha}\gamma'^2\delta^{\mu\nu}{}_{\rho\sigma},\\
&R^{ab}{}_{cd} = \left(e^{-2\beta} - e^{-2\alpha}\beta'^2\right)\delta^{ab}{}_{cd},\\
&R^{u\mu}{}_{u\nu} = -\delta^\mu_\nu e^{-\gamma-\alpha}\left(e^{\gamma-\alpha}\gamma'\right)',\\
&R^{ua}{}_{ub} = -\delta^a_b e^{-\beta-\alpha}\left(e^{\beta-\alpha}\beta'\right)',\\
&R^{a\mu}{}_{b\nu} = -\delta^\mu_\nu\delta^a_b e^{-2\alpha}\gamma'\beta',
\end{aligned} \quad (2)$$

where

$$\delta^{\mu\nu}{}_{\rho\sigma} = \delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho \quad (3)$$

and similarly for $\delta^{ab}{}_{cd}$. Greek indices $\mu, \nu, \ldots$ correspond to the d_0-dimensional space–time and Latin indices $a, b, \ldots$ to d_1 angular coordinates on $\mathbb{S}^{d_1}$. We mostly bear in mind the usual dimension $d_0 = 4$, but keep d_0 arbitrary for generality.

A necessary condition of regularity is the finiteness of the Kretschmann scalar

$$\mathcal{K} = R^{AB}{}_{CD}R^{CD}{}_{AB}.$$

(Capital indices $A, B, \ldots$ correspond to all D coordinates.) In our case, $\mathcal{K}$ is a sum of squares of all nonzero $R^{AB}{}_{CD}$. Hence, in regular configurations, all components of Riemann tensor (2) are finite.

For the Ricci tensor, we have

$$\begin{aligned}
&R^\nu_\mu = -\delta^\nu_\mu e^{-2\alpha}\left[\gamma'' + \gamma'(-\alpha' + d_0\gamma' + d_1\beta)\right],\\
&R^u_u = -e^{-2\alpha}\left[d_0(\gamma'' + \gamma'^2 - \alpha'\gamma') +\right.\\
&\left.+ d_1(\beta'' + \beta'^2 - \alpha'\beta')\right],\\
&R^n_m = e^{-2\beta}(d_1 - 1)\delta^n_m -\\
&- \delta^n_m e^{-2\alpha}\left[\beta'' + \beta'(-\alpha' + d_0\gamma' + d_1\beta')\right].
\end{aligned} \quad (4)$$

2.2. Topological defects

A global defect with a nonzero topological charge can be constructed as a multiplet of $d_1 + 1$ real scalar fields ϕ^k, in the same way as, e.g., in [14]. It comprises a hedgehog configuration in $\mathbb{R}_u \times \mathbb{S}^{d_1}$:

$$\phi^k = \phi(u)n^k(x^a),$$

where n^k is a unit vector in the $(d_1 + 1)$-dimensional Euclidean target space of the scalar fields:

$$n^k n^k = 1.$$

The total Lagrangian of the system is taken in the form

$$L = \frac{R}{2\varkappa^2} + \frac{1}{2}g^{AB}\partial_A\phi^k\partial_B\phi^k - V(\phi), \quad (5)$$

where R is the D-dimensional scalar curvature, $\varkappa^2$ is the D-dimensional gravitational constant, and V is a symmetry-breaking potential depending on $\phi^2(u) = \phi^a\phi^a$.

In the case where $d_1 = 0$, there is only one extra dimension. The topological defect is a flat domain wall. Combined with $d_0 = 4$, it is widely considered with reference to our Universe. Regular thick Minkowski branes supported by scalar fields with arbitrary potentials were analyzed in [10, 11], see also Sec. 3 below.

The case where $d_1 = 1$ is a global cosmic string with the winding number $n = 1$. If $d_0 = 2$, it is a cosmic string in four dimensions, whose gravitational properties are reviewed in [15]. The case $d_0 = 4$ corresponds to a string in extra dimensions.

The case where $d_1 = 2$ and $d_0 = 1$ is a global monopole in our 4-dimensional space–time. We have analyzed it in detail in [16]. The case where $d_1 > 2$ and $d_0 = 1$ is its multidimensional generalization to static spherically symmetric space–times with d_1-dimensional rather than two-dimensional coordinate spheres [14]. It was shown that such a heavy multidimensional global monopole leads to a multidimensional cosmology where the symmetry-breaking potential at late times can mimic both dark matter and dark energy.

In the case where $d_0 = 4$ and $d_1 > 2$, we have a multidimensional global monopole entirely in the extra space-like dimensions. Different models of this kind were studied in Refs. [12, 13, 17, 18]. In particular, such a monopole in extra dimensions was used in an attempt to explain the origin of inflation [17].

2.3. Field equations

We use the Einstein equations in the form

$$R_A^B = -\varkappa^2 \widetilde{T}_A^B, \quad \widetilde{T}_A^B = T_A^B - \frac{\delta_A^B}{D-2} T_C^C,$$

where T_A^B is the stress–energy tensor of the scalar field multiplet. For our hedgehog configuration,

$$\widetilde{T}_\mu^\nu = -\frac{2V\delta_\mu^\nu}{D-2},$$

$$\widetilde{T}_u^u = -\frac{2V}{D-2} - e^{-2\alpha}\phi'^2,$$

$$\widetilde{T}_a^b = -\frac{2V\delta_a^b}{D-2} - e^{-2\beta}\delta_a^b\phi^2.$$

So far, we did not specify the radial coordinate u. For our purposes, the most helpful is the Gaussian gauge, with the real distance l along the radial direction taken as a coordinate,

$$u \equiv l, \quad \alpha \equiv 0, \tag{6}$$

and the metric

$$ds^2 = e^{2\gamma(l)}\eta_{\mu\nu}dx^\mu dx^\nu - \left(dl^2 + e^{2\beta(l)}d\Omega^2\right). \tag{7}$$

Then two independent components of the Einstein equations take the form (the prime now denotes d/dl)

$$\gamma'' + d_0\gamma'^2 + d_1\beta'\gamma' = -\frac{2\varkappa^2}{D-2}V, \tag{8}$$

$$\beta'' + d_0\beta'\gamma' + d_1\beta'^2 = (d_1 - 1 - \varkappa^2\phi^2)e^{-2\beta} - \frac{2\varkappa^2}{D-2}V. \tag{9}$$

The Einstein equation

$$G_l^l = -\varkappa^2 T_l^l$$

(where G_A^B is the Einstein tensor) is free of second-order derivatives:

$$(d_0\gamma' + d_1\beta')^2 - d_0\gamma'^2 - d_1\beta'^2 = \\ = \varkappa^2(\phi'^2 - 2V) + d_1 e^{-2\beta}(d_1 - 1 - \varkappa^2\phi^2). \tag{10}$$

The scalar field equations

$$\nabla^A\nabla_A\phi^k + \frac{\partial V}{\partial\phi^k} = 0$$

combine to yield an equation for $\phi(l)$:

$$\phi'' + (d_0\gamma' + d_1\beta')\phi' - d_1 e^{-2\beta}\phi = \frac{dV}{d\phi}. \tag{11}$$

Due to the Bianchi identities, it is a consequence of Einstein equations (8)–(10). On the other hand, (10) is a first integral of Eqs. (8), (9), and (11).

In our analytical study, we do not specify any particular form of $V(\phi)$. However, we suppose that V has a maximum at $\phi = 0$ and a minimum at some $\phi = \eta > 0$, and hence $V'(0) = V'(\eta) = 0$. For convenience, we do not single out a cosmological constant, which may be identified with a constant component of the potential V or, in many cases, with its minimum value.

The parameter η (as the scalar field itself) has the dimension $[l^{-(D-2)/2}]$ and thus specifies a certain length scale $\eta^{-2/(D-2)}$ and energy scale $\eta^{2/(D-2)}$ (we use the natural units such that $c = \hbar = 1$.) In the conventional case $D = 4$, η has the dimension of energy and characterizes the SSB energy scale.

2.4. Regularity conditions. A regular center

For the geometry to be regular, we must require finite values of all Riemann tensor components (2). In Gaussian gauge (6), the regularity conditions simply state that

$$\beta', \quad \beta'', \quad \gamma', \quad \gamma'' \quad \text{are finite.} \tag{12}$$

For $d_1 > 0$, in addition to (12), a special regularity condition is needed at the center, which is a singular point of the spherical coordinates in $\mathbb{R}_u \times \mathbb{S}^{d_1}$. The center is a point where the radius $r \equiv e^\beta$ turns to zero. The regularity conditions there, also following from the finiteness of Riemann tensor components (2), are the same as in the usual static, spherically symmetric space–time: in terms of an arbitrary u coordinate, they are given by

$$\gamma = \gamma_c + O(r^2), \\ e^{\beta-\alpha}|\beta'| = 1 + O(r^2) \quad \text{as} \quad r \to 0, \tag{13}$$

where γ_c is a constant that can be set to zero by a proper choice of scales of the coordinates x^μ. The second condition in (13) follows, for $d_1 > 1$, from the finiteness of the Riemann tensor components $R^{ab}{}_{cd}$, see (2). Its geometric meaning is the property of being locally Euclidean at $r = 0$, which implies that $dr^2 = dl^2$, i.e., the correct circumference-to-radius ratio for small circles. In the special case where $d_1 = 1$, with the quotient space $\mathbb{R}_u \times \mathbb{S}^{d_1}$ two-dimensional, we evidently have $R^{ab}{}_{cd} \equiv 0$, but the second condition in (13) should still be imposed to avoid a conical singularity.

