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COULOMB CORRECTIONS TO BREMSSTRAHLUNG IN ELECTRIC FIELD OF HEAVY ATOM AT HIGH ENERGIES

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We consider the differential and partially integrated cross sections for bremsstrahlung from high-energy electrons in the atomic field, with this field taken into account exactly. We use the semiclassical electron Green's function and wave functions in an external electric field. It is shown that the Coulomb corrections to the differential cross section are very susceptible to screening. Nevertheless, the Coulomb corrections to the cross section summed over the final-electron states are independent of screening in the leading approximation in the small parameter $1/mr_{scr}$ (r_{scr} is the screening radius and m is the electron mass, $\hbar = c = 1$). We also consider bremsstrahlung from a finite-size electron beam on a heavy nucleus. The Coulomb corrections to the differential probability are also very susceptible to the beam shape, while the corrections to the probability integrated over momentum transfer are independent of it, apart from the trivial factor, which is the electron-beam density at zero impact parameter. For the Coulomb corrections to the bremsstrahlung spectrum, the next-to-leading terms with respect to the parameters m/ε (ε is the electron energy) and $1/mr_{scr}$ are obtained.

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1. INTRODUCTION

Bremsstrahlung in the electric field of atoms is a fundamental QED process. Its investigation, started in the 1930s, is important for various applications. In the Born approximation, both the differential cross section and the bremsstrahlung spectrum have been obtained for arbitrary electron energies and atomic form factors [1] (see also Ref. [2]). High-energy asymptotics of the bremsstrahlung cross section in a Coulomb field has been studied in detail in Ref. [3] exactly in the parameter $Z\alpha$ (where Z is the atomic number and $\alpha = 1/137$ is the fine-structure constant). In these papers, the differential cross sections and the bremsstrahlung spectrum have been obtained. For a screened Coulomb field, the high-energy asymptotics of the differential cross section was derived in Ref. [4]. The effect of screening on the spectrum was studied in Refs. [5, 6]. For the spectrum, it turned out that screening is essential only in the Born approximation. In other words, the Coulomb corrections to the spectrum are not significantly modified by screening. By definition, Coulomb corrections are the difference between the result obtained exactly in the external field and that obtained in the Born approximation. In the recent paper [7], it was claimed that Coulomb corrections to the differential cross section of the bremsstrahlung are also independent of screening.

In the present paper, we investigate the bremsstrahlung cross section in the electric field of a heavy atom. We assume that $\varepsilon, \varepsilon' \gg m$, where ε and ε' are the initial and final electron energies, respectively. In Sec. 2, we consider the differential cross section in detail in the leading approximation, i.e., neglecting corrections in the parameters m/ε and $1/mr_{scr}$. In contrast to the statement in Ref. [7], screening may strongly modify Coulomb corrections to the differential cross section. We demonstrate explicitly that this fact does

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not contradict the final-state integration theorem [5], which implies that Coulomb corrections to the spectrum are independent of screening. We also study the influence of the electron beam finite size on Coulomb corrections. Again, Coulomb corrections to the differential cross section are very sensitive to the shape of the electron beam, while the spectrum is independent of it, except for a trivial factor. In Sec. 3, we consider corrections to Coulomb corrections in the spectrum. It turns out that in the first nonvanishing order, they enter the spectrum as a sum of two terms. The first term is proportional to m/ε and is independent of screening. The second term is small in the parameter $1/mr_{scr}$ and is independent of the energy.

Our approach is based on the use of the semiclassical Green's function and the semiclassical wave function of the electron in an external field. Previously, this method was successfully applied to the investigation of the photoproduction process at high energy [8, 9].

2. DIFFERENTIAL CROSS SECTION

The cross section of the electron bremsstrahlung in the external field has the form

$$d\sigma^\gamma = \frac{\alpha}{(2\pi)^4\omega}\,d\mathbf{p}'\,d\mathbf{k}\,\delta(\varepsilon-\varepsilon'-\omega)|M|^2\,, \qquad (1)$$

where $\mathbf{k}$ is the photon momentum, $\mathbf{p}$ and $\mathbf{p}'$ are the respective initial and final electron momenta,

$$\omega=|\mathbf{k}|, \quad \varepsilon=\varepsilon_p=\sqrt{\mathbf{p}^2+m^2}, \quad \varepsilon'=\varepsilon_{p'}.$$

The matrix element M is given by

$$M=\int d\mathbf{r}\exp(-i\mathbf{k}\cdot\mathbf{r})\bar\psi^{(out)}_{P'}(\mathbf{r})\hat e^*\psi^{(in)}_P(\mathbf{r}), \qquad (2)$$

where $\psi^{(in)}_P$ and $\psi^{(out)}_P$ are the respective wave functions of the in- and out-state of the electron in an external field, containing the diverging and converging spherical waves and the plain wave with 4-momentum P in their asymptotics, $\hat e^*=e^*_\mu\gamma^\mu$, e_μ is the photon polarization 4-vector, and γ^μ are the Dirac matrices.

In Ref. [10], the semiclassical wave function of the electron in an arbitrary localized potential was found with the first correction in m/ε taken into account. In calculating bremsstrahlung and the e^+e^- photoproduction cross section in the leading approximation, the following form of the wave function can be used [10]:

$$\begin{gathered}\psi^{(in,\,out)}_P(\mathbf{r})=\pm\int\frac{d\mathbf{q}}{i\pi}\times\\ \times\exp\left[i\mathbf{p}\cdot\mathbf{r}\pm iq^2\mp i\lambda\int\limits_0^\infty dx\,V(\mathbf{r}_x)\right]\times\\ \times\left\{1\mp\frac{1}{2p}\int\limits_0^\infty dx\,\boldsymbol{\alpha}\cdot\nabla V(\mathbf{r}_x)\right\}u_P,\\ \mathbf{r}_x=\mathbf{r}\mp x\mathbf{n}+\mathbf{q}\sqrt{2|\mathbf{r}\cdot\mathbf{n}|/p}\,,\\ \lambda=\operatorname{sign}P^0\,,\quad \mathbf{n}=\mathbf{p}/p\,.\end{gathered} \qquad (3)$$

In this formula, $\mathbf{q}$ is a two-dimensional vector lying in the plane perpendicular to $\mathbf{p}$, the upper sign corresponds to $\psi^{(in)}_P$, and u_P is the conventional Dirac spinor. We recall that the wave function $\psi^{(in)}_{(-\varepsilon_p,-\mathbf{p})}$ corresponds to the positron in the final state with the 4-momentum $(\varepsilon_p,\mathbf{p})$. For a Coulomb field, wave function (3) coincides with the standard Furry–Sommerfeld–Maue wave function. When the angles between $\mathbf{p}$ and $\mathbf{r}$ in $\psi^{(in)}_P(\mathbf{r})$, and between $\mathbf{p}$ and $-\mathbf{r}$ in $\psi^{(out)}_P(\mathbf{r})$ are not small, it is possible to replace $\mathbf{r}_x$ in Eq. (3) by $\mathbf{R}_x=\mathbf{r}\mp x\mathbf{n}$. Then the integral over $\mathbf{q}$ can be taken, and we obtain the conventional eikonal wave function

$$\begin{gathered}\psi^{(in,\,out)}_{P,\,eik}(\mathbf{r})=\exp\left[i\mathbf{p}\cdot\mathbf{r}\mp i\lambda\int\limits_0^\infty dx\,V(\mathbf{R}_x)\right]\times\\ \times\left\{1\mp\frac{1}{2p}\int\limits_0^\infty dx\,\boldsymbol{\alpha}\cdot\nabla V(\mathbf{R}_x)\right\}u_P\,.\end{gathered} \qquad (4)$$

We direct the z axis along the vector $\boldsymbol{\nu}=\mathbf{k}/\omega$, then $\mathbf{r}=z\boldsymbol{\nu}+\boldsymbol{\rho}$. In this frame, the polar angles of $\mathbf{p}$ and $\mathbf{p}'$ are small. We split the integration region in Eq. (2) into two: $z>0$ and $z<0$. The corresponding contributions to M are denoted as M_+ and M_-, with $M=M_++M_-$. For $z>0$, the function $\psi^{(out)}_{p'}(\mathbf{r})$ has the eikonal form and we obtain

$$\begin{gathered}M_+=\int\limits_{z>0}d\mathbf{r}\int\frac{d\mathbf{q}}{i\pi}\exp\left\{i\mathbf{q}^2-i\boldsymbol{\Delta}\cdot\mathbf{r}-\right.\\ \left.-i\int\limits_0^\infty dx\left[V\left(\mathbf{r}-\mathbf{n}x+\mathbf{q}\sqrt{2z/p}\right)+V(\mathbf{r}+\mathbf{n}'x)\right]\right\}\times\\ \times\bar u_{p'}\left[\hat e^*-\frac{1}{2p}\int\limits_0^\infty dx\hat e^*\boldsymbol{\alpha}\cdot\nabla V\left(\mathbf{r}-\mathbf{n}x+\mathbf{q}\sqrt{2z/p}\right)-\right.\\ \left.-\frac{1}{2p'}\int\limits_0^\infty dx\boldsymbol{\alpha}\cdot\nabla V(\mathbf{r}+\mathbf{n}'x)\hat e^*\right]u_p\,,\end{gathered} \qquad (5)$$

where $\mathbf{\Delta} = \mathbf{p}' + \mathbf{k} - \mathbf{p}$ is the momentum transfer.

