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SPECTRUM OF QUANTIZED BLACK HOLE, CORRESPONDENCE PRINCIPLE, AND HOLOGRAPHIC BOUND

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An equidistant spectrum of the horizon area of a quantized black hole does not follow from the correspondence principle or from general statistical arguments. On the other hand, such a spectrum obtained in loop quantum gravity (LQG) either does not comply with the holographic bound or requires a special choice of the Barbero–Immirzi parameter for the horizon surface, distinct from its value for other quantized surfaces. The problem of distinguishability of edges in LQG is discussed, with the following conclusion. Only under the assumption of partial distinguishability of the edges, the microcanonical entropy of a black hole can be made both proportional to the horizon area and satisfying the holographic bound.

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1. The idea of quantizing the horizon area of black holes was put forward many years ago by Bekenstein in the pioneering article [1]. It was based on the intriguing observation, made by Christodoulou and Ruffini [2, 3]: the horizon area of a nonextremal black hole behaves in a sense as an adiabatic invariant. Of course, the quantization of an adiabatic invariant is perfectly natural, in accordance with the correspondence principle.

One more conjecture made in [1] is that the spectrum of a quantized horizon area is equidistant. The argument therein was that a periodic system is quantized by equating its adiabatic invariant to $2\pi\hbar n$, $n = 0, 1, 2, \ldots$.

Later, it was pointed out by Bekenstein [4] that the classical adiabatic invariance does not by itself guarantee the equidistance of the spectrum, at least because any function of an adiabatic invariant is itself an adiabatic invariant. But up to now, articles on the subject abound in assertions that the form

$$A = \beta\, l_p^2 n, \quad n = 1, 2, \ldots, \tag{1}$$

for the horizon area spectrum[1)] is dictated by the respectable correspondence principle. The list of these references is too long to be presented here.

We consider an instructive example of the situation where a nonequidistant spectrum arises in spite of the classical adiabatic invariance. We start with a classical spherical top of an angular momentum $\mathbf{J}$. Of course, the z-projection J_z of $\mathbf{J}$ is an adiabatic invariant. If the z axis is chosen along $\mathbf{J}$, the value of J_z is maximum, J, or $\hbar j$ in the quantum case. The classical angular momentum squared J^2 is also an adiabatic invariant, with the eigenvalues $\hbar^2 j(j+1)$ when quantized. We now try to use the operator $\hat{J}^2$ for the area quantization in quite natural units of l_p^2. For the horizon area A to be finite in the classical limit, the power of the quantum number j in the result for $j \gg 1$ should be the same as that of $\hbar$ in l_p^2 [5]. With $l_p^2 \sim \hbar$, we thus arrive at

$$A \sim l_p^2 \sqrt{j(j+1)}.$$

Because

$$\sqrt{j(j+1)} \to j + 1/2 \quad \text{for } j \gg 1,$$

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1) Here and below, $l_p^2 = \hbar k/c^3$ is the Planck length squared, $l_p = 1.6 \cdot 10^{-33}$ cm, k is the Newton gravitational constant; β is here some numerical factor.

we have returned to the equidistant spectrum in the classical limit. However, the equidistant spectrum can be avoided as follows. We assume that the horizon area consists of sites with area of the order of l_p^2, and to each site i ascribe its own quantum number j_i and the contribution $\sqrt{j_i(j_i+1)}$ to the area. Then the above formula changes to

$$A \sim l_p^2 \sum_i \sqrt{j_i(j_i+1)} \qquad (2)$$

(in fact, this formula for a quantized area arises as a special case in loop quantum gravity, see below). Of course, to retain a finite classical limit for A, we should require that

$$\sum_i \sqrt{j_i(j_i+1)} \gg 1.$$

But any of the j_i can be well comparable with unity. Therefore, in spite of the adiabatic invariance of A, its quantum spectrum (2) is not equidistant, although of course discrete.