It is natural to put $l = 0$ at a regular center, then l is the distance from the center.

Regularity of the Ricci tensor components $R^A_B = R^{AC}{}_{BC}$ implies regularity of the stress–energy tensor T^B_A, whence it follows that

$$|V| < \infty, \quad e^{-\beta}|\phi| < \infty, \quad e^{-\alpha}|\phi'| < \infty \tag{14}$$

at any regular point and with any radial coordinate.

2.5. Boundary conditions

Domain walls. For $d_1 = 0$, the metric in (1) or (7) describes a plane-symmetric five-dimensional space–time, the coordinate l ranges over the entire real axis, and the broken symmetry is $\mathbb{Z}_2$, the mirror symmetry with respect to the plane $l = 0$. The topological defect is a domain wall separating two vacua corresponding to two values of a single real scalar field ϕ, e.g., $\phi = \pm\eta$. Accordingly, we assume that $\phi(l)$ is an odd function, whereas $\gamma(l)$ and $V(\phi)$ are even functions, and the conditions at $l = 0$ are

$$\gamma(0) = \gamma'(0) = \phi(0) = 0. \tag{15}$$

We thus have three initial conditions for the third-order set of equations (8), (10) (Eq. (11) is their consequence), because the unknown function β is absent in this case.

Global strings and monopoles. For $d_1 > 0$, the regular center requirement leads to the following boundary conditions for Eqs. (8)–(10) at $l = 0$:

$$\phi(0) = \gamma(0) = \gamma'(0) = r(0) = 0, \quad r'(0) = 1. \tag{16}$$

We have five initial conditions for a fifth-order set of equations. However, $l = 0$, being a singular point of the spherical coordinate system (not to be confused with a space–time curvature singularity), is also a singular point of our set of equations. As a result, the requirements of the theorem on the solution existence and uniqueness for our set of ordinary differential equations are violated. It turns out that the derivative $\phi'(0)$ remains undetermined by (16). If we set $\phi'(0) = 0$, we obtain a trivial (symmetric) solution with $\phi \equiv 0$ and a configuration without a topological defect. In the case where $V(0) = 0$, we arrive at the flat D-dimensional metric: we then have $\gamma \equiv 0$ and $r \equiv l$ in (7). If, however, $V(0) \neq 0$, the corresponding exact solutions to the Einstein equations for $d_0 > 1$, $d_1 > 1$ are yet to be found. A direct inspection shows that it cannot be the de Sitter or AdS space: the constant curvature metrics are not solutions of the vacuum Einstein equations with a cosmological constant.

Nontrivial solutions exist if $\phi'(0) \neq 0$ and can correspond to SSB. We note that the very possibility of SSB appears as a result of violation of the solution uniqueness at $r = 0$ provided that a maximum of the potential $V(\phi)$ at $\phi = 0$ corresponds to the center. The lacking boundary condition that may lead to a unique solution can now follow from the requirement of regularity at the other extreme of the range of l, whose nature is in turn determined by the shape of the potential.

In what follows, assuming a regular center, we try to find all possible conditions at the other extreme of the range $\mathbb{R}_l$ of the Gaussian radial coordinate, providing the existence of globally regular models with metric (7). In other words, we seek solutions with asymptotic forms such that the quantities in (2) are finite. All the other regularity conditions, such as (14), then follow. In doing so, we do not restrict the possible shape of the potential $V(\phi)$ in advance. The cases under consideration are classified by the final values of $r = e^\beta$ (infinite, finite or zero) and V (positive, negative or zero). The scalar field ϕ is assumed to be finite everywhere.

Without loss of generality, we assume that $\phi'(0) > 0$ near $l = 0$, i.e., that ϕ increases, at least initially, as we recede from the center.

3. DOMAIN WALLS AND THE PROBLEM OF MATTER CONFINEMENT

Below, we mostly consider configurations with $d_1 \geq 2$ that correspond to a global monopole in the spherically symmetric space $\mathbb{R}_u \times \mathbb{S}^{d_1}$. Before that, we briefly discuss the problem of matter confinement on the brane and the complications involved the 5-dimensional case.

The metric coefficient $e^{2\gamma}$ in (1), sometimes called the warp factor, actually plays the role of a gravitational potential that determines an attractive or repulsive nature of gravity with respect to the brane. If it forms a potential well with a bottom on (or very near) the brane, there is a hope that matter, at least its low-energy modes, is to be trapped.

It has been shown, in particular, that spin-1/2 fields are localized due to an increasing warp factor in (1+4)- and (1+5)-dimensional models [19, 20]. It was also repeatedly claimed that in (1+4) dimensions, a brane with an exponentially decreasing warp factor (as, e.g., in the RS2 model) can trap spin 0 and 2 fields. Our calculation for a scalar field shows that this is not the case.

A gravitational trapping mechanism suggested in Refs. [21] was characterized there as a universal one, suitable for all fields. It is based on nonexponential warp factors, which increase with the distance from the brane and approach finite values at infinity. This mechanism was exemplified in [22] with a special choice of two so-called «smooth source functions» in the stress–energy tensor, describing a continuous distribution of some phenomenological matter and vanishing outside the brane.

Our analysis uses more natural assumptions: a scalar field system admitting SSB, without any special choice of the symmetry breaking potential, under the requirement of global regularity.

We briefly show, following Refs. [10, 11] (but in other coordinates), that this approach in (4+1) dimensions always leads to a decaying warp factor for any choice of $V(\phi)$ and that such a system cannot trap a test scalar field. We consider a domain wall in 5 dimensions, and hence $l \in \mathbb{R}$, we set $d_1 = 0$ in our equations, the unknown $\beta(l)$ is absent, and Eqs. (8) and (11) for γ and the single scalar field ϕ are given by

$$\gamma'' + d_0\gamma'^2 = -\frac{2\varkappa^2}{d_0 - 1}V, \qquad (17)$$

$$\phi'' + d_0\gamma'\phi' - \frac{dV}{d\phi} = 0. \qquad (18)$$

Their first integral in (10) reduces to

$$\gamma'^2 = -\frac{\varkappa^2}{d_0(d_0 - 1)}(2V - \phi'^2). \qquad (19)$$

The initial conditions at $l = 0$ corresponding to the $\mathbb{Z}_2$ symmetry (broken for the scalar field but preserved for the geometry) have form (15).

Eliminating V from (17) and (19) and integrating subject to (15), we obtain

$$(d_0 - 1)\gamma'(l) = -\varkappa \int_0^l \phi'^2 dl, \qquad (20)$$

and we conclude that $\gamma'(l)$ is negative at all $l > 0$ and describes gravitational repulsion from the brane; moreover, $e^{-\gamma}$ monotonically grows with growing l. The only possible regular solution corresponds to $|\gamma'(\infty)| < \infty$. Because $\gamma''(\infty) = 0$ in this case, it follows from Eq. (17) that $V(\infty) < 0$, corresponding to a negative cosmological constant $\Lambda = \varkappa^2 V(\infty)$. Hence, the only possible regular asymptotic form is AdS, with

$$e^\gamma \approx a e^{-hl}, \quad a, h = \text{const}, \quad h = \sqrt{-\Lambda/6}. \qquad (21)$$

The constant a depends on the particular shape of $V(\phi)$. At $l = \infty$, there is an AdS horizon ($e^\gamma = 0$), which, like a black hole horizon, attracts matter and prevents its trapping by the brane.