In Eq. (5), we have replaced $\sqrt{2|\mathbf{r}\cdot\mathbf{n}|/p}$ in the definition of $\mathbf{r}_x$ in Eq. (3) by $\sqrt{2z/p}$. It is easy to see that within our accuracy, we can also replace the quantity $V(\mathbf{r}+\mathbf{n}'x)$ in Eq. (5) by $V(\mathbf{r}+\mathbf{n}'x+\mathbf{q}\sqrt{2z/p})$ and consider the vector $\mathbf{q}$ to be perpendicular to z axis. After that, we shift $\boldsymbol{\rho}\to\boldsymbol{\rho}-\mathbf{q}\sqrt{2z/p}$ and take the integral over $\mathbf{q}$. We obtain

$$
\begin{aligned}
M_+ = \int\limits_{z>0} d\mathbf{r}\exp\Bigg\{ -i\frac{z}{2p}\Delta_\perp^2 - i\mathbf{\Delta}\cdot\mathbf{r} - \\
- i\int\limits_0^\infty dx[V(\mathbf{r}-\mathbf{n}x)+V(\mathbf{r}+\mathbf{n}'x)]\Bigg\}\times \\
\times\,\bar{u}_{p'}\Bigg[\hat{e}^* - \frac{1}{2p}\int\limits_0^\infty dx\,\hat{e}^*\boldsymbol{\alpha}\cdot\nabla V(\mathbf{r}-\mathbf{n}x) - \\
-\frac{1}{2p'}\int\limits_0^\infty dx\,\boldsymbol{\alpha}\cdot\nabla V(\mathbf{r}+\mathbf{n}'x)\hat{e}^*\Bigg]u_p. \quad (6)
\end{aligned}
$$

In the same way, we obtain

$$
\begin{aligned}
M_- = \int\limits_{z<0} d\mathbf{r}\exp\Bigg\{ i\frac{z}{2p'}\Delta_\perp^2 - i\mathbf{\Delta}\cdot\mathbf{r} - \\
- i\int\limits_0^\infty dx[V(\mathbf{r}-\mathbf{n}x)+V(\mathbf{r}+\mathbf{n}'x)]\Bigg\}\times \\
\times\,\bar{u}_{p'}\Bigg[\hat{e}^* - \frac{1}{2p}\int\limits_0^\infty dx\hat{e}^*\boldsymbol{\alpha}\cdot\nabla V(\mathbf{r}-\mathbf{n}x) - \\
-\frac{1}{2p'}\int\limits_0^\infty dx\,\boldsymbol{\alpha}\cdot\nabla V(\mathbf{r}+\mathbf{n}'x)\hat{e}^*\Bigg]u_p. \quad (7)
\end{aligned}
$$

There are two overlapping regions of the momentum transfer Δ,

$$
\begin{aligned}
&\text{I. } \Delta \ll \frac{m\omega}{\varepsilon}, \\
&\text{II. } \Delta \gg \Delta_{min} = \frac{m^2\omega}{2\varepsilon\varepsilon'}.
\end{aligned} \quad (8)
$$

In the first region, we can neglect the terms proportional to $\Delta_\perp^2$ in the exponents in Eqs. (6) and (7). Then the sum

$$M = M_+ + M_-$$

becomes

$$
\begin{aligned}
M = \int d\mathbf{r}\exp\Bigg\{ - i\mathbf{\Delta}\cdot\mathbf{r} - \\
-i\int\limits_0^\infty dx[V(\mathbf{r}-\mathbf{n}x)+V(\mathbf{r}+\mathbf{n}'x)]\Bigg\}\times \\
\times\,\bar{u}_{p'}\Bigg[\hat{e}^* - \frac{1}{2p}\int\limits_0^\infty dx\,\hat{e}^*\boldsymbol{\alpha}\cdot\nabla V(\mathbf{r}-\mathbf{n}x) - \\
-\frac{1}{2p'}\int\limits_0^\infty dx\,\boldsymbol{\alpha}\cdot\nabla V(\mathbf{r}+\mathbf{n}'x)\hat{e}^*\Bigg]u_p. \quad (9)
\end{aligned}
$$

We can make the replacement $\mathbf{n},\mathbf{n}'\to\boldsymbol{\nu}$ in the prefactor in Eq. (9). In the exponent, we must take the linear term of the expansion of the integral in $\mathbf{n}-\boldsymbol{\nu}$ and $\mathbf{n}'-\boldsymbol{\nu}$ into account. As a result, we have

$$
\begin{aligned}
M = \int d\mathbf{r}\exp\left[-i\mathbf{\Delta}\cdot\mathbf{r} - i\chi(\boldsymbol{\rho})\right]\times \\
\times\int\limits_0^\infty dy\,\bar{u}_{p'}\Big[\hat{e}^*[iy(\mathbf{n}-\boldsymbol{\nu})-\boldsymbol{\alpha}/2p]\cdot\nabla V(\mathbf{r}-\boldsymbol{\nu}y) + \\
+[-i\,y(\mathbf{n}'-\boldsymbol{\nu})-\boldsymbol{\alpha}/2p']\cdot\nabla V(\mathbf{r}+\boldsymbol{\nu}y)\hat{e}^*\Big]u_p\,, \\
\chi(\boldsymbol{\rho}) = \int\limits_{-\infty}^{\infty} dz V(\mathbf{r})\,.
\end{aligned} \quad (10)
$$

In the arguments of $V(\mathbf{r}\pm\boldsymbol{\nu}y)$, we make the substitutions $z\to z\mp y$. After that, we take the integral over y and obtain

$$
\begin{aligned}
M = \mathbf{A}(\mathbf{\Delta})\cdot\Bigg(\bar{u}_{p'}\Bigg[\frac{(\mathbf{n}-\mathbf{n}')\hat{e}^*}{\Delta_z^2} - \\
-\frac{\hat{e}^*\boldsymbol{\alpha}}{2p\Delta_z}+\frac{\boldsymbol{\alpha}\hat{e}^*}{2p'\Delta_z}\Bigg]u_p\Bigg)\,, \\
\mathbf{A}(\mathbf{\Delta}) = -i\int d\mathbf{r}\exp[-i\mathbf{\Delta}\cdot\mathbf{r}-i\chi(\boldsymbol{\rho})]\nabla_\rho V(\mathbf{r})\,.
\end{aligned} \quad (11)
$$

We now pass to the calculation of M in the second region, where $\Delta\gg\Delta_{min}$. In Eq. (6) for M_+, we can replace $\mathbf{n}'\to\mathbf{n}$ and $z\Delta_\perp^2/2p\to\tilde{z}\Delta_\perp^2/2p$, where $\tilde{z}=\mathbf{r}\cdot\mathbf{n}$. Because the polar angle of $\mathbf{n}$ is small, we can integrate in Eq. (6) over the half-space $\tilde{z}>0$. After the integration over $\tilde{z}$, we obtain

$$
\begin{aligned}
M_+ = -i\int d\boldsymbol{\rho}\exp\left[-i\mathbf{\Delta}\cdot\boldsymbol{\rho}-i\chi(\boldsymbol{\rho})\right]\times \\
\times\frac{\bar{u}_{p'}\hat{e}^*\left[2p+\boldsymbol{\alpha}\cdot\mathbf{\Delta}_\perp\right]u_p}{2p\mathbf{\Delta}\cdot\mathbf{n}+\Delta_\perp^2}\,. \quad (12)
\end{aligned}
$$

The calculation of M_- is performed quite similarly. As a result, we have

$$M = -i\int d\boldsymbol{\rho}\exp\left[-i\boldsymbol{\Delta}\cdot\boldsymbol{\rho} - i\chi(\boldsymbol{\rho})\right]\times \\ \times\bar{u}_{p'}\left[\frac{\hat{e}^*(2p+\boldsymbol{\alpha}\cdot\boldsymbol{\Delta}_\perp)}{2p\boldsymbol{\Delta}\cdot\mathbf{n}+\Delta_\perp^2} - \frac{(2p'+\boldsymbol{\alpha}\cdot\boldsymbol{\Delta}_\perp)\hat{e}^*}{2p'\boldsymbol{\Delta}\cdot\mathbf{n}'-\Delta_\perp^2}\right]u_p. \quad (13)$$

Now we can write the representation for M that is valid in both regions,

$$M = \frac{\varepsilon\varepsilon'}{\omega}\mathbf{A}(\boldsymbol{\Delta})\cdot\left\{\bar{u}_{p'}\left[-2\hat{e}^*\frac{\mathbf{p}_\perp+\mathbf{p}'_\perp}{\delta\delta'} + \right.\right. \\ \left.\left. + \frac{\hat{e}^*\boldsymbol{\alpha}}{\varepsilon\delta'} - \frac{\boldsymbol{\alpha}\hat{e}^*}{\varepsilon'\delta}\right]u_p\right\}, \quad (14) \\ \delta = m^2 + \mathbf{p}_\perp^2, \quad \delta' = m^2 + \mathbf{p}'^2_\perp.$$

Within our accuracy, this expression coincides with Eq. (11) in region I and with Eq. (13) in region II. Using the explicit form of the Dirac spinors, we finally obtain

$$M = \frac{1}{2\delta\delta'}\mathbf{A}(\boldsymbol{\Delta})\cdot\left\{\varphi'^\dagger\left[(\mathbf{p}_\perp+\mathbf{p}'_\perp)\times\right.\right. \\ \times\left(\frac{\varepsilon+\varepsilon'}{\omega}\mathbf{e}^*\cdot(\mathbf{p}_\perp+\mathbf{p}'_\perp) - i[\boldsymbol{\sigma}\times\mathbf{e}^*]\cdot(\mathbf{p}_\perp+\mathbf{p}'_\perp) + \right. \\ \left. + 2im[\boldsymbol{\sigma}\times\mathbf{e}^*]_z\right) - (\delta+\delta')\times \\ \left.\left.\times\left(\frac{\varepsilon+\varepsilon'}{\omega}\mathbf{e}^* - i[\boldsymbol{\sigma}\times\mathbf{e}^*]_\perp\right)\right]\varphi\right\}. \quad (15)$$

This expression is in agreement with that obtained in [4] by another method. We emphasize that the potential enters amplitude (15) only via $\mathbf{A}(\boldsymbol{\Delta})$.

