ЖЭТФ, 2004, том 125, вып. 6, стр. 1219–1228

QUANTUM LONG-RANGE INTERACTIONS IN GENERAL RELATIVITY

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Submitted 11 February 2004

We consider one-loop effects in general relativity that result in quantum long-range corrections to the Newton law, as well as to the gravitational spin-dependent and velocity-dependent interactions. Some contributions to these effects can be interpreted as quantum corrections to the Schwarzschild and Kerr metrics.

PACS: 04.60.-m

1. INTRODUCTION

It has been recognized long ago that quantum effects in general relativity can generate long-range corrections to the Newton law. Such corrections due to the photon and massless neutrino contributions to the graviton polarization operator were calculated in [1–4]. The corresponding quantum correction to the Newton potential between two bodies with masses m_1 and m_2 is

$$U_{\gamma\nu} = -\frac{4+N_\nu}{15\pi}\,\frac{k^2\hbar m_1 m_2}{c^3 r^3}\,, \tag{1}$$

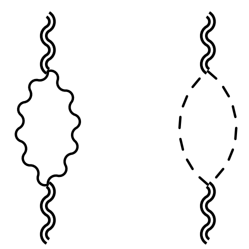
where N_ν is the number of massless two-component neutrinos and k is the Newton gravitational constant.

The reason why the problem allows a closed solution is as follows. The Fourier transform of $1/r^3$ is

$$\int d\mathbf{r}\frac{\exp(-i\mathbf{q}\cdot\mathbf{r})}{r^3} = -2\pi\ln q^2. \tag{2}$$

This singularity in the momentum transfer $\mathbf{q}$ implies that the discussed correction can be generated only by diagrams with two massless particles in the t-channel. The number of such diagrams of the second order in k is finite, and their logarithmic part in q^2 can be calculated unambiguously.

Analogous diagrams with gravitons and ghosts in the loop, Fig. 1*a, b*, were considered in Refs. [1, 5–7].



a *b*

Fig. 1. Graviton loop

(Here and below, wavy lines refer to quantum fluctuations of metric, double wavy lines denote a background gravitational field; dashed lines here refer to ghosts.) Clearly, other diagrams with two gravitons in the t-channel also contribute to the discussed correction proportional to $1/r^3$. This was pointed out long ago in [8], where all relevant diagrams were explicitly indicated.

The problem of quantum corrections to the Newton law is certainly interesting from the theoretical standpoint. It was addressed later in [9–15]. Unfortunately, as demonstrated in [16], none of these attempts was satisfactory.

The problem was then considered quantitatively in our previous paper [16]. Therein, all relevant diagrams, except one (see Fig. 4*b* below), were calculated correctly. In a recent paper [17], this last diagram is cal-

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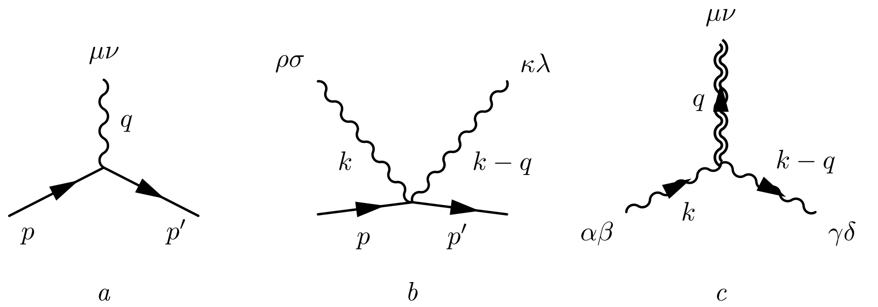


Fig. 2. Gravitational vertices

culated correctly[1)], and our results for all other contributions are confirmed.

The content of our present work is as follows. Using the background field technique [7], we construct invariant operators that describe quantum power-law corrections in general relativity. In the limit as one of the interacting particles is heavy, some of the derived corrections can be interpreted as quantum corrections to the Schwarzschild and Kerr metrics. Here our results differ essentially from those in [18].