We show this for $d_0 = 4$ and a test scalar field χ with the Lagrangian

$$L_\chi = \frac{1}{2}\partial_A\chi^*\partial^A\chi - \frac{1}{2}m_0^2\chi^*\chi - \frac{1}{2}\lambda\phi^2\chi^*\chi, \qquad (22)$$

where χ^* is the complex conjugate field and the last term describes a possible interaction between χ and the wall scalar field ϕ; λ is the coupling constant. The field $\chi(x^A)$ satisfies the linear homogeneous (modified Fock – Klein – Gordon) equation

$$\frac{1}{\sqrt{g}}\partial_A\left(\sqrt{g}g^{AB}\partial_B\chi\right) + (\lambda\phi^2 + m_0^2)\chi = 0. \qquad (23)$$

Its coefficients depend on l only, and $\chi(x^A)$ may be sought in the form

$$\chi(x^A) = X(l)\exp(-ip_\mu x^\mu), \quad \mu = 0, 1, 2, 3, \qquad (24)$$

where $p_\mu = (E,\ \mathbf{p})$ is a constant 4-momentum. The function $X(l)$ determines the χ field distribution across the brane and satisfies the equation

$$X'' + 4\gamma'X' + \left[e^{-2\gamma}(E^2 - \mathbf{p}^2) - \lambda\phi^2 - m_0^2\right]X = 0. \qquad (25)$$

The χ field is able to describe particles localized on the brane only if its stress–energy tensor $T_\mu^\nu[\chi]$ is finite in the whole 5-space and decays sufficiently rapidly at large l. As an evident necessary condition of localization, the χ field energy per unit 3-volume of the brane must be finite, i.e.,

$$E_{tot}[\chi] = \int_{-\infty}^{\infty} T_t^t\sqrt{g}dl = \int_0^\infty e^{4\gamma}\Big[e^{-2\gamma}(E^2 + \mathbf{p}^2)X^2 + (m_0^2 + \lambda\phi^2)X^2 + X'^2\Big]dl < \infty. \qquad (26)$$

Inequality (26) implies a finite norm of the χ field defined as

$$\|\chi\|^2 = \int_{-\infty}^{\infty} \sqrt{g}\,\chi^*\chi\,dl = \int_{-\infty}^{\infty} e^{4\gamma}\,X^2\,dl. \qquad (27)$$

At large l, because $e^{-2\gamma} \to \infty$, the terms with λ and m_0 in Eq. (25) can be neglected, and the equation determining the behavior of χ at large l can be written as

$$X'' - 4h\,X' + P^2 e^{2hl}\,X = 0, \quad P^2 = \frac{E^2 - \mathbf{p}^2}{a^2h^2}. \qquad (28)$$

It is solved in terms of Bessel functions, and the solution has the asymptotic form

$$X = Ce^{3hl/2}\sin(Pe^{hl} + \varphi_0), \quad z \to \infty, \qquad (29)$$

where C and φ_0 are integration constants. We see that quantity (29) not only is nonvanishing as $l \to \infty$ but even oscillates with an increasing amplitude. As a result, the stress–energy tensor components $T_\mu^\nu[\chi]$ are infinite at $l = \infty$. Moreover, integral (26) behaves as $\int e^{hl}dl$ and diverges. However, normalization integral (27) converges because the integrand behaves as e^{-hl}. This result is sometimes treated as a sufficient condition for localization, but, in our view, it is not true because the very existence of the brane configuration is put to doubt if the test field stress–energy tensor is infinite somewhere.

Thus, a test scalar field with any mass tends to infinity as $l \to \infty$ and develops an infinite stress–energy tensor; even its interaction with the ϕ field that supports the brane does not improve the situation. We conclude that a single extra dimension is insufficient for providing gravitational attraction of matter to a regular isolated brane.

4. A SEARCH FOR REGULAR ASYMPTOTIC REGIMES

We now consider field equations (8)–(11) for global monopoles, assuming $d_1 \geq 2$. The string case $d_1 = 1$ is left aside because it has some peculiarities that require a special study.

4.1. Solutions with the $r \to \infty$ asymptotic regime

We denote

$$\overline{V} = \frac{2\varkappa^2 V}{D-2}, \quad \overline{V}_\infty = \overline{V}\Big|_{r\to\infty}. \qquad (30)$$

Evidently, $l \to \infty$ as $r \to \infty$ because otherwise we would have $\beta' \to \infty$, violating the regularity conditions. The derivatives β' and γ' should tend to certain constant values, to be denoted by β'_∞ and γ'_∞, respectively. Both β'' and γ'' vanish as $l \to \infty$. Moreover, the second term in the right-hand side of Eq. (9) also vanishes. Therefore, in the leading order of magnitude, Eqs. (8) and (9) take the form

$$\begin{aligned} \gamma'_\infty(d_0\gamma'_\infty + d_1\beta'_\infty) &= -\overline{V}_\infty, \\ \beta'_\infty(d_0\gamma'_\infty + d_1\beta'_\infty) &= -\overline{V}_\infty. \end{aligned} \qquad (31)$$

We consider the cases where $\overline{V}_\infty \neq 0$ and $\overline{V}_\infty = 0$ separately.

A1: $\overline{V}_\infty \neq 0$. Equations (31) immediately give

$$\beta'_\infty = \gamma'_\infty = \sqrt{-\overline{V}_\infty/(D-1)}, \quad \overline{V}_\infty < 0. \qquad (32)$$

An evident necessary condition of the existence of regular configurations is $\overline{V}_\infty \leq 0$. We thus obtain

$$e^\beta \sim e^\gamma \sim e^{\beta'_\infty l},$$

and the metric takes the asymptotic form

$$ds^2 \approx C_1 e^{2\beta'_\infty l}\eta_{\mu\nu}dx^\mu dx^\nu - dl^2 - C_2 e^{2\beta'_\infty l}d\Omega^2, \quad (33)$$

with some positive constants C_1 and C_2. Equation (10) holds automatically if $\phi'(\infty) = 0$, as should be the case if we assume a finite asymptotic value of ϕ. Finally, in Eq. (11), all terms except $dV/d\phi$ manifestly vanish as $l \to \infty$, and hence $dV/d\phi$ also vanishes, which should be the case if the field ϕ reaches an extremum of the potential V.

The finiteness condition for ϕ as $l \to \infty$ separates a family of regular solutions among the continuum of integral curves leaving the regular center with different slopes $\phi'(0)$. As is confirmed by numerical experiments, if the potential has only one extremum (minimum) $\overline{V}_\infty < 0$, then there can be only one regular solution with $r \to \infty$, $l \to \infty$. However, there can be numerous regular solutions if the potential has several extremum points $\overline{V}_\infty < 0$.

In particular, if the initial maximum of the potential is located below the zero level, $V(0) \leq 0$, then there can be a continuum of regular integral curves starting from the regular center and returning to $\phi = 0$ at $l \to \infty$. As can be verified numerically (see Sec. 4), there is a bunch of such curves parameterized by $\phi'(0) \in (0,\ \phi'_s)$, where $\phi'(0) = \phi'_s$ corresponds to a limiting regular curve (separatrix), also starting at $\phi(0) = 0$ but ending at the minimum $V(\eta)$.

The metric in (33) solves the Einstein equations with the stress–energy tensor $T_A^B = \delta_A^B V_\infty$ having the structure of a (negative) cosmological term. Moreover, according to (2), the Riemann tensor has the structure of a constant-curvature space at large l. In other words, such solutions have an anti-de Sitter (AdS_D) asymptotic form far from the center. But the metric in (33) is not a solution of our equations in the whole space even in the case where $\phi = \mathrm{const}$. As was already mentioned, for $d_0 > 1$ and $d_1 > 1$, constant-curvature metrics (dS_D and AdS_D) are not solutions of the vacuum Einstein equations with a cosmological constant.

A2: $\overline{V}_\infty = 0$. Equations (31) are solved either by

$$\beta'_\infty = \gamma'_\infty = 0$$

or by

$$d_0\gamma'_\infty + d_1\beta'_\infty = 0.$$

But when we substitute the second condition in Eq. (10), taking into account that $\phi' \to 0$ at large l, we obtain

$$d_0{\gamma'_\infty}^2 + d_1{\beta'_\infty}^2 = 0$$

and return to

$$\beta'_\infty = \gamma'_\infty = 0.$$

Thus, both β' and γ' vanish at infinity, and we can try to seek them as expansions in inverse powers of l:

$$\beta' = \frac{\beta_1}{l} + \frac{\beta_2}{l^2} + \ldots, \quad \gamma' = \frac{\gamma_1}{l} + \frac{\gamma_2}{l^2} + \ldots \qquad (34)$$

Then $O(l^{-2})$ is the leading order in the Einstein equations, and, to avoid a contradiction,

$$r^{-2} = e^{-2\beta}$$

should be of the order $O(l^{-2})$ or smaller. Moreover, because we assume that ϕ tends to a finite value $\phi_\infty > 0$, we have $\phi' = o(1/l)$, and scalar field equation (11) shows that

$$\frac{dV}{d\phi} = O(l^{-2})$$

or smaller, i.e., ϕ_∞ should be an extremum of $V(\phi)$. If $\phi(l)$ grows monotonically to $\phi_\infty > 0$, then ϕ_∞ is a minimum of V because, according to (11),

$$\frac{dV}{d\phi} < 0 \quad \text{as} \quad \phi \to \phi_\infty.$$

However, if $V(0) = 0$, one cannot exclude that ϕ returns to zero as $l \to \infty$, see item c) below.