2.1. Coulomb corrections to the differential cross section in a screened Coulomb potential

We discuss Coulomb corrections to the differential cross section of bremsstrahlung. We recall that these corrections are the difference between the exact (in the external field strength) cross section and that obtained in the Born approximation, which is proportional to $[|\mathbf{A}(\boldsymbol{\Delta})|^2 - |\mathbf{A}_B(\boldsymbol{\Delta})|^2]$ with $\mathbf{A}(\boldsymbol{\Delta})$ from Eq. (11) and

$$\mathbf{A}_B(\boldsymbol{\Delta}) = -i\int d\mathbf{r}\exp[-i\boldsymbol{\Delta}\cdot\mathbf{r}]\nabla_\rho V(\mathbf{r}) = \\ = \boldsymbol{\Delta}_\perp\int d\mathbf{r}\exp[-i\boldsymbol{\Delta}\cdot\mathbf{r}]V(\mathbf{r}). \quad (16)$$

The screening modifies the Coulomb potential of the nucleus at distances

$$r_{scr} \gg \lambda_C = 1/m.$$

In the region

$$\Delta \gg \max(\Delta_{min}, r_{scr}^{-1}),$$

the quantities $\mathbf{A}(\boldsymbol{\Delta})$ and $\mathbf{A}_B(\boldsymbol{\Delta})$ are of the form

$$\mathbf{A}(\boldsymbol{\Delta}) = \mathbf{A}_B(\boldsymbol{\Delta})\frac{\Gamma(1-iZ\alpha)}{\Gamma(1+iZ\alpha)}\left(\frac{4}{\Delta_\perp^2}\right)^{-iZ\alpha} = \\ = -\boldsymbol{\Delta}_\perp\pi Z\alpha\frac{\Gamma(1-iZ\alpha)}{\Gamma(1+iZ\alpha)}\left(\frac{4}{\Delta_\perp^2}\right)^{1-iZ\alpha}. \quad (17)$$

Therefore,

$$|A(\boldsymbol{\Delta})|^2 = |A_B(\boldsymbol{\Delta})|^2 \quad \text{for} \quad \Delta \gg \max(\Delta_{min}, r_{scr}^{-1})$$

and Coulomb corrections to the differential cross section vanish in this region in the leading approximation. Hence, Coulomb corrections are important only in the region

$$\Delta \lesssim \max(\Delta_{min}, r_{scr}^{-1}) \ll m.$$

In this region, we can use Eq. (11) for the matrix element. For the Coulomb corrections, substituting Eq. (11) in Eq. (1), using the relation

$$d\Omega_{\mathbf{p}'}d\Omega_{\mathbf{k}} = \frac{d\phi\, d\boldsymbol{\Delta}_\perp d\Delta_z}{\omega\varepsilon\varepsilon'},$$

and integrating over the azimuthal angle ϕ and summating over polarizations, we obtain

$$d\sigma_C^\gamma = \frac{\alpha d\omega\, d\boldsymbol{\Delta}_\perp d\Delta_z}{16\pi^3\varepsilon^3\varepsilon'\Delta_z^2}\left[\varepsilon^2+\varepsilon'^2+2\frac{m^2\omega}{\Delta_z}+\frac{m^4\omega^2}{\varepsilon\varepsilon'\Delta_z^2}\right]\times \\ \times\left[|\mathbf{A}(\boldsymbol{\Delta})|^2 - |\mathbf{A}_B(\boldsymbol{\Delta})|^2\right]. \quad (18)$$

We note that in this formula, we can assume that the z axis is directed along the vector $\mathbf{p}$. Then Δ_z is negative and

$$|\Delta_z| \geqslant \Delta_{min} = \frac{m^2\omega}{2\varepsilon\varepsilon'}.$$

The potential $V(\mathbf{r})$ and the transverse momentum transfer $\boldsymbol{\Delta}_\perp$ enter Eq. (18) only as the factor dR,

$$dR = d\boldsymbol{\Delta}_\perp\left[|\mathbf{A}(\boldsymbol{\Delta})|^2 - |\mathbf{A}_B(\boldsymbol{\Delta})|^2\right]. \quad (19)$$

It follows from the definition of $\mathbf{A}(\boldsymbol{\Delta})$ that for $r_{scr} \gg |\Delta_z|^{-1}$, screening can be neglected. However, it is obvious from Eq. (19) that screening drastically modifies the $\boldsymbol{\Delta}_\perp$-dependence of the differential cross section for $r_{scr} \lesssim |\Delta_z|^{-1}$. We illustrate this statement with the example of the Yukawa potential

$$V(r) = -Z\alpha\exp[-\beta r]/r.$$

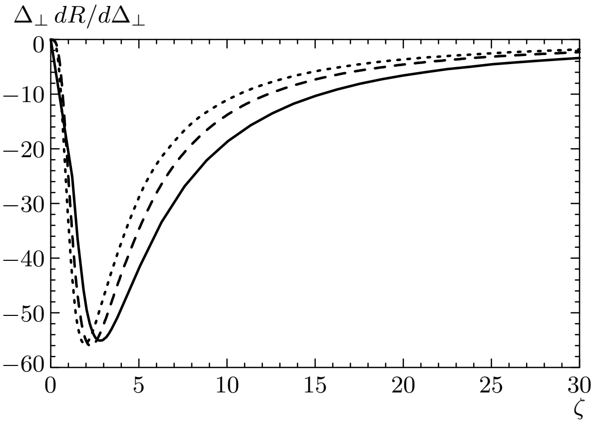

Fig. 1. The quantity $\Delta_\perp\, dR/d\Delta_\perp$ as a function of ζ for $Z = 80$ and $\gamma = 1$ (solid curve), $\gamma = 0.5$ (dashed curve), and $\gamma = 0.01$ (dotted curve). The variable ζ is defined in Eq. (20)

After the straightforward calculation, we have

$$\Delta_\perp \frac{dR}{d\Delta_\perp} = 32\pi^3 (Z\alpha)^2 \times \times \left[\zeta^2 \left| \int_0^\infty dx\, x J_1(x\zeta) K_1(x) \times \times \exp[2iZ\alpha K_0(\gamma x)] \right|^2 - \frac{\zeta^4}{(1+\zeta^2)^2} \right], \quad (20)$$
$$\zeta = \frac{\Delta_\perp}{\sqrt{\Delta_z^2 + \beta^2}}, \quad \gamma = \frac{\beta}{\sqrt{\Delta_z^2 + \beta^2}}.$$

We emphasize that $\Delta_\perp$ enters the right-hand side of Eq. (20) only via the variable ζ, and hence $\sqrt{\Delta_z^2 + \beta^2}$ is the characteristic scale of the distribution (20). For $\beta \gg |\Delta_z|$, this scale is entirely determined by the screening radius $r_{scr} = \beta^{-1}$. In this case, the $\Delta_\perp$-distribution is much wider than that in the absence of screening. We therefore conclude that in contrast to the statement in Ref. [7], Coulomb corrections to the differential cross section strongly depend on screening. We note that screening also affects the shape of the $\Delta_\perp$-distribution (20) via the parameter γ, which varies from 0 to 1. In Fig. 1, we show the dependence of $\Delta_\perp\, dR/d\Delta_\perp$ on the scaling variable ζ for $Z = 80$ and different values of the parameter γ.

We note that in contrast to bremsstrahlung, Coulomb corrections to the differential cross section of e^+e^- photoproduction in the atomic field are important only in the region $\Delta_\perp \sim m$, where screening may be neglected [4].

2.2. Integrated cross section

It was shown in Ref. [5] that Coulomb corrections to the cross section of bremsstrahlung integrated over $\mathbf{\Delta}_\perp$ are independent of screening in the leading approximation. The statement was based on the possibility to obtain this cross section from that for the e^+e^- photoproduction. In this subsection, we perform the explicit integration of $d\sigma_C^\gamma$, Eq. (18), over $\mathbf{\Delta}_\perp$. We show that the strong influence of screening on the shape of $d\sigma_C^\gamma$ does not contradict the statement in Ref. [5]. Our consideration is quite similar to that used in Ref. [11] in the calculation of Coulomb corrections to the e^+e^- pair production in ultrarelativistic heavy-ion collisions.