We also demonstrate in an elementary way that to our accuracy, the spin-independent part of the discussed corrections for spinor particles coincides with the corrections for scalar particles. In particular, this implies that the obtained quantum corrections to the Schwarzschild metric are universal, i.e., independent of the spin of the central body. For some loop diagrams relevant to the problem, the mentioned coincidence of the spin-independent contributions of spinor particles with the corresponding results for scalar ones was proved previously in [18] by direct calculation.

With the effective operators constructed, we not only derive the corrections to the Newton law easily, but also obtain quantum corrections to other gravitational effects: spin-dependent and velocity-dependent interactions. In the present paper, we mainly consider the case of scalar particles. By spin, we therefore mean the internal angular momentum of a compound particle with scalar constituents.

We also comment on the problem of the classical relativistic corrections to the Newton law. Our conclusions here agree completely with the results in [19–21] (see also the textbook [22, § 106]), but on some point we disagree essentially with the statements in [17].

1) Both previous results for this contribution, by Donoghue [10] and by us [16], were incorrect.

2. PROPAGATORS AND VERTICES

Below, we use the units where $c = 1$ and $\hbar = 1$. Our metric signature is $\mathrm{diag}(1, -1, -1, -1)$.

The graviton operator $h_{\mu\nu}$ describes quantum fluctuations of the metric $g_{\mu\nu}$ in the background metric $g^0_{\mu\nu}$,

$$g_{\mu\nu} = g^0_{\mu\nu} + \varkappa h_{\mu\nu}, \quad \varkappa^2 = 32\pi k = 32\pi l_p^2. \tag{3}$$

We use the gauge condition

$$h^\mu_{\nu;\mu} - \frac{1}{2} h^\mu_{\mu;\nu} = 0 \tag{4}$$

for $h_{\mu\nu}$, where the indices of $h_{\mu\nu}$ are raised with the background metric $g^0_{\mu\nu}$, and the covariant derivatives are taken in the background field $g^0_{\mu\nu}$. The free graviton propagator is

$$\begin{gathered} D_{\mu\nu,\alpha\beta}(q) = i\frac{P_{\mu\nu,\alpha\beta}}{q^2 + i0}, \\ P_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\nu\alpha}\delta_{\mu\beta} - \delta_{\mu\nu}\delta_{\alpha\beta}\right). \end{gathered} \tag{5}$$

The tensor $P_{\mu\nu,\alpha\beta}$ is conveniently represented as [7]

$$P_{\mu\nu,\alpha\beta} = I_{\mu\nu,\alpha\beta} - \frac{1}{2}\,\delta_{\mu\nu}\delta_{\alpha\beta}\,,$$

where

$$I_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\nu\alpha}\delta_{\mu\beta}\right)$$

is a sort of unit operator with the property

$$I_{\mu\nu,\alpha\beta}t_{\alpha\beta} = t_{\mu\nu}$$

for any symmetric tensor $t_{\alpha\beta}$. We note the useful identity

$$P_{\alpha\beta,\kappa\lambda}P_{\kappa\lambda,\gamma\delta} = I_{\alpha\beta,\gamma\delta}\,. \tag{6}$$

The propagators of scalar and spinor particles are the usual ones,

$$D(p) = i\frac{1}{p^2 - m^2 + i0} \quad \text{and} \quad G(p) = i\frac{1}{\hat{p} - m + i0}$$

respectively.

The single-graviton vertex for both scalar and spinor particles (see Fig. 2a) are related to the energy-momentum tensor $T_{\alpha\beta}(p,p')$ of the corresponding particle as

$$V_{\alpha\beta}(p,p') = -i\,\frac{\varkappa}{2}\,T_{\alpha\beta}(p,p')\,. \tag{7}$$

The explicit expressions for the scalar and spinor particle vertices are

$$V^{(0)}_{\alpha\beta}(p,p') = -i\frac{\varkappa}{2}\left[p_\alpha p'_\beta + p'_\alpha p_\beta - \delta_{\alpha\beta}(pp'-m^2)\right] \tag{8}$$

and

$$V^{(1/2)}_{\mu\nu} = = -\frac{i\,\varkappa}{4}\,\bar u\,(p')\left[I_{\mu\nu\alpha\beta}P_\alpha\gamma_\beta - \delta_{\mu\nu}(\hat P - 2m)\right]u(p) \tag{9}$$

respectively; here, $P = p + p'$.