In the case where $\phi \to \phi_\infty > 0$, because

$$V_\infty = dV/d\phi(\phi_\infty) = 0,$$

$V(\phi)$ is decomposed as

$$V(\phi) = \frac{1}{2}V_{\phi\phi}(\phi_\infty)(\phi - \phi_\infty)^2 + \ldots, \qquad (35)$$

where

$$V_{\phi\phi} = \frac{d^2V}{d\phi^2},$$

and therefore

$$V = o(l^{-2}).$$

As a result, Eqs. (8)–(10) lead to

$$\gamma_1(-1 + d_0\gamma_1 + d_1\beta_1) = 0, \qquad (36)$$

$$\beta_1(-1 + d_0\gamma_1 + d_1\beta_1) = \frac{l^2}{r^2}(d_1 - 1 - \varkappa^2\phi^2_\infty), \qquad (37)$$

$$(d_0\gamma_1 + d_1\beta_1)^2 - d_0\gamma_1^2 - d_0\beta_1^2 = = d_1\frac{l^2}{r^2}(d_1 - 1 - \varkappa^2\phi^2_\infty). \qquad (38)$$

Now, it can be easily verified that we must necessarily set $\beta_1 = 1$. Indeed, for any $\beta_1 \neq 0$, we have

$$r = e^\beta \sim l^{\beta_1}.$$

Therefore, $\beta_1 < 1$ is excluded because it leads to $r \ll l$, contrary to the above requirement. But if we suppose that $\beta_1 > 1$, then $l^2/r^2 \to 0$ as $l \to \infty$, and Eq. (38) leads either (if $\gamma_1 = 0$) to

$$\beta_1^2 = 0$$

or (if $\gamma_1 \neq 0$ and then $d_0\gamma_1 + d_1\beta_1 = 1$) to

$$d_0\gamma_1^2 + d_1\beta_1^2 = 1.$$

Both possibilities contradict the assumption that $\beta_1 > 1$.

Thus, $\beta_1 = 1$, and hence

$$r \approx kl, \quad k = \mathrm{const} > 0,$$

at large l.

Equation (36) now leaves two possibilities,

$$\gamma_1 = 0$$

and

$$\gamma_1 = -\frac{d_1 - 1}{d_0},$$

and we consider them separately in items a) and b). Item c) describes the case where expansions (34) do not work.

a) If $\gamma_1 = 0$, then Eq. (37) yields

$$k^2 = 1 - \frac{\varkappa^2\phi^2_\infty}{d_1 - 1}, \qquad (39)$$

and Eq. (10) in the same order is satisfied automatically. The metric takes the asymptotic form

$$ds^2 = e^{2\gamma_\infty}\eta_{\mu\nu}dx^\mu dx^\nu - dl^2 - k^2 l^2\, d\Omega^2, \tag{40}$$

where γ_∞ is a constant (we cannot turn it to zero by rescaling the coordinates x^μ because such an operation has already been done for making $\gamma = 0$ at the center).

Thus, the whole metric has a flat asymptotic form, up to a solid angle deficit in the spherical part due to $k^2 \neq 1$. Such a deficit is a common feature of topological defects in the cases where they have (almost) flat asymptotic forms. Its appearance due to cosmic strings and global monopoles in space–times without extra dimensions is discussed in detail in [8]. For a global monopole in extra dimensions in the particular case where $d_0 = 4$ and $d_1 = 2$, it was treated by Benson and Cho [18]. We stress that the situation of a quasi-flat asymptotic form with a solid angle deficit is not general. It occurs only for potentials with zero value at the minimum,

$$V(\phi_\infty) = 0,$$

and even in that case, not always, see item B below. Namely, this geometry requires

$$|\phi_\infty| < \phi_{cr} := \frac{\sqrt{d_0 - 1}}{\varkappa}, \tag{41}$$

i.e., ϕ_∞ should be smaller than the critical value ϕ_{cr} related to the D-dimensional Planck length. As ϕ_∞ approaches ϕ_{cr}, $k \to 0$, the deficit absorbs the whole solid angle, and the above geometry disappears.

Scalar equation (11) shows how ϕ approaches ϕ_∞: in the leading order, we have

$$-\frac{d_1}{k^2 l^2} = V_{\phi\phi}(\phi_\infty)(\phi - \phi_\infty). \tag{42}$$

Assuming

$$V_{\phi\phi}(\phi_\infty) \neq 0,$$

we obtain

$$\phi - \phi_\infty \sim 1/l^2.$$

b) If $\gamma_1 = -(d_1 - 1)/d_0$, then Eq. (37) leads to

$$\varkappa^2 \phi_\infty^2 = d_1 - 1,$$

i.e.,

$$\phi_\infty = \phi_{cr},$$

while a substitution in (10) gives

$$(d_1 - 1)(d_0 + d_1 - 1) = 0,$$

contrary to our assumption that $d_1 - 1 > 1$. Therefore, this possibility does not lead to a regular asymptotic regime.

c) If $V(0) = 0$, then a regular integral curve, starting at $l = 0$ and $\phi = 0$, can finish again with $\phi \to 0$ as $l \to \infty$. For large l and r, scalar field equation (11) for $|\phi| \ll 1$ reduces to

$$\phi'' + (d_0\gamma' + d_1\beta')\phi' - V_2\phi = 0, \tag{43}$$

where

$$V_2 = V_{\phi\phi}(0).$$

Because $\phi = 0$ is a maximum of $V(\phi)$ by assumption, we assume that $V_2 < 0$.

If we further assume that the function

$$s(l) = e^{d_0\gamma + d_1\beta}$$

satisfies the condition

$$s''/s \to 0 \quad \text{as} \quad l \to \infty$$

(which is the case, e.g., for any power-behaved function), the solution of Eq. (43) is an oscillating function at large l,

$$\phi \approx \phi_0 e^{-(d_0\gamma + d_1\beta)/2} \cos\left[\sqrt{|V_2|}(l - l_0)\right], \quad l \to \infty, \tag{44}$$

where ϕ_0 and l_0 are arbitrary constants. Substituting this in Eq. (8) and averaging $\cos^2 \to 1/2$, we obtain

$$e^{d_0\gamma} \approx \frac{d_0\varkappa^2|V_2|\phi_0^2}{2(D-2)}\int^l \frac{l\,dl}{r^{d_1}}, \quad l \to \infty. \tag{45}$$

It is easy to verify that for $d_1 > 2$, when the integral in (45) converges, the asymptotic form of the solution for $r = e^\beta$ and γ is $r \approx l$ and

$$\gamma = \gamma_\infty - \gamma_1/l^{d_1 - 2}, \quad \gamma_1,\ \gamma_\infty = \text{const},$$

i.e., we have a flat asymptotic regime.

In the special case where $d_1 = 2$, the integral diverges logarithmically, and the solution may be approximated as (again)

$$r \approx l$$

and

$$e^\gamma \approx \text{const} \cdot \ln l.$$

This «logarithmic» asymptotic form resembles the behavior of cylindrically symmetric solutions in standard general relativity.

4.2. Solutions with the $r \to r_* > 0$ asymptotic regime

Evidently, a regular solution cannot terminate at finite r and $l < \infty$. Therefore, we seek a regular asymptotic regime as $l \to \infty$, where r and ϕ tend to finite limits, r_* and ϕ_*, and hence the quantities β', β'', ϕ', and ϕ'' vanish.

Moreover, in a regular solution, γ' should tend to a finite limit as $l \to \infty$, and hence $\gamma'' \to 0$. As a result, Eqs. (8) and (9) at large l lead to

$$d_0\gamma'^2 = -\overline{V}_* = \frac{1}{r_*^2}(\varkappa^2\phi_*^2 - d_1 + 1), \qquad (46)$$

where $\overline{V}_* = \overline{V}(\phi_*)$. We see that $\overline{V}_* \le 0$ and, in addition, the scalar field should be critical or larger, $\phi_* \ge \phi_{cr}$. According to (46), at large l,

$$\pm\gamma' \approx h := \sqrt{-\overline{V}_*/d_0} \ge 0, \qquad (47)$$

and Eq. (10), as in the previous cases, simply verifies that the solution is correct in the leading order. The scalar field equation gives a finite asymptotic value of $V_\phi \equiv dV/d\phi$:

$$V_\phi(\phi_*) = -d_1\phi_* r_*^{-2}. \qquad (48)$$

This value is negative if $\phi_* > 0$.

We obtain different asymptotic regimes for negative, positive, and zero values of γ'.

B1: $e^\gamma \sim e^{-hl}$, $h > 0$. The metric has the asymptotic form

$$ds^2 = C^2 e^{-2hl}\eta_{\mu\nu}dx^\mu dx^\nu - dl^2 - r_*^2\, d\Omega^2. \qquad (49)$$

The extra-dimensional part of the metric again describes an infinitely long cylindrical tube, but now the vanishing function $g_{tt} = e^{2\gamma}$ resembles a horizon. The substitution $e^{-hl} = \rho$ (converting $l = \infty$ to a finite coordinate value, $\rho = 0$) brings metric (49) to the form

$$ds^2 = C^2\rho^2\eta_{\mu\nu}dx^\mu dx^\nu - \frac{d\rho^2}{h^2\rho^2} - r_*^2\, d\Omega^2. \qquad (50)$$

Therefore, $\rho = 0$ is a second-order Killing horizon in the 2-dimensional subspace parameterized by t and ρ, it is of the same nature as, e.g., the extreme Reissner–Nordström black hole horizon, or the AdS horizon in the second Randall–Sundrum braneworld model. A peculiarity of the present horizon is that the spatial part of the metric, which at large l takes the form $\rho^2(d\mathbf{x})^2$, is degenerate at $\rho = 0$. The volume of the d_0-dimensional space–time vanishes as $l \to \infty$. And it remains degenerate even if we pass to Kruskal-like coordinates in the (t, ρ) subspace. But the D-dimensional curvature is finite there, indicating that the transition to negative values of ρ (where the old coordinate l no longer works) is meaningful[1)].