We consider the quantity

$$R = \int dR = \int d\mathbf{\Delta}_\perp \left[|\mathbf{A}(\mathbf{\Delta})|^2 - |\mathbf{A}_B(\mathbf{\Delta})|^2 \right]. \quad (21)$$

This integral converges due to the compensation in the integrand, and the main contribution comes from the region

$$\Delta_\perp \lesssim \max(|\Delta_z|, r_{scr}^{-1}),$$

see Eq. (17). Substituting the integral representation for $\mathbf{A}(\mathbf{\Delta})$, Eq. (11), and for $\mathbf{A}_B(\mathbf{\Delta})$, Eq. (16), in Eq. (21), we have

$$R = \int d\mathbf{\Delta}_\perp \iint d\mathbf{r}_1\, d\mathbf{r}_2 \times \times \exp[i\mathbf{\Delta}\cdot(\mathbf{r}_1 - \mathbf{r}_2)] \{\exp[i\chi(\boldsymbol{\rho}_1) - i\chi(\boldsymbol{\rho}_2)] - 1\} \times \times [\nabla_{1\perp} V(\mathbf{r}_1)] \cdot [\nabla_{2\perp} V(\mathbf{r}_2)]. \quad (22)$$

It is necessary to treat this repeated integral with care. If one naively changes the order of integration over $\mathbf{\Delta}_\perp$ and $\mathbf{r}_{1,2}$, the integration over $\mathbf{\Delta}_\perp$ in infinite limits leads to $\delta(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)$. Then the quantity R vanishes after the integration over $\boldsymbol{\rho}_1$, which is not correct. Such an erroneous change of the order of integrations was made in Ref. [4] in explicitly verifying that the integrated cross section is independent of screening. Although this independence itself takes place, the proof of this fact given in Ref. [4] and widely cited in textbooks is not consistent. The correct integration in Eq. (22) can be performed as follows. We restrict the region of integration over $\mathbf{\Delta}_\perp$ by the condition

$$\Delta_\perp < Q,$$

where

$$Q \gg \max(|\Delta_z|, r_{scr}^{-1}).$$

In this region, integral (21) is saturated and hence the result of integration must be independent of Q. We can then change the order of integrations over $\mathbf{r}_{1,2}$ and $\mathbf{\Delta}_\perp$ in Eq. (22) and take the integral over $\mathbf{\Delta}_\perp$:

$$R = 2\pi Q \int\int d\mathbf{r}_1\, d\mathbf{r}_2 \frac{J_1(Q|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)}{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|} \times \\ \times \exp[i\Delta_z(z_1-z_2)]\,\{\exp[i\chi(\boldsymbol{\rho}_1)-i\chi(\boldsymbol{\rho}_2)]-1\} \times \\ \times [\nabla_{1\perp}V(\mathbf{r}_1)]\cdot[\nabla_{2\perp}V(\mathbf{r}_2)]\,. \quad (23)$$

It is seen from this formula that the main contribution to the integral is given by the region $|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|\sim 1/Q$. If $\rho_{1,2}\gg 1/Q$ and $|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|\sim 1/Q$, then

$$|\exp[i\chi(\boldsymbol{\rho}_1)-i\chi(\boldsymbol{\rho}_2)]-1|\ll 1$$

and the integrand is suppressed. Therefore, integral (23) is determined by the region where both $\rho_1 \sim 1/Q$ and $\rho_2 \sim 1/Q$. Due to the factor $\nabla_{1\perp}V(\mathbf{r}_1)\nabla_{2\perp}V(\mathbf{r}_2)$ in the integrand, $z_{1,2}\sim 1/Q$ also. If $r\ll r_{scr}$, then

$$V(\mathbf{r})\approx -Z\alpha/r$$

and

$$\chi(\boldsymbol{\rho})\approx 2Z\alpha(\ln\rho+\mathrm{const}).$$

In addition, for $r_{1,2}\ll|\Delta_z|^{-1}$, we can omit the factor $\exp[i\Delta_z(z_1-z_2)]$ in (23). We then perform the substitution $\mathbf{r}_{1,2}\to\mathbf{r}_{1,2}/Q$ and obtain

$$R = 8\pi(Z\alpha)^2\iint d\boldsymbol{\rho}_1\, d\boldsymbol{\rho}_2 \frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)J_1(|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)}{\rho_1^2\rho_2^2|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|}\times \\ \times\left\{\left(\frac{\rho_2}{\rho_1}\right)^{2iZ\alpha}-1\right\}. \quad (24)$$

We emphasize that this formula does not contain Q. Using the identity

$$\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)J_1(|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)}{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|} = \\ = \frac{\rho_1\rho_2}{\rho_1^2-\rho_2^2}\left(\rho_1\frac{\partial}{\partial\rho_2}-\rho_2\frac{\partial}{\partial\rho_1}\right)J_0(|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|), \quad (25)$$

and the relation

$$\int_0^{2\pi} d\phi\, J_0\left(\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\phi}\right) = \\ = 2\pi J_0(\rho_1)J_0(\rho_2)\,, \quad (26)$$

which follows from the summation theorem for the Bessel functions, we have

$$R = 32\pi^3(Z\alpha)^2\times \\ \times\int_0^\infty\int_0^\infty \frac{d\rho_1\, d\rho_2}{\rho_1^2-\rho_2^2}\,[\rho_2 J_0(\rho_2)J_1(\rho_1)-\rho_1 J_0(\rho_1)J_1(\rho_2)]\times \\ \times\left\{\left(\frac{\rho_2}{\rho_1}\right)^{2iZ\alpha}-1\right\}. \quad (27)$$

Making the change of variables $\rho_{1,2}=r\exp(\pm t/4)$ and integrating over r, we finally obtain

$$R = 32\pi^3(Z\alpha)^2\int_0^\infty dt\frac{\cos(Z\alpha t)-1}{\exp(t)-1} = \\ = -32\pi^3(Z\alpha)^2[\mathrm{Re}\,\psi(1+iZ\alpha)+C] = \\ = -32\pi^3(Z\alpha)^2 f(Z\alpha)\,, \quad (28)$$

where C is the Euler constant and

$$\psi(x)=d\ln\Gamma(x)/dx.$$

Using this formula and taking the integral over Δ_z from $-\infty$ to $-\Delta_{min}$ in Eq. (18), we reproduce the well-known result obtained in Ref. [3]. We note that the value of R following from the numerical integration of Eq. (20) over $\Delta_\perp$ agrees with the universal result (28).

Thus, we come to a remarkable conclusion: Coulomb corrections to the cross section integrated over $\mathbf{\Delta}_\perp$ are independent of screening, although the main contribution to the integral comes from the region

$$\Delta_\perp \lesssim \max(\Delta_{min},\, r_{scr}^{-1}),$$

where, for $\Delta_{min}\ll r_{scr}^{-1}$, the differential cross section is essentially modified by screening. We emphasize that this result is valid in the leading approximation with respect to the parameters $m/\varepsilon\ll 1$ and $\lambda_C/r_{scr}\ll 1$. In the next section, we show that in the limit $m/\varepsilon\to 0$, the screening contributes to $d\sigma_C^\gamma/d\omega$ only as a correction in the parameter λ_C/r_{scr}.

2.3. Beam-size effect on Coulomb corrections

It is interesting to consider the effect of a finite transverse size b of the electron beam on Coulomb corrections to bremsstrahlung in a Coulomb field of a heavy nucleus. This consideration should be performed in terms of the probability dW rather than the cross section. Similarly to the effect of screening, the finite beam size can lead to a substantial modification of Coulomb corrections to the differential probability

dW_C, while Coulomb corrections to the probability integrated over $\mathbf{\Delta}$ is a universal function. To illustrate this statement, we consider bremsstrahlung from the electron described in the initial state by the wave function of the form

$$\psi(\mathbf{r}) = \int d\Omega_{\mathbf{p}}\, h(\mathbf{p})\psi_P^{(in)}(\mathbf{r}), \tag{29}$$

where the function $h(\mathbf{p})$ is peaked at $\mathbf{p} = \mathbf{p}_0$. If the width δp of the peak satisfies the condition

$$\delta p \ll \sqrt{\Delta_{min}\varepsilon} \lesssim m,$$

then

$$\psi(\mathbf{r}) \approx \int d\Omega_{\mathbf{p}}\, h(\mathbf{p}) \exp[i(\mathbf{p}-\mathbf{p}_0)\cdot\boldsymbol{\rho}]\psi_{P_0}^{(in)}(\mathbf{r}) = \\ = \phi(\boldsymbol{\rho})\psi_{P_0}^{(in)}(\mathbf{r}), \tag{30}$$

where the function $\phi(\boldsymbol{\rho})$ is normalized as

$$\int d\boldsymbol{\rho}|\phi(\boldsymbol{\rho})|^2 = 1$$

and has the width

$$b \gg 1/\sqrt{\Delta_{min}\varepsilon} \gtrsim \lambda_C.$$

The quantity dW_C is given by the right-hand side of formula (18), where the functions $\mathbf{A}(\mathbf{\Delta})$ and $\mathbf{A}_B(\mathbf{\Delta})$ are given by Eq. (11) and Eq. (16) with the additional factor $\phi(\boldsymbol{\rho})$ in the integrands. Substituting

$$V(r) = -Z\alpha/r,$$

we have

$$\begin{aligned} \mathbf{A}(\mathbf{\Delta}) &= -2iZ\alpha\Delta_z \int d\boldsymbol{\rho}\phi(\boldsymbol{\rho}) \times \\ &\times \exp[-i\mathbf{\Delta}_\perp\cdot\boldsymbol{\rho}]K_1(\Delta_z\rho)\boldsymbol{\rho}/\rho^{1+2iZ\alpha}, \\ \mathbf{A}_B(\mathbf{\Delta}) &= -2iZ\alpha\Delta_z \int d\boldsymbol{\rho}\phi(\boldsymbol{\rho}) \times \\ &\times \exp[-i\mathbf{\Delta}_\perp\cdot\boldsymbol{\rho}]K_1(\Delta_z\rho)\boldsymbol{\rho}/\rho. \end{aligned} \tag{31}$$

If

$$b \gg |\Delta_z|^{-1} \sim \Delta_{min}^{-1},$$

then we can simply replace $\phi(\boldsymbol{\rho}) \to \phi(0)$ in Eq. (31), such the differential distribution does not change compared with the case of a plain wave. Therefore, we consider the case $b \ll \Delta_{min}^{-1}$, where the finiteness of the beam size is very important. In this case we can replace $K_1(\Delta_z\rho) \to (\Delta_z\rho)^{-1}$ in Eq. (31).