One more quite popular argument in favor of equidistant spectrum (1) is as follows [4, 6, 7]. On the one hand, the entropy S of a horizon is related to its area A by the Bekenstein–Hawking formula

$$A = 4l_p^2 S. \qquad (3)$$

On the other hand, the entropy is nothing but $\ln g(n)$, where the statistical weight $g(n)$ of any quantum state n is an integer. In [4, 6, 7], the requirement of integer $g(n)$ is taken literally, and results after simple reasoning not only in equidistant spectrum (1), but also in the following allowed values for the numerical factor β in this spectrum:

$$\beta = 4\ln k, \quad k = 2, 3, \ldots .$$

We can imagine, however, that with some model for S, $g(n)$ is given by a noninteger $K+\delta$, $0<\delta<1$, instead of an integer value K. Then the entropy is

$$S = \ln(K+\delta) = \ln K + \delta/K.$$

Now, the typical value of the black hole entropy

$$S = \ln K = \frac{A}{4l_p^2}$$

is huge, roughly 10^{76}. Therefore, the correction δ/K is absolutely negligible compared to $S=\ln K$. Moreover, it is far below any conceivable accuracy of a description of entropy, and can therefore be safely omitted and forgotten. As usual for macroscopic objects, the fact that the statistical weight is an integer has no consequences for the entropy.

Thus, contrary to the popular belief, the equidistance of the spectrum for the horizon area does not follow from the correspondence principle and/or from general statistical arguments.

2. This does not mean, however, that any model leading to an equidistant spectrum for the quantized horizon area should be automatically rejected. Quite simple and elegant version of such a model, so-called «it from bit», was formulated for a Schwarzschild black hole by Wheeler [8]. The assumption is that the horizon surface consists of ν patches, each of them supplied with an «angular momentum» quantum number j with two possible projections $\pm 1/2$. The total number K of degenerate quantum states of this system is

$$K = 2^\nu. \qquad (4)$$

Then the entropy of the black hole is

$$S_{1/2} = \ln K = \nu \ln 2. \qquad (5)$$

With Bekenstein–Hawking relation (3), one obtains the following equidistant formula for the area spectrum:

$$A_{1/2} = 4\ln 2\, l_p^2\, \nu. \qquad (6)$$

This model of a quantized Schwarzschild black hole looks by itself flawless.

This result was later derived in Ref. [9] in the framework of loop quantum gravity (LQG) [10–14]. We discuss below whether the «it from bit» picture, if considered as a special case of the area quantization in LQG, can be reconciled with the holographic bound [15–17].

More generally, a quantized surface in LQG is described as follows. One ascribes a set of punctures to the surface. Each puncture is supplied with two integer or half-integer «angular momenta» j^u and j^d,

$$j^u,\, j^d = 1/2, 1, 3/2, \ldots . \qquad (7)$$

j^u and j^d are related to edges directed up and down the normal to the surface, respectively, and add up to the angular momentum j^{ud},

$$\mathbf{j}^{ud} = \mathbf{j}^u + \mathbf{j}^d, \quad |j^u - j^d| \le j^{ud} \le j^u + j^d. \qquad (8)$$

The area of the surface is

$$A = \beta l_p^2 \sum_i \sqrt{2j_i^u(j_i^u+1) + 2j_i^d(j_i^d+1) - j_i^{ud}(j_i^{ud}+1)}. \qquad (9)$$

The overall numerical factor β in (9) cannot be determined without an additional physical input. This ambiguity originates from a free (so-called Barbero–Immirzi) parameter [18, 19] that corresponds to a family of inequivalent quantum theories, all of them being viable without such an input.

The result (6) was obtained in [9] under the additional condition that the gravitational field on the horizon is described by the $U(1)$ Chern–Simons theory. Formula (6) is a special case of general one (9) with all j^d vanishing and all j^u equal to 1/2 (or vice versa). As regards the overall factor β, its value here is[2)]

$$\beta = \frac{8\ln 2}{\sqrt{3}} . \tag{10}$$

We now turn to the holographic bound [15–17]. According to it, the entropy S of any spherically symmetric system confined inside a sphere of area A is bounded as

$$S \le \frac{A}{4l_p^2} , \tag{11}$$

with the equality attained only for a system that is a black hole.