One more observation can be made. According to (46), the potential V is necessarily negative at large l. On the other hand, Eq. (8) may be rewritten in integral form:

$$e^{d_0\gamma + d_1\beta}\gamma' = -\int_0^l e^{d_0\gamma + d_1\beta}\,\overline{V}\, dl. \qquad (51)$$

The lower limit of the integral corresponds to a regular center, where the left-hand side of (51) vanishes. As $l \to \infty$, it also vanishes due to $\gamma \to -\infty$. Thus, the integral in the right-hand side, taken from zero to infinity, is zero. This means that the potential $V(\phi)$ has alternate sign and is positive in a certain part of the range $(0,\ \phi_*)$.

Thus, purely scalar solutions of the monopole type may contain second-order horizons. The degenerate nature of the spatial metric at the horizon does not lead to a curvature singularity, and the solutions may be continued in a Kruskal-like manner. Nevertheless, we do not consider these solutions as describing viable monopole configurations because the zero volume of the corresponding spatial section makes the density of any additional (test) matter infinite. It is then impossible to neglect its back reaction, which evidently destroys such a configuration.

B2: $e^\gamma \sim e^{hl}$, $h > 0$. The metric has the asymptotic form

$$ds^2 = C^2 e^{2hl}\eta_{\mu\nu}dx^\mu dx^\nu - dl^2 - r_*^2 d\Omega^2, \qquad C = \text{const} > 0. \qquad (52)$$

Thus, in the spherically symmetric extra-dimensional part of the metric, we have an infinitely long d_1-

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[1)] One may wonder why we here do not obtain simple (first-order) horizons, like those in the Schwarzschild and de Sitter metrics, while such horizons generically appeared in the special case $d_0 = 1$, which corresponds to spherically symmetric global monopoles in general relativity, considered in detail in Refs. [14, 16].

The reason is that for $d_0 = 1$, $\delta^{\mu\nu}{}_{\rho\sigma}$ in (3) is zero, and the corresponding component of the Riemann tensor is also zero regardless of the values of γ'. In terms of the Gaussian coordinate l, a simple horizon occurs at some finite $l = l_h$ near which $g_{tt} = e^{2\gamma} \sim (l - l_h)^2$, such that $\gamma' \to \infty$. When $d_0 = 1$, this does not lead to a singularity because only the combinations $\gamma'' + \gamma'^2$ and $\beta'\gamma'$ are then required to be (and actually are) finite. In the case where $d_0 > 1$, instead of a horizon, we would have a curvature singularity at finite l, a situation excluded from the present study.

We thus have a general result for the metric in (1): for $d_0 > 1$, horizons can only be of order 2 and higher.

dimensional cylindrical «tube» with an infinitely growing gravitational potential $g_{tt} = e^{2\gamma}$.

With this cylindrical asymptotic form, according to (47) and (48), the potential V tends to a negative value and has a negative slope. Moreover, the integral in Eq. (51) is negative and diverges at large l due to growing e^{γ}.

Regular solutions with $\gamma'(\infty) > 0$ naturally arise if the potential $V(\phi)$ is negative everywhere. We note, however, that when $V(0)$ is above zero, by (51), the function $\gamma(l)$ decreases near the center ($l = 0$) due to $V > 0$, and grows at large l. It therefore has a minimum at some $l > 0$.

B3: $\overline{V}_* = 0$. This case contains one more asymptotic regime where the extra space ends with a regular tube.

Indeed, we can once again use expansions (34), but now with ϕ_* instead of ϕ_∞ and $\beta_1 = 0$ in accordance with $r \to r_*$. Equation (9) (order $O(1)$) shows that

$$\varkappa^2 \phi_*^2 = d_1 - 1,$$

i.e.,

$$\phi_* = \phi_{cr}.$$

Equation (11) (order $O(1)$) gives a finite value of the derivative

$$dV/d\phi(\phi_*) = -d_1 \phi_*/r_*^2.$$

Further, Eq. (8) (order $O(l^{-2})$) yields

$$\gamma_1(d_0\gamma_1 - 1)/l^2 = -\overline{V},$$

showing that

$$V = O(l^{-2})$$

(or even smaller). Because

$$V = (dV/d\phi(\phi_*))(\phi - \phi_*) + o(\phi - \phi_*),$$

we have to conclude that

$$\phi - \phi_* = O(l^{-2})$$

or smaller.

Now, assuming

$$V(\phi) = V_2/l^2 + \dots,$$

we can find V_2 directly as the leading term in

$$(dV/d\phi(\phi_*))(\phi - \phi_*)$$

and, independently, from Eq. (9) (order $O(l^{-2})$), obtaining the two expressions

$$V_2 = -d_1 \frac{\phi_* \phi_2}{r_*^2}$$

and

$$V_2 = -(D-2)\frac{\phi_* \phi_2}{r_*^2},$$

whence it follows that $d_1 = D - 2$, or $d_0 = 1$. Such a «critical» asymptotic regime ($\phi \to \phi_{cr}$, $g_{tt} \to 0$, and $r \to \text{const}$) was indeed found for $d_0 = 1$ in our papers [14, 16] describing (d_1+2)-dimensional spherically symmetric global monopoles, but, as we see, it does not exist in the case under consideration, $d_0 > 1$.

The only remaining possibility is that

$$\phi - \phi_* = o(l^{-2})$$

and

$$\gamma \to \gamma_* = \text{const},$$

i.e., a solution tending at large l to the following simple «flux-tube» solution, valid for any d_0 and d_1:

$$\begin{aligned} &r = \text{const}, \quad \gamma = \text{const}, \quad \phi = \phi_{cr}, \\ &V = 0, \quad dV/d\phi = -d_1\phi_{cr}/r^2. \end{aligned} \tag{53}$$

Such a solution can exist if the potential $V(\phi)$ has the properties

$$V(\phi_{cr}) = 0$$

and

$$dV/d\phi(\phi_{cr}) < 0,$$

and the last equality in (53) then relates the constant radius r to $dV/d\phi(\phi_{cr})$.

4.3. Solutions with the $r \to 0$ asymptotic regime

The limit $r \to 0$ means a center, and for it to be regular, conditions (12) must hold, and hence, for our system, initial conditions (16) with $l = 0$ should be replaced, e.g., with $l = l_0 > 0$.

We now recall that conditions (16) determine the solution of the field equations for a given potential $V(\phi)$ up to the value of ϕ'. In particular, if there is one more center at $l = l_0$, then, starting from it and choosing

$$\phi'(l_0) = -\phi'(0),$$

we obtain the same solution in terms of $l_0 - l$ instead of l. We thus obtain a solution with two regular centers that is symmetric with respect to the middle point $l = l_0/2$, to be called the equator. To be smooth there, it must satisfy the conditions

$$\beta' = \gamma' = \phi' = 0 \quad \text{at} \quad l = l_0/2, \tag{54}$$

which implicitly restrict the shape of the potential. Given a potential $V(\phi)$, conditions (54) create, in general, three relations among l_0, $\phi'(0)$, and the free parameters of $V(\phi)$ (if any). Eliminating l_0 and $\phi(0)$, we must obtain a single «fine tuning» condition for the parameters of the potential.

A necessary condition for the existence of such a solution is that $V(\phi)$ has a variable sign. This follows from Eq. (51) by integration over the segment $(0, l_0/2)$: the integral vanishes because $\gamma' = 0$ at both ends.

Moreover, as follows from Eqs. (9) and (10) with (54),

$$r_e^{-2}(d_1 - 1 - \varkappa^2\phi_e^2) = \frac{D-2}{d_1}\overline{V}_e = \beta_e'' + \overline{V}_e, \quad (55)$$

leading to

$$d_1\beta_e'' = (d_0 - 1)\overline{V}_e$$

(where the index «e» refers to values at the equator). If $r = e^\beta$ is assumed to grow monotonically from zero to its maximum value at the equator, we have $\beta_e'' < 0$, and hence $\overline{V}_e < 0$, and (55) implies that $\phi_e > \phi_{cr}$, i.e., the scalar field at the equator must exceed its critical value.

The existence of asymmetric solutions with two regular centers, corresponding to

$$\phi'(l_0) \neq -\phi'(0),$$

is also possible. In this case, there would be no equator in general, because β and ϕ would have maxima at different l; moreover, in general, we would have

$$\gamma(l_0) \neq \gamma(0) = 0,$$

and $\gamma(l)$ could even have no extremum. But because $\gamma' = 0$ at both centers, the integral in (51) taken from 0 to l_0 should vanish, and hence, again, V would have alternating sign.