Substituting the functions $\mathbf{A}(\mathbf{\Delta}_\perp)$ and $\mathbf{A}_B(\mathbf{\Delta}_\perp)$ from Eq. (31) in dR defined by Eq. (19) and repeating all the steps of the derivation of

$$R = \int dR$$

in the previous subsection, we obtain

$$R = -32\pi^3(Z\alpha)^2 f(Z\alpha)|\phi(0)|^2. \tag{32}$$

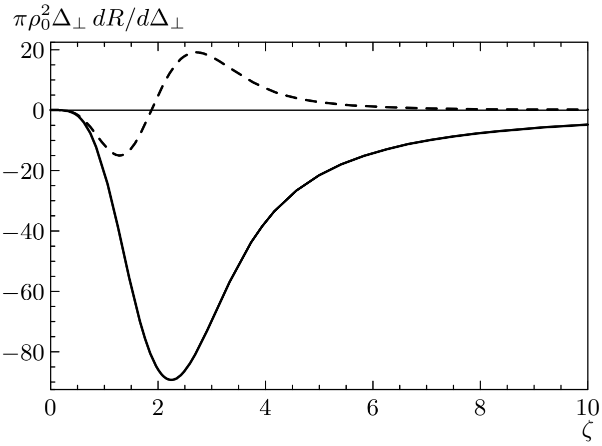

Fig. 2. The quantity $\Delta_\perp\, dR/d\Delta_\perp$ in the units $(\pi\rho_0^2)^{-1}$ as a function of $\zeta = \rho_0\Delta_\perp$ for $Z = 80$ and $\phi(\boldsymbol{\rho}) = \phi_0(\rho)$ (solid curve), $\phi(\boldsymbol{\rho}) = \phi_1(\rho)$ (dashed curve). The functions $\phi_{0,1}$ are defined in Eq. (33)

We see that Coulomb corrections to the integrated probability depend on the shape of the wave packet only through the factor $|\phi(0)|^2$ corresponding to the electron density at zero impact parameter. Therefore, their dependence on $Z\alpha$ coincides with that in the case of a plain wave (24). However, the shape of $\phi(\boldsymbol{\rho})$ can essentially modify the $\mathbf{\Delta}_\perp$-dependence of dW_C. As an illustration, in Fig. 2, we show the dependence of $\Delta_\perp dR/d\Delta_\perp$ on ζ for $Z = 80$ and $\phi(\boldsymbol{\rho}) = \phi_0(\rho)$ (solid curve) and $\phi(\boldsymbol{\rho}) = \phi_1(\rho)$ (dashed curve), where

$$\begin{aligned} \phi_0(\boldsymbol{\rho}) &= \frac{\exp[-\rho^2/2\rho_0^2]}{\sqrt{\pi\rho_0^2}}, \\ \phi_1(\boldsymbol{\rho}) &= \frac{(\rho/\rho_0)^2\exp[-\rho^2/2\rho_0^2]}{\sqrt{2\pi\rho_0^2}}, \quad \zeta = \rho_0\Delta_\perp. \end{aligned} \tag{33}$$

It is seen that the behavior of $\Delta_\perp dR/d\Delta_\perp$ differs drastically for the two cases considered. In accordance with Eq. (32),

$$R = -32\pi^3(Z\alpha)^2 f(Z\alpha)/\pi\rho_0^2$$

for $\phi(\boldsymbol{\rho}) = \phi_0(\rho)$ and

$$R = 0$$

for $\phi(\boldsymbol{\rho}) = \phi_1(\rho)$. We note that in the latter case, the function $\Delta_\perp dR/d\Delta_\perp$ itself is different from zero.

3. NEXT-TO-LEADING TERMS IN THE BREMSSTRAHLUNG SPECTRUM

As is known [5], the modification of the high-energy asymptotics of Coulomb corrections to the spectrum due to the screening effect is small. Below, we show that the same is also true for the next term in m/ε. In this section, we explicitly calculate the screening correction in the leading term of the high-energy asymptotics and neglect screening in calculating the next-to-leading term in m/ε. In other words, we calculate the first corrections in the small parameters m/ε and $1/mr_{scr}$ to the bremsstrahlung spectrum

$$\frac{d\sigma^\gamma}{d\omega} = \frac{\alpha\omega p'\varepsilon'}{2(2\pi)^4}\int d\Omega_{\mathbf{p}'}\, d\Omega_{\mathbf{k}} \sum_{\lambda_e,\lambda'_e,\lambda_\gamma} |M|^2\,, \quad (34)$$

with the amplitude M given by Eq. (2) and summation performed over the polarizations of all particles. It is convenient to calculate $d\sigma^\gamma/d\omega$ using the Green's function $G(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)$ of the Dirac equation in an external field. This Green's function can be represented as

$$G(\mathbf{r}_2,\mathbf{r}_1|\,\varepsilon) = \sum_{\lambda_e,n}\frac{\psi_n(\mathbf{r}_2)\bar\psi_n(\mathbf{r}_1)}{\varepsilon-\varepsilon_n+i0} + \sum_{\lambda_e}\int\frac{d\mathbf{p}}{(2\pi)^3}\left[\frac{\psi_P(\mathbf{r}_2)\bar\psi_P(\mathbf{r}_1)}{\varepsilon-\varepsilon_p+i0} + \frac{\psi_{-P}(\mathbf{r}_2)\bar\psi_{-P}(\mathbf{r}_1)}{\varepsilon+\varepsilon_p-i0}\right], \quad (35)$$

where ψ_n is the discrete-spectrum wave function, ε_n is the corresponding binding energy, and $P=(\varepsilon_p,\mathbf{p})$. The set of either in- or out-wave functions can be used in Eq. (35). The regularization of denominators in Eq. (35) corresponds to the Feynman rule. From Eq. (35),

$$\sum_{\lambda_e}\int d\Omega_{\mathbf{p}}\;\psi_P^{(in)}(\mathbf{r}_1)\bar\psi_P^{(in)}(\mathbf{r}_2) = \sum_{\lambda_e}\int d\Omega_{\mathbf{p}}\;\psi_P^{(out)}(\mathbf{r}_1)\bar\psi_P^{(out)}(\mathbf{r}_2) = i\frac{(2\pi)^2}{\varepsilon_p p}\delta G(\mathbf{r}_1,\mathbf{r}_2|\varepsilon_p)\,, \quad (36)$$

where $\Omega_{\mathbf{p}}$ is the solid angle of $\mathbf{p}$ and $\delta G = G-\tilde G$. The function $\tilde G$ is obtained from (35) by the replacement $i\cdot 0 \leftrightarrow -i\cdot 0$. Because the bremsstrahlung spectrum is independent of the direction of the vector $\mathbf{p}$, we can average the right-hand side of Eq. (34) over the angles of this vector. Using Eq. (36), we then obtain

$$\frac{d\sigma^\gamma}{d\omega} = -\frac{\alpha\omega}{2\varepsilon p}\int\frac{d\Omega_k}{4\pi}\iint d\mathbf{r}_1\,d\mathbf{r}_2\,\exp(-i\mathbf{k}\cdot\mathbf{r})\times \sum_{\lambda_\gamma}\mathrm{Sp}\left\{\delta G(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)\,\hat e\,\delta G(\mathbf{r}_1,\mathbf{r}_2|\varepsilon')\,\hat e\right\}\,, \quad (37)$$

where $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$ and $\varepsilon'=\varepsilon-\omega$ is the energy of the final electron. Here and below, we use the linear polarization basis ($\mathbf{e}^*=\mathbf{e}$). We note that the integration over $d\Omega_{\mathbf{k}}$ is trivial because the integrand is independent of the angles of $\mathbf{k}$, and we therefore omit the integral $\int d\Omega_k/4\pi$ below. It is convenient to represent $d\sigma^\gamma/d\omega$ in another form using the Green's function $D(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)$ of the squared Dirac equation,

$$G(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = \left[\gamma^0(\varepsilon - V(\mathbf{r}_2)) - \boldsymbol{\gamma}\cdot\mathbf{p}_2 + m\right]\times D(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)\,,\quad \mathbf{p}_2=-i\nabla_2. \quad (38)$$

Performing transformations as in Refs. [12, 9], we can rewrite Eq. (37) as

$$\frac{d\sigma^\gamma}{d\omega} = -\frac{\alpha\omega}{4\varepsilon p}\iint d\mathbf{r}_1\,d\mathbf{r}_2\exp(-i\mathbf{k}\cdot\mathbf{r})\times \sum_{\lambda_\gamma}\mathrm{Sp}\{[(2\mathbf{e}\cdot\mathbf{p}_2-\hat e\hat k)\delta D(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)]\times [(2\mathbf{e}\cdot\mathbf{p}_1+\hat e\hat k)\delta D(\mathbf{r}_1,\mathbf{r}_2|\varepsilon')]\}. \quad (39)$$