The contact interaction of a scalar particle with two gravitons (see Fig. 2b) is

$$V^{(0)}_{\varkappa\lambda,\rho\sigma} = i\varkappa^2\left[I_{\kappa\lambda,\alpha\delta}I_{\delta\beta,\rho\sigma}(p_\alpha p'_\beta + p'_\alpha p_\beta) - \right. - \frac{1}{2}\delta_{\kappa\lambda}I_{\rho\sigma,\alpha\beta} + \delta_{\rho\sigma}I_{\kappa\lambda,\alpha\beta}p_\alpha p'_\beta + \left. + \frac{(p'-p)^2}{4}\left(I_{\kappa\lambda,\rho\sigma} - \frac{1}{2}\delta_{\kappa\lambda}\delta_{\rho\sigma}\right)\right]. \tag{10}$$

To our accuracy, we can neglect the last term with $(p'-p)^2 = q^2$ in this expression, and rewrite the vertex conveniently as

$$V^{(0)}_{\varkappa\lambda,\rho\sigma} = = i\varkappa^2\left[I_{\varkappa\lambda,\alpha\delta}I_{\delta\beta,\rho\sigma}T_{\alpha\beta} - \frac{1}{4}(\delta_{\varkappa\lambda}T_{\rho\sigma} + \delta_{\rho\sigma}T_{\varkappa\lambda})\right]. \tag{11}$$

We use the two-graviton vertices on mass shell only. Therefore, the terms with the Kronecker δ entering the energy-momentum tensor in the last expression are also proportional to q^2, and hence can be neglected.

The contact two-graviton interaction of a spinor particle (see Fig. 2b) can be written on mass shell as

$$V^{(1/2)}_{\varkappa\lambda,\,\rho\sigma} = i\frac{\varkappa^2}{8}\left[\frac{3}{2}\left(I_{\varkappa\lambda,\,\mu\beta}I_{\rho\sigma,\,\beta\alpha} + I_{\rho\sigma,\,\mu\beta}I_{\varkappa\lambda,\,\beta\alpha}\right)P_\mu - \right. \left. -\delta_{\varkappa\lambda}I_{\rho\sigma,\,\mu\alpha}P_\mu - \delta_{\rho\sigma}I_{\varkappa\lambda,\,\mu\alpha}P_\mu\right]\bar u\,(p')\gamma^\alpha u(p) = = i\varkappa^2\left[\frac{3}{4}\,I_{\varkappa\lambda,\alpha\delta}I_{\delta\beta,\ \rho\sigma}T_{\alpha\beta} - \right. \left. - \frac{1}{4}(\delta_{\varkappa\lambda}T_{\rho\sigma} + \delta_{\rho\sigma}T_{\varkappa\lambda})\right]. \tag{12}$$

As regards the 3-graviton vertex (see Fig. 2c), which has the most complicated form, we follow [7, 17] in representing it as