The whole configuration with two regular centers has the topology $\mathbb{M}^{d_0} \times \mathbb{S}^{d_1+1}$, with closed extra dimensions in the spirit of Kaluza–Klein models. The main difference from them is that all variables now essentially depend on the extra coordinate l.

The main properties of all regular asymptotic regimes found, which lead to a classification of possible global monopole configurations in extra dimensions, are summarized in the Table. The word «attraction» corresponds to an increasing warp factor far from the brane.

5. SCALAR FIELD TRAPPING BY GLOBAL MONOPOLES

We consider a test scalar field with Lagrangian (22) in the background of global monopole configurations described in Sec. 4. After variable separation (24), the field equation for a $\mathbf{p}$-mode of the scalar field χ becomes

$$X'' + (d_0\gamma' + d_1\beta')X' + (e^{-2\gamma}p^2 - \mu^2)X = 0, \quad (56)$$

where

$$p^2 = p_\mu p^\mu = E^2 - \mathbf{p}^2$$

is the d_0-momentum squared and

$$\mu^2 = m_0^2 + \lambda\phi^2$$

is the effective mass squared. The trapping criterion consists, as before, in the requirements that the χ field stress–energy tensor must vanish far from the brane and the total χ field energy per unit volume of the brane must be finite, i.e.,

$$E_{tot}[\chi] = \int \sqrt{g} d^{d_1+1}x \times \\ \times \left[e^{-2\gamma}(E^2 + \mathbf{p}^2)X^2 + \mu^2X^2 + X'^2\right]dl < \infty. \quad (57)$$

The first requirement means that each term in the square brackets in (57) must vanish at large l.

We now check whether these requirements can be met at different kinds of asymptotic regimes listed in the Table.

A1: attracting AdS asymptotic regime $\beta \sim \gamma \sim kl$, $k > 0$. At large l, Eq. (56) reduces to the equation with constant coefficients

$$X'' + (D-1)X - \mu^2X = 0,$$

and its solution vanishing as $l \to \infty$ is

$$X \sim e^{-al}, \\ a = \frac{1}{2}\left[(D-1)k + \sqrt{(D-1)^2k^2 + 4\mu^2}\right]. \quad (58)$$

It is straightforward to verify that the trapping requirements are satisfied for all momenta $\mathbf{p}$ and all $\mu^2 \geq 0$.

A2(a): a quasi-flat asymptotic regime with a solid angle deficit. At large l, Eq. (25) reduces to

$$X'' + d_1X/l + P^2X = 0,$$

where

$$P^2 = p^2e^{-2\gamma_\infty} - \mu^2$$

and γ_∞ is the limiting value of γ at $l = \infty$. In terms of

$$Y = l^{d_1/2}X,$$

Classification of global monopole solutions for arbitrary $V(\phi)$ by asymptotic types. Attraction or repulsion is understood with respect to the center

Notation	r	$V(\phi)$	ϕ	γ	Asymptotic type
A1	∞	$V(\eta) < 0$	$\eta < \phi_{cr}$	∞	AdS, attraction
A2(a)	∞	0	$\eta < \phi_{cr}$	const	flat, solid angle deficit
A2(c), $d_1 > 2$	∞	0	0	const	flat
A2(c), $d_1 = 2$	∞	0	0	∞	«logarithmic», attraction
B1	r_*	$V_* < 0$	$\phi_* > \phi_{cr}$	$-\infty$	double horizon, repulsion
B2	r_*	$V_* < 0$	$\phi_* > \phi_{cr}$	∞	attracting tube
B3	r_*	0	$\phi_* = \phi_{cr}$	const	trivial tube
C	0	$V(0)$	0	const	second center

this equation is (at large l) rewritten as

$$Y'' + P^2 Y = 0,$$

while trapping condition (57) implies that

$$\int l^{d_1} X^2(l)\, dl < \infty.$$

Therefore, only an exponentially falling $Y(l)$ is suitable. In other words, the trapping condition is $P^2 < 0$, or

$$p^2 < m_{cr}^2 := \mu^2 e^{2\gamma_\infty}, \qquad (59)$$

where now

$$\mu^2 = m_0^2 + \lambda^2\eta^2.$$

We note that

$$p^2 = E^2 - \mathbf{p}^2$$

is nothing else but the observable mass of a free χ-particle if the observer watches its motion in the Minkowski section $l = 0$ of our manifold, i.e., on the brane. Hence, condition (59) means that the brane traps all scalar particles of masses smaller than the critical value m_{cr} depending on the model parameters.

A2(c), $d_1 > 2$**:** this case differs from the previous one only by the asymptotic value of ϕ, which is now zero, and hence $\mu = m_0$.

A2(c), $d_1 = 2$**:** a «logarithmic» asymptotic regime, $e^\gamma \sim \ln l$. Because $e^{-2\gamma} \sim 1/(\ln l)^2 \to 0$, the term with p^2 drops out from Eq. (56), which then leads to the decreasing solution

$$X \sim l^{-1} e^{-\mu l},$$

and a χ-particle is trapped provided $\mu = m_0 > 0$.

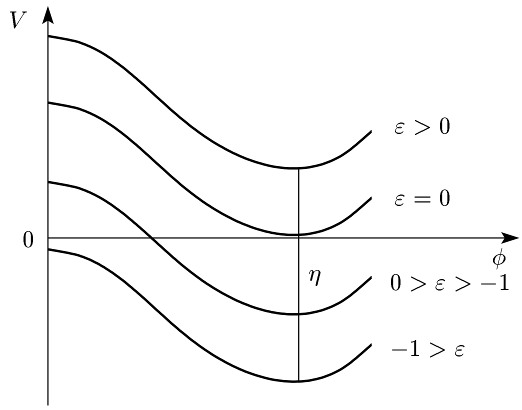

Fig. 1. Mexican hat potential

B1: a horizon. As was remarked previously, we do not regard this configuration viable and omit it from our discussion.

B2: an attracting tube, $r \to r_*$ and $\gamma \approx hl$, $h > 0$ as $l \to \infty$. Equation (56) takes the form

$$X'' + d_0 h X' - \mu^2 X = 0$$

and has the decreasing solutions

$$X \sim e^{-al}, \quad a = \frac{1}{2}\left(d_0 h + \sqrt{d_0^2 h^2 + 4\mu^2}\right). \qquad (60)$$

As in item A1, it is easy to verify that the trapping conditions hold provided $\mu^2 > 0$.

B3: a trivial tube, both β and γ tend to constants as $l \to \infty$. In Eq. (56), the term with X' drops out at large l, and an exponentially decreasing solution exists under condition (59) where

$$\mu^2 = m_0^2 + \lambda^2 \phi_{cr}^2.$$

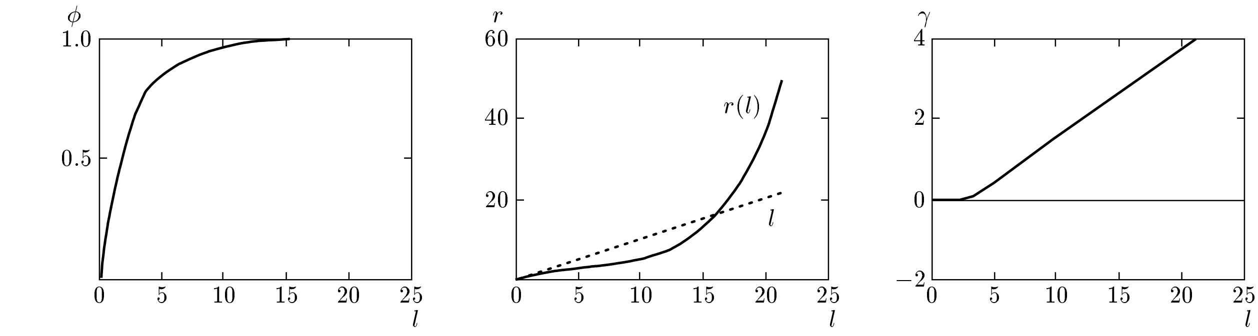

Fig. 2. A regular solution with an AdS asymptotic regime (type A1) for the potential (61) with $\varkappa\eta^2 = 5$, $\varepsilon = -0.75$, $d_0 = 4$, $d_1 = 3$

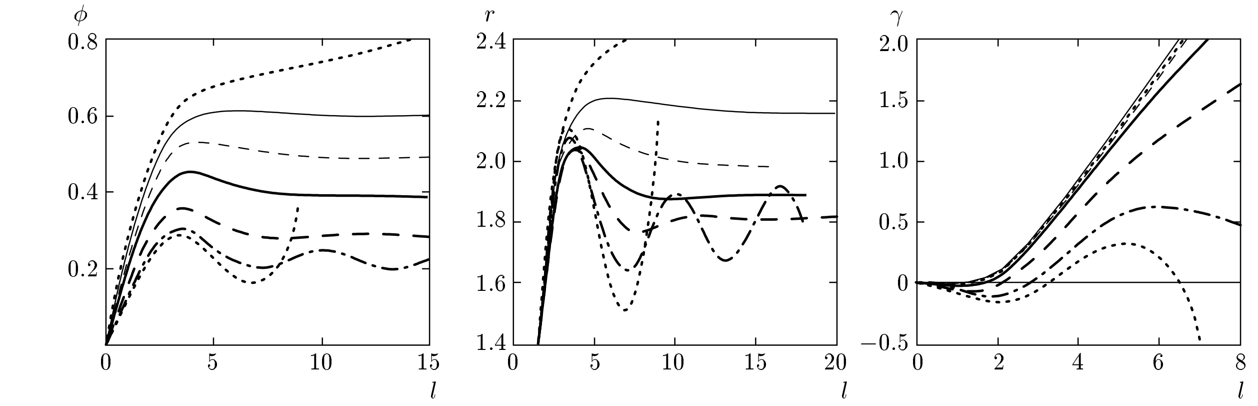

Fig. 3. Regular (except the dotted curves) solutions with the B2 asymptotic regime (attracting tube), such that $r \to r_* < \infty$ and $\gamma'_\infty > 0$

C: these configurations have no large l asymptotic regimes and are not interpreted in terms of branes.