For the first two terms of the high-energy asymptotic expansion of the spectrum, the leading contribution to the integral in Eqs. (37) and (39) is given by the region

$$r = |\mathbf{r}_2-\mathbf{r}_1| \sim \frac{1}{\Delta_{min}} = \frac{2\varepsilon\varepsilon'}{\omega m^2} \gg \frac{1}{m}.$$

This estimate is in accordance with the uncertainty relation. Substituting $\delta D = D-\tilde D$ in Eq. (39), we obtain four terms. Within our accuracy, the terms containing $D(\varepsilon)D(\varepsilon')$ and $\tilde D(\varepsilon)\tilde D(\varepsilon')$ can be omitted and we have

$$\frac{d\sigma^\gamma}{d\omega} = \frac{\alpha\omega}{2\varepsilon p}\,\mathrm{Re}\iint d\mathbf{r}_1\,d\mathbf{r}_2\exp(-i\mathbf{k}\cdot\mathbf{r})\times \sum_{\lambda_\gamma}\mathrm{Sp}\{[(2\mathbf{e}\cdot\mathbf{p}_2-\hat e\hat k)D(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)]\times [(2\mathbf{e}\cdot\mathbf{p}_1+\hat e\hat k)\tilde D(\mathbf{r}_1,\mathbf{r}_2|\varepsilon')]\}. \quad (40)$$

Here and below, we assume the subtraction from the integrand of its value at $Z\alpha=0$. For calculations in the leading approximation in m/ε, the function $D(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)$ can be used in the form [12]

$$D(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = \left[1+\frac{\boldsymbol{\alpha}\cdot(\mathbf{p}_1+\mathbf{p}_2)}{2\varepsilon}\right]D^{(0)}(\mathbf{r}_2,\mathbf{r}_1|\varepsilon), \quad (41)$$

where $D^{(0)}(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)$ is the semiclassical Green's function of the Klein–Gordon equation in the external field. The function $\tilde{D}$ is obtained from Eq. (41) by the replacement $D^{(0)} \to D^{(0)*}$. Representation (41) can be directly used for the calculation of the screening correction to the spectrum. It is shown below that it can be used for the calculation of the correction in m/ε as well.

Substituting Eq. (41) in Eq. (40) and taking the trace, we obtain

$$\begin{gathered}\frac{d\sigma^\gamma}{d\omega} = \frac{2\alpha\omega}{\varepsilon^2}\,\mathrm{Re}\iint d\mathbf{r}_1\,d\mathbf{r}_2\exp(-i\mathbf{k}\cdot\mathbf{r}) \times \\ \times\sum_{\lambda_\gamma}\Big\{4[\mathbf{e}\cdot\mathbf{p}_2D_2^{(0)}][\mathbf{e}\cdot\mathbf{p}_1D_1^{(0)}] + \\ + \frac{\omega^2}{\varepsilon\varepsilon'}[\mathbf{e}\cdot(\mathbf{p}_1+\mathbf{p}_2)D_2^{(0)}][\mathbf{e}\cdot(\mathbf{p}_1+\mathbf{p}_2)D_1^{(0)}]\Big\}, \\ D_2^{(0)} = D^{(0)}(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)\,,\quad D_1^{(0)} = D^{(0)*}(\mathbf{r}_1,\mathbf{r}_2|\varepsilon')\,.\end{gathered} \tag{42}$$

In deriving Eq. (42) we integrated the terms containing second derivatives of $D^{(0)}$ by parts. We are interested in the Coulomb corrections that can be obtained from Eq. (42) by the additional subtraction of the Born term ($\propto (Z\alpha)^2$) from the integrand.

3.1. Next-to-leading term in m/ε for Coulomb corrections to the spectrum

We start with Eq. (40) and introduce the variables

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1,\quad \boldsymbol{\rho} = \frac{\mathbf{r}\times[\mathbf{r}_1\times\mathbf{r}_2]}{r^2},\quad z = -\frac{\mathbf{r}\cdot\mathbf{r}_1}{r^2}\,. \tag{43}$$

We note that the variable $\boldsymbol{\rho}$ in this section has quite different meaning than the variable $\boldsymbol{\rho}$ in the representation for $\mathbf{A}(\boldsymbol{\Delta})$ in the previous section, see Eq. (11). The analysis performed shows that the leading contribution to the term under discussion originates from the region $\rho \sim 1/m$ and $\theta, \psi \sim m/\varepsilon \ll 1$, where θ is the angle between the vectors $\mathbf{r}_2$ and $-\mathbf{r}_1$, and ψ is the angle between the vectors $\mathbf{r}$ and $\mathbf{k}$. Screening can then be neglected and we can use the semiclassical Green's function D in a Coulomb field obtained in Ref. [9],

$$\begin{gathered}D(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = \frac{i\kappa e^{i\kappa r}}{8\pi^2 r_1r_2}\int d\mathbf{q}\exp\left[i\frac{\kappa r q^2}{2r_1r_2}\right] \times \\ \times\left(\frac{2\sqrt{r_1r_2}}{|\mathbf{q}-\boldsymbol{\rho}|}\right)^{2iZ\alpha\lambda}\left\{\left(1+\frac{\lambda r}{2r_1r_2}\boldsymbol{\alpha}\cdot\mathbf{q}\right)\times \right. \\ \times\left(1+i\frac{\pi(Z\alpha)^2}{2\kappa|\mathbf{q}-\boldsymbol{\rho}|}\right) - \\ \left. -\frac{\pi(Z\alpha)^2}{4\kappa^2}(\gamma^0\lambda - \boldsymbol{\gamma}\cdot\mathbf{r}/r)\frac{\boldsymbol{\gamma}\cdot(\mathbf{q}-\boldsymbol{\rho})}{|\mathbf{q}-\boldsymbol{\rho}|^3}\right\}, \\ \lambda = \mathrm{sign}\,\varepsilon\,,\quad \kappa = \sqrt{\varepsilon^2-m^2}\,,\quad \boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma},\end{gathered} \tag{44}$$

where $\mathbf{q}$ is a two-dimensional vector in the plane perpendicular to $\mathbf{r}$. We note that because the angle θ is small, we can assume that the variable z belongs to the interval $(0,1)$ and $r_1 = rz$, $r_2 = r(1-z)$. The function $\tilde{D}$ entering Eq. (40) is obtained from Eq. (44) by the replacement $\kappa \to -\kappa$ and $\lambda \to -\lambda$. The contribution of the last term in braces in Eq. (44) vanishes after taking the trace in Eq. (40). Therefore, this term can be omitted in the problem under consideration. The remaining terms in Eq. (44) can be represented in form (41) with

$$\begin{gathered}D^{(0)}(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = \frac{i\kappa e^{i\kappa r}}{8\pi^2r_1r_2}\int d\mathbf{q}\exp\left[i\frac{\kappa rq^2}{2r_1r_2}\right]\times \\ \times\left(\frac{2\sqrt{r_1r_2}}{|\mathbf{q}-\boldsymbol{\rho}|}\right)^{2iZ\alpha\lambda}\left(1+i\frac{\pi(Z\alpha)^2}{2\kappa|\mathbf{q}-\boldsymbol{\rho}|}\right).\end{gathered} \tag{45}$$

Then, using the relation

$$\begin{gathered}(\mathbf{e}\cdot\mathbf{p}_{1,2})D^{(0)}(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = \frac{i\kappa^2e^{i\kappa r}}{8\pi^2r_1r_2}\times \\ \times\int d\mathbf{q}\exp\left[i\frac{\kappa rq^2}{2r_1r_2}\right]\left(\frac{2\sqrt{r_1r_2}}{|\mathbf{q}-\boldsymbol{\rho}|}\right)^{2iZ\alpha\lambda}\times \\ \times\left(1+i\frac{\pi(Z\alpha)^2}{2\kappa|\mathbf{q}-\boldsymbol{\rho}|}\right)\left(\mp\frac{\mathbf{e}\cdot\mathbf{r}}{r}+\frac{\mathbf{e}\cdot\mathbf{q}}{r_{1,2}}\right),\end{gathered} \tag{46}$$

and passing from the variables $\mathbf{r}_{1,2}$ to the variables $\mathbf{r}$, $\boldsymbol{\rho}$, and z, we obtain from (42) that

$$\begin{gathered}\frac{d\sigma_C^\gamma}{d\omega} = -\frac{\alpha\omega\varepsilon'}{32\pi^4\varepsilon}\,\mathrm{Re}\int\frac{d\mathbf{r}}{r^5}\int_0^1\frac{dz}{z^2(1-z)^2}\times \\ \times\iiint d\mathbf{q}_1\,d\mathbf{q}_2\,d\boldsymbol{\rho}\times \\ \times\exp\left[\frac{i\omega r}{2}\left(\psi^2+\frac{m^2}{\varepsilon\varepsilon'}\right)+i\frac{\varepsilon q_1^2-\varepsilon' q_2^2}{2rz(1-z)}\right]\times \\ \times\left\{\left(\frac{Q_2}{Q_1}\right)^{2iZ\alpha}-1+2(Z\alpha)^2\ln^2\frac{Q_2}{Q_1}+\frac{i\pi(Z\alpha)^2}{2}\times\right. \\ \left.\times\left[\left(\frac{Q_2}{Q_1}\right)^{2iZ\alpha}-1\right]\left(\frac{1}{\varepsilon Q_1}-\frac{1}{\varepsilon' Q_2}\right)\right\}\times \\ \times\sum_{\lambda_\gamma}\left\{4\varepsilon\varepsilon'\left(-\mathbf{e}\cdot\mathbf{r}+\frac{\mathbf{e}\cdot\mathbf{q}_1}{1-z}\right)\left(\mathbf{e}\cdot\mathbf{r}+\frac{\mathbf{e}\cdot\mathbf{q}_2}{z}\right)+\right. \\ \left.+\frac{\omega^2}{z^2(1-z)^2}(\mathbf{e}\cdot\mathbf{q}_1)(\mathbf{e}\cdot\mathbf{q}_2)\right\},\end{gathered} \tag{47}$$