A simple intuitive argument confirming this bound is as follows [17]. We consider the discussed system collapsing into a black hole. During the collapse, the entropy increases from S to S_{bh}, and the resulting horizon area A_{bh} is certainly smaller than the initial confining area A. Now, with Bekenstein–Hawking relation (3) for a black hole taken into account, we arrive, through the obvious chain of (in)equalities

$$S \le S_{bh} = \frac{A_{bh}}{4l_p^2} \le \frac{A}{4l_p^2} ,$$

at the discussed bound (11).

The result (11) can be formulated differently. Among spherical surfaces of a given area, the surface of a black hole horizon has the largest entropy.

On the other hand, it is only natural that the entropy of an eternal black hole in equilibrium is maximum. This was used by Vaz and Witten [20] in a model of the quantum black hole originating from a dust collapse. The idea was then employed by us [21, 22] in the problem of quantizing the horizon of a black hole in LQG. In particular, the coefficient β was calculated in Ref. [22] in the case where the area of a black hole horizon is given by the general formula (9) of LQG, as well as under some more special assumptions on the values of j^u, j^d, and j^{ud}. Moreover, it was demonstrated in Ref. [22] for a rather general class of the horizon quantization schemes that the maximum entropy of a quantized surface is proportional to its area.

We sketch the proof of this result (for more technical details, see [22]). Here and below, we consider the microcanonical entropy S of a surface (although with fixed area instead of fixed energy). It is defined as the logarithm of the number of states of this surface with a fixed area A, i.e., with a fixed sum

$$N = \sum_i \sqrt{2j_i^u(j_i^u+1)+2j_i^d(j_i^d+1)-j_i^{ud}(j_i^{ud}+1)}. \tag{12}$$

Let ν_{im} be the number of punctures with a given set of momenta j_i^u, j_i^d, j_i^{ud}, and a given projection m of j_i^{ud}. The total number of punctures is

$$\nu = \sum_{i,m} \nu_{im}.$$

We assume that the edges with the same set of the quantum numbers i,m (i.e., with the same j_i^u, j_i^d, j_i^{ud}, and m) are indistinguishable, and therefore interchanging them does not result in new states. All other permutations, those among the edges with differing i,m, do create new states, and hence such edges, with differing i,m, are distinguishable,

We note that the «it from bit» values (4) and (5) for the number of states and entropy also follow from this assumption. Indeed, let ν be the total number of patches with $j = 1/2$ and let ν_+ and $\nu_- = \nu - \nu_+$ patches have the respective projections $+1/2$ and $-1/2$. Then the number of the corresponding states is obviously given by

$$\frac{\nu!}{\nu_+!\,(\nu-\nu_+)!} ,$$

and the total number of states is

$$K = \sum_{\nu_+=0}^{\nu} \frac{\nu!}{\nu_+!\,(\nu-\nu_+)!} = 2^\nu,$$

in agreement with (4).

Thus, the entropy is

$$S = \ln\left[\nu\,!\prod_{i,m}\frac{1}{\nu_{im}\,!}\right] . \tag{13}$$

The structure of expressions (9) and (13) is so different that the entropy certainly cannot be proportional to the area in the general case. However, this is the case for the maximum entropy in the classical limit.

[2)] The common convention for the numerical factor in formula (9) is $8\pi\beta$; with it, the parameter β is smaller than ours by the factor 8π.

By combinatorial reasons, it is natural to expect that the absolute maximum of entropy is reached when all values of quantum numbers $j_i^{u,d,ud}$ are present. We also assume that in the classical limit, the typical values of puncture numbers ν_{im} are large. Then, with the Stirling formula for factorials, expression (13) becomes

$$S=\left(\sum_{i,m}\nu_{im}\right)\ln\left(\sum_{i',m'}\nu_{i'm'}\right)-\\ -\sum_{i,m}\left(\nu_{im}\ln\nu_{im}\right). \quad (14)$$