A conclusion is that scalar particles of any mass and momentum are trapped by global monopoles with A1 and B2 asymptotic regimes with exponentially growing warp factors and A2(c) with a logarithmic asymptotic regime; they are trapped under restrictions (59) on the particle's observable mass by monopoles with A2 and B3 asymptotic regimes whose warp factors tend to constant limits far from the brane.

6. NUMERICAL RESULTS: MEXICAN HAT POTENTIAL

In this section, we present the results of our numerical calculations, which confirm the classification of regular solutions given above. We have used the «Mexican hat» potential in the form (Fig. 1)

$$V = \frac{\lambda\eta^4}{4}\left[\varepsilon + \left(1 - \frac{\phi^2}{\eta^2}\right)^2\right]. \qquad (61)$$

It has two extremum points in the range $\phi \geq 0$: a maximum at $\phi = 0$ and a minimum at $\phi = \eta$. The SSB energy scale is characterized by $\eta^{2/(D-2)}$, while $\sqrt{\lambda}\eta$ determines, as usual, a length scale. The nonconventional parameter ε introduced in (61), moves the potential up and down, which is equivalent to adding a cosmological constant to the usual Mexican hat potential.

Given potential (61), the nature of the solutions essentially depends on its two dimensionless parameters: ε, fixing the extremal values of the potential with respect to zero, and $\varkappa^2\eta^2$, characterizing the gravitational field strength: as we remember from Sec. 4, the asymptotic regime $r \to \infty$ only exists when $\phi_\infty < \phi_{cr}$, which is the same as

$$\varkappa^2\eta^2 < d_1 - 1.$$

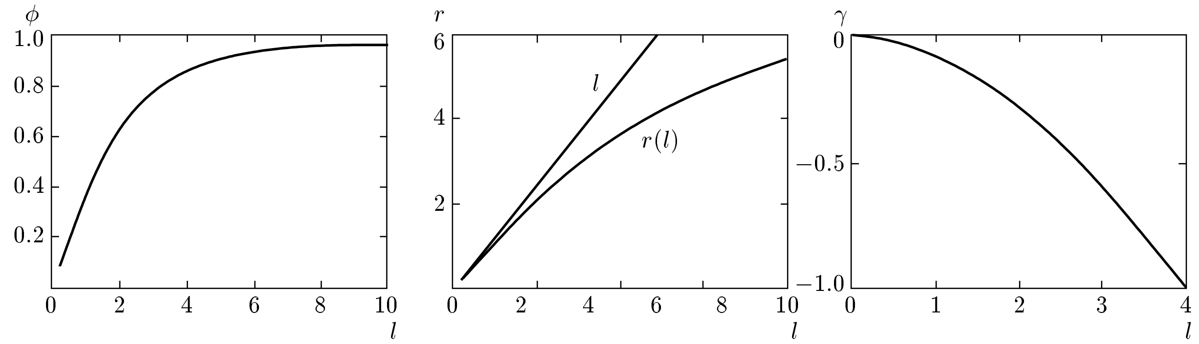

Fig. 4. A regular solution with the asymptotic regime $r \to r_* < \infty$ and $\gamma'_\infty < 0$ (case B1, horizon)

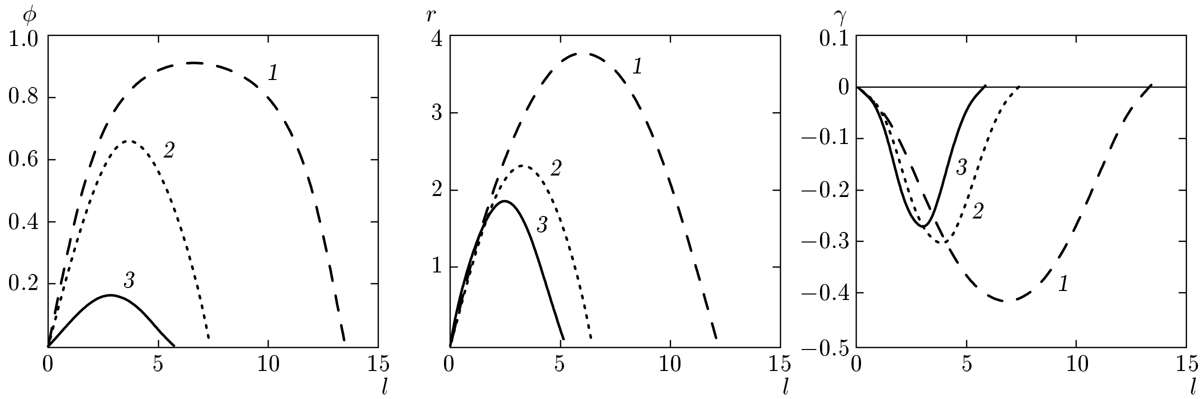

Fig. 5. Type-C solutions with two regular centers ($r \to 0$, $\phi \to 0$, $\gamma' \to 0$ as $l \to l_0$)

If $\varepsilon > 0$, potential (61) is always positive, and, in accordance with item A1, regular solutions are absent.

In the conventional case where $\varepsilon = 0$, in the range

$$0 < \varkappa^2\eta^2 < d_1 - 1,$$

there are asymptotically flat regular solutions with a solid angle deficit (class A2).

The most complex case $0 > \varepsilon > -1$ contains a variety of possibilities. Regular solutions with the asymptotic behavior $r \to \infty$ as $l \to \infty$ having $\gamma'_\infty > 0$ (case A1) exist in some range $0 < \eta < \eta_s$, where the separating value η_s depends on d_0, d_1, and ε. As an example, such a regular solution with $\varkappa^2\eta^2 = 5$, $\varepsilon = -0.75$, $d_0 = 4$, and $d_1 = 3$ is presented in Fig. 2.

Depending on the parameters of the potential, there are regular solutions with the asymptotic regime $r \to r_* < \infty$ and $\gamma'_\infty > 0$ (case B2) in some range $\eta_{s1} < \eta < \eta_{s2}$, see Fig. 3. Here, $\varepsilon = -0.9$, $d_0 = 4$, $d_1 = 3$. The curves are given for $\varkappa^2\eta^2 = 10$, 12, 15, 20, 30, 40, and 45 (from top down). The dotted curves ($\varkappa^2\eta^2 = 10$ and $\varkappa^2\eta^2 = 45$) correspond to singular configurations. It follows that for $\varepsilon = -0.9$, $d_0 = 4$, $d_1 = 3$, the lower bound of this parameter leading to regular models is somewhere between 10 and 12, while the upper bound is between 30 and 45.

An example of a regular solution with the asymptotic regime $r \to r_* < \infty$ and $\gamma'_\infty < 0$ (class B1), corresponding to a second-order Killing horizon, is shown in Fig. 4.

The value $\varkappa\eta^2 = 17.37$ is fine-tuned to the parameters $\varepsilon = -0.75$, $d_0 = 4$, $d_1 = 2$ of this particular solution.

Other examples of fine-tuned regular solutions, namely, type C with two regular centers ($r \to 0$, $\phi \to 0$, $\gamma' \to 0$ at $l \to l_0$), are presented in Fig. 5.

For all the three curves, $d_0 = 4$ and $d_1 = 2$. The curves (*1*, *2*, and *3*) correspond to $\varepsilon = -0.15$, -0.5, and -0.9626, respectively. The fine-tuned values of $\varkappa^2\eta^2$ are approximately 2.637, 6.17, and 100.