where $Q_{1,2} = |\mathbf{q}_{1,2}-\boldsymbol{\rho}|$. The integral over $\boldsymbol{\rho}$ can be taken using the relations (see Appendix B in [9])

$$
\begin{aligned}
f(Z\alpha) &= \frac{1}{2\pi(Z\alpha)^2 q^2}\times \\
&\times \int d\boldsymbol{\rho}\left[\left(\frac{Q_2}{Q_1}\right)^{2iZ\alpha} - 1 + 2(Z\alpha)^2 \ln^2\frac{Q_2}{Q_1}\right] = \\
&= \mathrm{Re}[\psi(1+iZ\alpha)+C], \qquad (48)\\
g(Z\alpha) &= \frac{i}{4\pi q}\int \frac{d\boldsymbol{\rho}}{Q_2}\left[\left(\frac{Q_2}{Q_1}\right)^{2iZ\alpha} - 1\right] = \\
&= Z\alpha\,\frac{\Gamma(1-iZ\alpha)\Gamma(1/2+iZ\alpha)}{\Gamma(1+iZ\alpha)\Gamma(1/2-iZ\alpha)},
\end{aligned}
$$

where

$$\psi(t) = d\ln\Gamma(t)/dt,$$

$C = 0.577\ldots$ is the Euler constant, and $q = |\mathbf{q}_1 - \mathbf{q}_2|$. We next perform summation over the photon polarization, pass to the variables

$$\tilde{\mathbf{q}} = \mathbf{q}_1 + \mathbf{q}_2, \quad \mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2,$$

and take all integrals in the following order: $d\Omega_{\mathbf{r}}$, $d\tilde{\mathbf{q}}$, $d\mathbf{q}$, dr, dz. The final result for Coulomb corrections to the bremsstrahlung spectrum is given by

$$
\begin{aligned}
y\frac{d\sigma_C^\gamma}{dy} &= -4\sigma_0\left[\left(y^2+\frac{4}{3}(1-y)\right)f(Z\alpha) - \right.\\
&\left. - \frac{\pi^3(2-y)m}{8(1-y)\varepsilon}\left(y^2+\frac{3}{2}(1-y)\right)\mathrm{Re}\,g(Z\alpha)\right], \qquad (49)\\
y &= \omega/\varepsilon, \quad \sigma_0 = \alpha(Z\alpha)^2/m^2.
\end{aligned}
$$

In this formula, the term proportional to $f(Z\alpha)$ corresponds to the leading approximation [3] and the term proportional to $\mathrm{Re}\,g(Z\alpha)$ is an $O(m/\varepsilon)$-correction. In our recent paper [9], this result was obtained by means of the substitution rules from the spectrum of pair production by photon in a Coulomb field. Formula (49) describes bremsstrahlung from electrons. For the spectrum of photons emitted by positrons, it is necessary to change the sign of $Z\alpha$ in (49). The $O(m/\varepsilon)$-correction becomes especially important in the hard part of the spectrum, as can be seen in Fig. 3, where $\sigma_0^{-1} y d\sigma_C^\gamma/dy$ with the correction (solid line) and without it (dashed line) are shown for $Z = 82$ and $\varepsilon = 50$ MeV. We note that in the whole range of y, the relative magnitude of the correction is appreciably larger than m/ε due to the presence of a large numerical coefficient.

3.2. Screening corrections

In this subsection, we calculate the screening correction to the high-energy asymptotics of $d\sigma_C^\gamma/d\omega$, considering λ_C/r_{scr} as a small parameter.

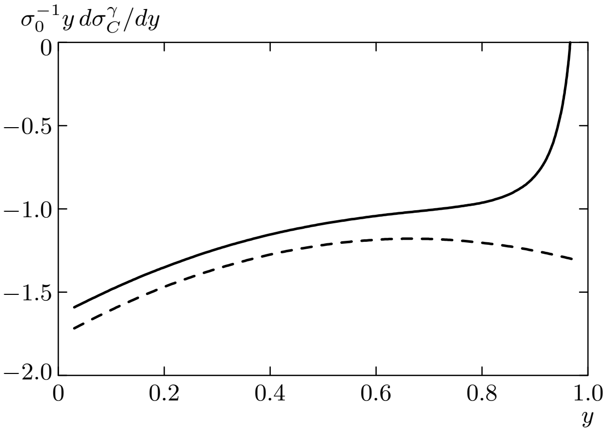

Fig. 3. The dependence of $\sigma_0^{-1} y d\sigma_C^\gamma/dy$ on y, see (49), for $Z = 82$, $\varepsilon = 50$ MeV. Dashed curve: leading approximation; solid curve: first correction is taken into account

We start from Eq. (42) and use the semiclassical Green's function $D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\varepsilon)$ for an arbitrary localized potential $V(\mathbf{r})$. This Green's function was obtained in [10] with the first correction in m/ε taken into account. The leading term has the form (see also [12])

$$
\begin{aligned}
D^{(0)}(\mathbf{r}_2, \mathbf{r}_1|\varepsilon) &= \frac{i\kappa e^{i\kappa r}}{8\pi^2 r_1 r_2}\times \\
&\times\int d\mathbf{q}\exp\left[i\frac{\kappa r\, q^2}{2r_1 r_2} - i\lambda r\int_0^1 dx\, V(\mathbf{r}_1 + x\mathbf{r} - \mathbf{q})\right]. \qquad (50)
\end{aligned}
$$

Similarly to Eq. (47), we obtain

$$
\begin{aligned}
\frac{d\sigma_C^\gamma}{d\omega} &= -\frac{\alpha\omega\varepsilon'}{32\pi^4\varepsilon}\mathrm{Re}\int\frac{d\mathbf{r}}{r^5}\int_0^1 \frac{dz}{z^2(1-z)^2}\times \\
&\times \iiint d\mathbf{q}_1\, d\mathbf{q}_2\, d\boldsymbol{\rho}\,\times \\
&\times\exp\left[i\Phi + \frac{i\omega r}{2}\left(\psi^2 + \frac{m^2}{\varepsilon\varepsilon'}\right) + i\frac{\varepsilon q_1^2 - \varepsilon' q_2^2}{2rz(1-z)}\right]\times \\
&\times\sum_{\lambda_\gamma}\left\{4\varepsilon\varepsilon'\left(-\mathbf{e}\cdot\mathbf{r} + \frac{\mathbf{e}\cdot\mathbf{q}_1}{1-z}\right)\left(\mathbf{e}\cdot\mathbf{r} + \frac{\mathbf{e}\cdot\mathbf{q}_2}{z}\right) + \right.\\
&\left. + \frac{\omega^2}{z^2(1-z)^2}(\mathbf{e}\cdot\mathbf{q}_1)(\mathbf{e}\cdot\mathbf{q}_2)\right\}, \qquad (51)
\end{aligned}
$$

where

$$\Phi = r\int_0^1 dx[V(\mathbf{r}_1 + x\mathbf{r} - \mathbf{q}_2) - V(\mathbf{r}_1 + x\mathbf{r} - \mathbf{q}_1)]. \qquad (52)$$

As we see in what follows, it is meaningful to retain the screening correction only in the case where

$r_{scr} \ll \Delta_{min}^{-1}$, which is considered below. Then the main contribution to integral (51) comes from the region

$$1/m \lesssim \rho \lesssim r_{scr} \ll r$$

and

$$q_{1,2} \sim 1/m.$$

Under these conditions, the narrow region

$$\delta x = \rho/r \ll 1$$

around the point

$$x_0 = -\frac{\mathbf{r}_1 \cdot \mathbf{r}}{r^2} = z$$

is important in the integration over x in Eq. (52). Therefore, we can perform this integration from $-\infty$ to ∞. The phase Φ then becomes

$$\Phi = 2Z\alpha \ln(Q_2/Q_1) + \Phi^{(scr)} = \\ = 2Z\alpha \ln(Q_2/Q_1) + r\int_{-\infty}^{\infty} dx[\delta V(\mathbf{r}_1 + x\mathbf{r} - \mathbf{q}_2) - \\ - \delta V(\mathbf{r}_1 + x\mathbf{r} - \mathbf{q}_1)], \quad (53)$$

where $\delta V(\mathbf{r})$ is the difference between the atomic potential and the Coulomb potential of a nucleus. The notation in Eq. (51) and in Eq. (53) is the same as in Eq. (47). It is seen that

$$\Phi_{scr} \sim \rho\, \delta V(\rho) \sim \frac{Z\alpha\, \delta V(\rho)}{V(\rho)} \ll 1 \quad \text{for} \approx \rho \sim m$$

and

$$\Phi_{scr} \sim \frac{q_{1,2}}{\rho} \sim \frac{1}{m\rho} \ll 1 \quad \text{for} \quad \rho \sim r_{scr} \gg \frac{1}{m}.$$