We seek the extremum of expression (14) under the condition

$$N=\sum_i \nu_{im}\, r_i\; = \;\text{const}, \quad (15)$$

where each partial contribution

$$r_i=\sqrt{2j_i^u(j_i^u+1)+2j_i^d(j_i^d+1)-j_i^{ud}(j_i^{ud}+1)}$$

is independent of m. The problem reduces to the solution of the system of equations

$$\ln\left(\sum_{i',m'}\nu_{i'm'}\right)-\ln\nu_{im}=\mu r_i\,, \quad (16)$$

or

$$\nu_{im}=e^{-\mu r_i}\sum_{i',m'}\nu_{i'm'}\;=\nu\, e^{-\mu r_i}. \quad (17)$$

Here, μ is the Lagrange multiplier for constraint (15). Summing expressions (17) over i and m, we arrive at the equation for μ,

$$\sum_{i,m}e^{-\mu r_i}=\sum_i g_i\, e^{-\mu r_i}=1; \quad (18)$$

the statistical weight

$$g_i=2j_i^{ud}+1$$

of a puncture arises here because r_i are independent of m. On the other hand, multiplying Eq. (16) by ν_{im} and summing over i and m, with constraint (15), we arrive at the following result for the maximum entropy for a given value of the sum N, or the black hole area A:

$$S_{\max}=\mu\, N\;=\;\frac{\mu}{\beta l_p^2}\,A. \quad (19)$$

One more curious feature of the obtained picture is worth noting: it gives a sort of the Boltzmann distribution for the occupation numbers (see (17)). In this distribution, the partial contributions r_i to the area are analogues of energies and the Lagrange multiplier μ corresponds (up to a factor) to the inverse temperature.

It should be emphasized that relation (19) is true not only in LQG, but applies to a more general class of approaches to quantization of surfaces. The following assumption is necessary here: the surface should consist of patches of different sorts, such that there are ν_{im} patches of each sort i,m, with a generalized effective quantum number r_i and a statistical weight g_i. Equally necessary is the above assumption on the distinguishability of the patches.

Thus, the maximum entropy of a surface is proportional to its area in the classical limit. This proportionality certainly occurs for a classical black hole. This is one more strong argument in favor of the assumption that the black hole entropy is maximum.

We now return to the result in Ref. [9]. If we assume that the value (10) of the parameter β is universal (i.e., is not special to black holes, but refers to any quantized spherical surface), then the value in (5) is not the maximum one in LQG for a surface of area (6). This looks quite natural: with the transition from the unique choice made in Ref. [9],

$$j^{u(d)}=1/2, \quad j^{d(u)}=0,$$

to a more extended and rich one, the number of degenerate quantum states should, generally speaking, increase. Together with this number, its logarithm, which is the entropy of a quantized surface, increases as well.

We start the proof of the above statement with rewriting formula (5) as

$$S_{1/2}=\ln 2\sqrt{\frac{4}{3}}\,N=0.80N, \quad N=\sqrt{\frac{3}{4}}\,\nu\,. \quad (20)$$

From now on, we consider this value of N fixed.

We start with a relatively simple example where

$$j^{d(u)}=0,$$

and hence the general formula (9) for a surface area reduces to

$$A=\beta\, l_p^2\sum_i\sqrt{j_i(j_i+1)}\;=\\ =\beta\, l_p^2\sum_{j=1/2}^{\infty}\sqrt{j(j+1)}\,\nu_j\,, \quad j=j^{u(d)} \quad (21)$$

(and coincides with our naive model (2)). We find the maximum entropy of such a surface for the fixed value of

$$N = \sum_{j=1/2}^{\infty} \sqrt{j(j+1)}\, \nu_j\,, \qquad (22)$$

which should be equal to the «it from bit» one, $\nu\sqrt{3/4}$. Here, the statistical weight of a puncture with the quantum number j is

$$g_j = 2j + 1,$$

and Eq. (18) can be rewritten as

$$\sum_{p=1}^{\infty}(p+1)\, z^{\sqrt{p(p+2)}} = 1, \quad p = 2j, \quad z = e^{-\mu/2}. \quad (23)$$

Its solution is

$$\mu = -2\ln z = 1.722$$

(see Ref. [22]) and the maximum entropy

$$S_{max,1} = 1.72\, N \qquad (24)$$

then exceeds the result in (20).