In the case $\varepsilon \le -1$, the maximum $V(0) \le 0$ is at or below the zero level, and there is a possibility for the integral curves to start and finish at the same value $\phi(0) = \phi(\infty) = 0$. We then observe a whole family of

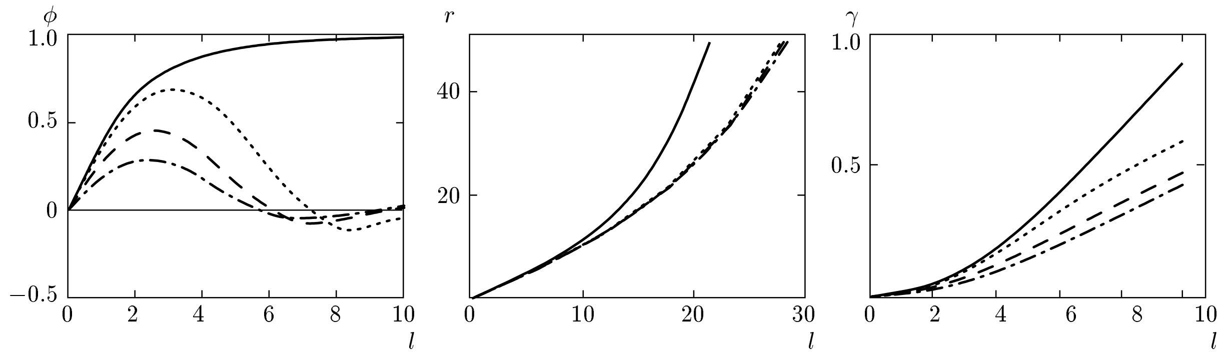

Fig. 6. Regular solutions starting and terminating at $\phi(0) = \phi(\infty) = 0$. The limiting solid curve with $\phi'(0) = \phi'_c = 0.4401425$ (separatrix) terminates at $\phi(\infty) = \eta$

such regular curves in the range $0 < \phi'(0) < \phi'_c$, see Fig. 6.

For the particular example presented ($\varepsilon = -1.5$, $\varkappa\eta^2 = 1$, $d_0 = 4$, $d_1 = 3$), the values of $\phi'(0)$ for the dotted curves ending with $\phi = 0$ are 0.2, 0.3, and 0.4 (from bottom up). The limiting solid curve with $\phi'(0) = \phi'_c = 0.4401425$ (separatrix) is a regular solution ending at the minimum of the potential: $\phi \to \eta$ as $l \to \infty$.

The Mexican hat potential (61), with its only two extrema at $\phi = 0$ and $\phi = \eta$, cannot demonstrate the whole variety of solutions that appear with more sophisticated potentials having additional maxima and/or minima. Thus, for instance, class-A solutions may have a large-r asymptotic regime at any such extremum.

7. CONCLUDING REMARKS

We have obtained as many as seven classes of regular solutions of the field equations describing a Minkowski thick brane with a global monopole in extra dimensions, see the Table.

Some of these classes, namely, A1 with an AdS asymptotic form and B2 ending with an attracting tube, have the exponentially growing warp factor $e^{2\gamma}$ at large l and are shown to trap linear test scalar fields modes of any mass and momentum.

Others — A2(a) and A2(c) for $d_1 > 2$, ending with a flat metric at large l — have a warp factor tending to a constant whose value is determined by the shape of the potential $V(\phi)$. They are also shown to trap a test scalar field but the observable mass of the field is restricted from above by a value depending on the particular model of the global monopole.

Lastly, for $d_1 = 2$, i.e., a three-dimensional global monopole in the extra dimensions, class A2(c) solutions have a logarithmically growing warp factor. All test scalar field modes are trapped by this configuration, but the slow growth of $\gamma(l)$ probably means that the test field is strongly smeared over the extra dimensions.

All such configurations, in sharp contrast to RS2-like domain walls in 5 dimensions, are able to trap scalar matter. It is certainly necessary to check whether nonzero-spin fields are trapped as well and Newton's law of gravity holds on the brane in conformity with the experiment. We hope to consider these subjects in our future publications.

In addition to the trapping problem, a shortcoming of RS2-type Minkowski branes is that they are necessarily fine-tuned. Many of the global monopole solutions, at least those existing in the weak gravity regime (class A), are free of this shortcoming and are thus better for thick brane model building.

Some results and conclusions in this paper were previously given in Refs. [12, 13]. The main difference of our approach from theirs is their boundary condition, which is $\phi = \eta$ in our notation. This excludes the cases where the solution ends at a maximum or slope of the potential, such as, e.g., symmetric solutions with two regular centers. Another difference is that they consider solutions with an exponentially decreasing warp factor as those leading to matter confinement on the brane. In our view, such solutions with second-order horizons do not represent viable models of a braneworld. We conclude that the present paper gives the most complete classification of all regular solutions for global monopoles in extra dimensions, which, even without gauge fields, seem to be promising as braneworld models.

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REFERENCES

1. K. Akama, Lect. Notes Phys. **176**, 267 (1982); V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125**, 136 (1983); M. Visser, Phys. Lett. B **159**, 22 (1985); I. Antoniadis, Phys. Lett. B **246**, 377 (1985); M. Pavšič, Phys. Lett. A **116**, 1 (1986); E-print archives gr-qc/0101075; Nuovo Cim. A **95**, 297 (1986); E. J. Squires, Phys. Lett. B **167**, 286 (1986); G. W. Gibbons and D. L. Wiltshire, Nucl. Phys. B **287**, 717 (1987).
2. P. Horava and E. Witten, Nucl. Phys. B **460**, 506 (1996); **475**, 94 (1996); E. Witten, Nucl. Phys. B **471**, 135 (1996).
3. V. A. Rubakov, Uspekhi Fiz. Nauk **171**, 913 (2001); F. Quevedo, Class. Quantum Grav. **19**, 5721 (2002); S. Nojiri, S. D. Odintsov, and S. Ogushi, Int. J. Mod. Phys. A **17**, 4809 (2002); P. Brax and C. van de Bruck, E-print archives hep-th/0303095; R. Maartens, Living Rev. Rel. **7**, 7 (2004); E-print archives gr-qc/0312059; P. Kanti, Int. J. Mod. Phys. A **19**, 4899 (2004); E-print archives hep-ph/0402168.
4. L. Rundall and R. Sundrum, Phys. Rev. Lett. **83**, 3690 (1999).
5. D. A. Kirzhnits, Pis'ma v Zh. Eksp. Teor. Fiz. **15**, 745 (1972).
6. Ya. B. Zel'dovich, I. Yu. Kobzarev, and L. B. Okun', Zh. Eksp. Teor. Fiz. **67**, 3 (1974).
7. A. H. Guth, Phys. Rev. D **23**, 347 (1981); A. D. Linde, Phys. Lett. B **114**, 431 (1982); V. A. Berezin, V. A. Kuzmin, and I. I. Tkachev, Phys. Lett. B **120**, 91 (1983); **124**, 479 (1983).
8. A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects*, Cambridge Univ. Press, Cambridge (1994).
9. C. Csaki, J. Erlich, T. J. Hollowood, and Y. Shirman, Nucl. Phys. B **581**, 309 (2000); D. Bazeia, F. A. Brito, and J. R. Nascimento, Phys. Rev. D **68**, 085007 (2003); S. Kobayashi, K. Koyama, and J. Soda, Phys. Rev. D **65**, 064014 (2002); A. Melfo, N. Pantoja, and A. Skirzewski, Phys. Rev. D **67**, 105003 (2003).
10. K. A. Bronnikov and B. E. Meierovich, Grav. & Cosmol. **9**, 313 (2003); E-print archives gr-qc/0402030.
11. S. T. Abdyrakhmanov, K. A. Bronnikov, and B. E. Meierovich, Grav. & Cosmol. **11**, 82 (2005); E-print archives gr-qc/0503055.
12. I. Olasagasti and A. Vilenkin, Phys. Rev. D **62**, 044014 (2000); E-print archives hep-th/0003300.
13. E. Roessl and M. Shaposhnikov, Phys. Rev. D **66**, 084008 (2002); E-print archives hep-th/0205320.
14. K. A. Bronnikov and B. E. Meierovich, Zh. Eksp. Teor. Fiz. **124**, 5 (2003); E-print archives gr-qc/0301084.
15. B. E. Meierovich, Uspekhi Fiz. Nauk **44**, 981 (2001).
16. K. A. Bronnikov, B. E. Meierovich, and E. R. Podolyak, Zh. Eksp. Teor. Fiz. **122**, 459 (2002).
17. I. Cho and A. Vilenkin, E-print archives gr-qc/9708005.
18. K. Benson and I. Cho, E-print archives hep-th/0104067.
19. B. Bajc and G. Gabadadze, Phys. Lett. B **474**, 282 (2000).
20. I. Oda, Phys. Rev. D **62**, 126009 (2000).
21. M. Gogberashvili and P. Midodashvili, Phys. Lett. B **515**, 447 (2001); Europhys. Lett. **61**, 308 (2003); M. Gogberashvili and D. Singleton, E-print archives hep-th/030524; I. Oda, Phys. Lett. B **571**, 235 (2003).
22. M. Gogberashvili and D. Singleton, Phys. Lett. B **582**, 95 (2004).