Therefore, expression (51) can be expanded in $\Phi^{(scr)}$. In our calculation of the screening correction $d\sigma_C^{\gamma(scr)}/d\omega$, we retain the linear term of the expansion in $\Phi^{(scr)}$. The function $\delta V(\boldsymbol{R})$ can be expressed via the atomic electron form factor $F(\mathbf{Q})$ as

$$\delta V(\boldsymbol{R}) = \int \frac{d\mathbf{Q}}{(2\pi)^3} \exp(i\mathbf{Q} \cdot \mathbf{R}) F(\mathbf{Q}) \frac{4\pi Z\alpha}{Q^2}. \quad (54)$$

Substituting this formula in Eq. (53) and taking the integral over x from $-\infty$ to ∞, we obtain

$$\Phi^{(scr)} = \int \frac{d\mathbf{Q}_\perp}{(2\pi)^2} [\exp(i\mathbf{Q}_\perp \cdot (\boldsymbol{\rho} - \mathbf{q}_2)) - \\ - \exp(i\mathbf{Q}_\perp \cdot (\boldsymbol{\rho} - \mathbf{q}_1))] F(\mathbf{Q}_\perp) \frac{4\pi Z\alpha}{Q_\perp^2}, \quad (55)$$

where $\mathbf{Q}_\perp$ is a two-dimensional vector lying in the plane perpendicular to $\mathbf{r}$. We next use the identity (see Eqs. (22) and (23) in [13])

$$\int d\boldsymbol{\rho} \left(\frac{|\boldsymbol{\rho} - \mathbf{q}_2|}{|\boldsymbol{\rho} - \mathbf{q}_1|}\right)^{2iZ\alpha} \exp[i\mathbf{Q}_\perp \cdot (\boldsymbol{\rho} - \mathbf{q}_{1,2})] = \\ = \frac{q^2}{4Q_\perp^2} \int d\mathbf{f} \left(\frac{f_2}{f_1}\right)^{2iZ\alpha} \exp[i\mathbf{q} \cdot \mathbf{f}_{1,2}/2], \quad (56)$$

where

$$\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2, \quad \mathbf{f}_{1,2} = \mathbf{f} \mp \mathbf{Q}_\perp.$$

Expanding the exponential in Eq. (51) with respect to $\Phi^{(scr)}$ and using relation (56), we take the integrals over $\mathbf{q}_{1,2}$, $\mathbf{r}$, and z and obtain

$$y\frac{d\sigma_C^{\gamma(scr)}}{dy} = \frac{4\alpha(Z\alpha)}{\pi} \mathrm{Im} \int \frac{d\mathbf{Q}_\perp}{Q_\perp^4} F(\mathbf{Q}_\perp) \times \\ \times \int \frac{d\mathbf{f}}{2\pi} \left[\left(\frac{f_2}{f_1}\right)^{2iZ\alpha} - 2iZ\alpha \ln\frac{f_2}{f_1}\right] \times \\ \times \left[\frac{S(\xi_1)}{f_1^2} - \frac{S(\xi_2)}{f_2^2}\right], \\ S(\mu) = \frac{(\mu - 1)}{\mu^2} \times \\ \times \left\{\frac{1}{2\sqrt{\mu}}\left[y^2(3-\mu) + (y-1)(\mu^2 + 2\mu - 3)\right] \times \right. \\ \times \ln\left[\frac{\sqrt{\mu}+1}{\sqrt{\mu}-1}\right] - \\ \left. - 3y^2 - (y-1)(\mu - 3)\right\}, \\ y = \frac{\omega}{\varepsilon}, \quad \xi_{1,2} = 1 + \frac{16m^2}{f_{1,2}^2}. \quad (57)$$

Using the trick introduced in [13], we can rewrite this formula in another form. We multiply the integrand in (57) by

$$1 \equiv \int_{-1}^{1} dx\, \delta\left(x - \frac{2\mathbf{f} \cdot \mathbf{Q}_\perp}{\mathbf{f}^2 + \mathbf{Q}_\perp^2}\right) = (\mathbf{f}^2 + \mathbf{Q}_\perp^2) \times \\ \times \int_{-1}^{1} \frac{dx}{|x|} \delta((\mathbf{f} - \mathbf{Q}_\perp/x)^2 - \mathbf{Q}_\perp^2(1/x^2 - 1)), \quad (58)$$

change the order of integrations over $\mathbf{f}$ and x, and make the shift

$$\mathbf{f} \to \mathbf{f} + \mathbf{Q}_\perp/x.$$

After that, the integration over f can be easily per-

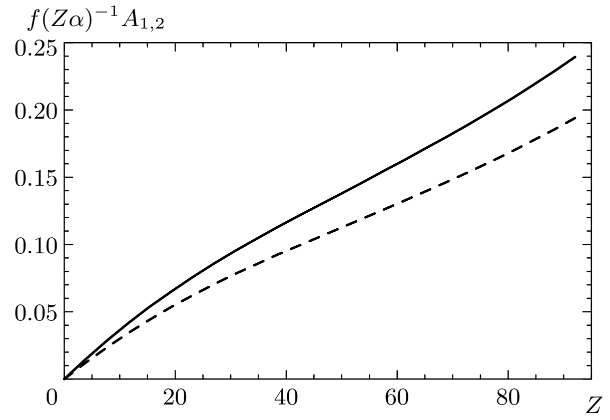

Fig. 4. The dependence of $A_1/f(Z\alpha)$ (solid curve) and $A_2/f(Z\alpha)$ (dashed curve) on Z

formed. Then we make the substitution $x = \operatorname{th}\tau$ and obtain

$$y\frac{d\sigma_C^{(scr)}}{dy} = 16\sigma_0 m^2 \int_0^\infty \frac{d\mathbf{Q}_\perp}{2\pi}\,\frac{F(\mathbf{Q}_\perp)}{Q_\perp^4} \times \\ \times \int_0^\infty \frac{d\tau}{\operatorname{sh}\tau}\left[\frac{\sin(2Z\alpha\tau)}{2Z\alpha} - \tau\right] \times \\ \times \int_0^{2\pi} \frac{d\varphi}{2\pi}\left[e^{\tau}S(\mu_2) - e^{-\tau}S(\mu_1)\right], \\ \mu_{1,2} = 1 + \frac{8m^2 e^{\mp\tau}\operatorname{sh}^2\tau}{Q_\perp^2(\operatorname{ch}\tau + \cos\varphi)}. \tag{59}$$

According to Eq. (57), the correction $yd\sigma_C^{\gamma(scr)}/dy$ has the form

$$y\frac{d\sigma_C^{\gamma(scr)}}{dy} = \sigma_0\left[A_1(1-y) + A_2 y^2\right]. \tag{60}$$

Shown in Fig. 4 is the Z dependence of the ratio $A_{1,2}/f(Z\alpha)$ calculated numerically with the use of form factors from [14]. For the less realistic Yukawa potential, we can perform analytic calculations of the functions A_i. It turns out that their dependence on the parameter $\beta = \lambda_c/r_{scr}$ has the form

$$A_i = (Z\alpha)^2\beta^2\left(a_i \ln^2\beta + b_i\ln\beta + c_i\right), \tag{61}$$

where b_i and c_i are some functions of $Z\alpha$, while a_i does not depend on $Z\alpha$. Recalling that β is proportional to $Z^{1/3}$ in the Thomas–Fermi model, we see that A_i depend on Z mainly via the factor

$$(Z\alpha)^2\beta^2 \propto (Z\alpha)^2 Z^{2/3}.$$

Therefore, it is quite natural that $yd\sigma_C^{\gamma(scr)}/dy$ calculated with the use of the exact form factors is well fitted by the expression

$$y\frac{d\sigma_C^{\gamma(scr)}}{dy} \approx 8.6\cdot 10^{-3}\sigma_0(Z\alpha)^2 \times \\ \times Z^{2/3}[1.2(1-y) + y^2]. \tag{62}$$

In fact, the accuracy of this fit for all Z is better than a few percent.

It follows from Eq. (61) that for $r_{scr} \gtrsim \Delta_{min}^{-1}$, the factor β^2 in the screening correction is extremely small, $\beta^2 \lesssim (m/\varepsilon)^2$. The terms of such an order were systematically neglected in our consideration. Hence, within our accuracy, the account of the screening correction is meaningful only for $r_{scr} \ll \Delta_{min}^{-1}$.

4. CONCLUSION

We have performed a detailed analysis of Coulomb corrections both to the differential and the integrated cross sections of bremsstrahlung in an atomic field. We have calculated the next-to-leading term in the high-energy asymptotics of the bremsstrahlung spectrum. Similar to the leading term of the high-energy asymptotics of Coulomb corrections to the spectrum, this term is independent of screening in the leading order in the parameter λ_c/r_{scr}. We have also calculated the first correction to the spectrum in the parameter λ_c/r_{scr}.

We have shown that in contrast with Coulomb corrections to the spectrum, Coulomb corrections to the differential cross section strongly depend on screening even in the leading approximation. This dependence is very important in the region that gives the main contribution to the integral over $\Delta_\perp$. We have performed the explicit integration over $\Delta_\perp$ of $d\sigma_C^\gamma$ for arbitrary screening and have verified the independence of the final result from screening.

We also examined the effect of the finite beam size on Coulomb corrections to bremsstrahlung in a Coulomb field of a heavy nucleus. Similar to the effect of screening, the finiteness of the beam size leads to a strong modification of Coulomb corrections to the differential probability, while the probability integrated over $\Delta_\perp$ depends only on the density of the electron beam at zero impact parameter.

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