As expected, in the general case, with N given by formula (12) with all the values of j_i^u, j_i^d, j_i^{ud} allowed and

$$g_i = 2j_i^{ud} + 1,$$

the maximum entropy is even larger [22],

$$S_{max} = 3.12\, N. \qquad (25)$$

Thus, the conflict is obvious between the holographic bound and the result (20) found within the LQG approach in [9].

One might try to avoid the conflict by assuming that value (10) of the Barbero–Immirzi parameter β is special for black holes only, while for other quantized surfaces, β is smaller. However, such a way out would be unattractive and unnatural.

3. We now return to the essential assumption made in the previous section: the edges with the same set of the quantum numbers i, m are identical, the edges with differing i, m are distinguishable. In principle, one might try to modify this assumption of partial distinguishability of edges in two opposite ways.

One possibility, which might look quite appealing, is that of complete indistinguishability of edges. It means that no permutation of any edges results in new states. To simplify the discussion, we confine ourself to expression (21) for the horizon area, instead of the most general one (9). Then, the total number of angular momentum states created by

$$\nu_j = \sum_m \nu_{jm}$$

indistinguishable edges of a given j with all $2j+1$ projections allowed, from $-j$ to j, is[3)]

$$K_j = \frac{(\nu_j + 2j)!}{\nu_j!\,(2j)!}\,. \qquad (26)$$

Those partial contributions

$$s_j = \ln K_j$$

to the black hole entropy

$$S = \sum_j s_j$$

that can potentially dominate the numerically large entropy may correspond to the three cases: $j \ll \nu_j$, $j \gg \nu_j$, and $j \sim \nu_j \gg 1$. These contributions are as follows:

$$\begin{aligned} &j \ll \nu_j, && s_j \approx 2j\ln\nu_j;\\ &j \gg \nu_j, && s_j \approx \nu_j \ln j;\\ &j \sim \nu_j \gg 1, && s_j \sim 4j\ln 2. \end{aligned}$$

In all the three cases, the partial contributions to the entropy S are much smaller parametrically than the corresponding contributions

$$a_j \sim j\nu_j$$

to the area

$$A = \sum_j a_j.$$

Therefore, $S \ll A$ in all these cases, and hence with indistinguishable edges of the same j, one cannot make the entropy of a black hole proportional to its area. This was pointed out earlier in Refs. [23, 24].

We now consider the last conceivable option, that of completely distinguishable edges. In this case, the total number of states is just $K = \nu\,!$, instead of (13), with the microcanonical entropy

$$S = \nu\ln\nu.$$

In principle, this entropy can be made proportional to the black hole area A. The model (which does not

3) Perhaps, the simplest derivation of this formula is as follows. Effectively, we here seek the number of ways of distributing ν_j identical balls into $2j+1$ boxes. Then the line of reasoning presented in [27, § 54] results in formula (26). I am grateful to V. F. Dmitriev for bringing to my attention that formula (26) can be derived in this simple way.

look natural, however) could be as follows. We choose a large quantum number $J \gg 1$ and assume that the horizon area A is saturated by the edges with j in the interval $J < j < 2J$ and with «occupation numbers» $\nu_j \sim \ln J$. Then the estimates for both S and A are of the order of $J \ln J$, and the proportionality between the entropy and the area can be attained.

However, although the entropy can be proportional to the area under the assumption of complete distinguishability, the maximum entropy for a given area is much larger than the area itself. Obviously, the maximum entropy for fixed

$$A \sim \sum_j \sqrt{j(j+1)}\, \nu_j$$

is here attained with all j's being as small as possible, e.g., $1/2$ or 1. In the classical limit $\nu \gg 1$, the entropy of a black hole then grows faster than its area, $A \sim \nu$, while

$$S = \nu \ln \nu \sim A \ln A.$$

Thus, the assumption of complete distinguishability is in conflict with the holographic bound, and therefore should be discarded.

There is no disagreement between this our conclusion and that in Refs. [23, 25, 26]: what is called complete distinguishability therein corresponds to our partial distinguishability.

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