$$\begin{aligned}
&V_{\mu\nu,\alpha\beta,\gamma\delta} = \\
&-i\,\frac{\varkappa}{2}\sum_i {}^i v_{\mu\nu,\alpha\beta,\gamma\delta}, \\
&{}^1 v_{\mu\nu,\alpha\beta,\gamma\delta} = P_{\alpha\beta,\gamma\delta}\times \\
&\times\left[k_\mu k_\nu + (k-q)_\mu(k-q)_\nu + q_\mu q_\nu - \frac{3}{2}\delta_{\mu\nu}q^2\right], \\
&{}^2 v_{\mu\nu,\alpha\beta,\gamma\delta} = \\
&= 2q_\varkappa q_\lambda[I_{\varkappa\lambda,\alpha\beta}I_{\mu\nu,\gamma\delta} + I_{\varkappa\lambda,\gamma\delta}I_{\mu\nu,\alpha\beta} - \\
&- I_{\varkappa\mu,\alpha\beta}I_{\lambda\nu,\gamma\delta} - I_{\varkappa\nu,\alpha\beta}I_{\lambda\mu,\gamma\delta}], \\
&{}^3 v_{\mu\nu,\alpha\beta,\gamma\delta} = q_\varkappa q_\mu(\delta_{\alpha\beta}I_{\varkappa\nu,\gamma\delta} + \\
&+ \delta_{\gamma\delta}I_{\varkappa\nu,\alpha\beta}) + q_\varkappa q_\nu(\delta_{\alpha\beta}I_{\varkappa\mu,\gamma\delta} + \delta_{\gamma\delta}I_{\varkappa\mu,\alpha\beta}) - \\
&- q^2(\delta_{\alpha\beta}I_{\mu\nu,\gamma\delta} + \delta_{\gamma\delta}I_{\mu\nu,\alpha\beta}) - \\
&- \delta_{\mu\nu}q_\varkappa q_\lambda(\delta_{\alpha\beta}I_{\gamma\delta,\varkappa\lambda} + \delta_{\gamma\delta}I_{\alpha\beta,\varkappa\lambda}), \\
&{}^4 v_{\mu\nu,\alpha\beta,\gamma\delta} = 2q_\varkappa\times \\
&\times[I_{\varkappa\lambda,\alpha\beta}I_{\gamma\delta,\nu\lambda}(k-q)_\mu + I_{\varkappa\lambda,\alpha\beta}I_{\gamma\delta,\mu\lambda}(k-q)_\nu - \\
&- I_{\varkappa\lambda,\gamma\delta}I_{\alpha\beta,\nu\lambda}k_\mu - I_{\varkappa\lambda,\gamma\delta}I_{\alpha\beta,\mu\lambda}k_\nu] + \\
&+ q^2(I_{\lambda\mu,\alpha\beta}I_{\gamma\delta,\lambda\nu} + I_{\lambda\nu,\alpha\beta}I_{\gamma\delta,\lambda\mu}) + \\
&+ \delta_{\mu\nu}q_\varkappa q_\lambda(I_{\alpha\beta,\varkappa\rho}I_{\rho\lambda,\gamma\delta} + I_{\gamma\delta,\varkappa\rho}I_{\rho\lambda,\alpha\beta}), \\
&{}^5 v_{\mu\nu,\alpha\beta,\gamma\delta} = [k^2 + (k-q)^2]\times \\
&\times\left(I_{\lambda\mu,\alpha\beta}I_{\gamma\delta,\lambda\nu} - \frac{1}{2}\delta_{\mu\nu}P_{\alpha\beta,\gamma\delta}\right) - \\
&- k^2\delta_{\gamma\delta}I_{\mu\nu,\alpha\beta} - (k-q)^2\delta_{\alpha\beta}I_{\mu\nu,\gamma\delta}.
\end{aligned} \tag{13}$$

In this vertex, we can also neglect the last structure ${}^5 v_{\mu\nu,\alpha\beta,\gamma\delta}$ to our accuracy.

3. UNIVERSALITY OF SPIN-INDEPENDENT EFFECTS

We first address the lowest-order s- and u-pole diagrams for graviton scattering, presented in Fig. 3a, b.

We start with a scalar particle. The terms with the Kronecker δ in single-graviton vertices (8) then cancel the s- and u-pole denominators. It can be easily demonstrated that in the sum of the two diagrams, the arising contact contributions combine into

$$V^{(0)\prime}_{\alpha\beta,\,\gamma\delta} = i\,\frac{\varkappa^2}{4}\left[\delta_{\alpha\beta}(p_\gamma p'_\delta + p'_\gamma p_\delta) + \right. \left. + \delta_{\gamma\delta}(p_\alpha p'_\beta + p'_\alpha p_\beta)\right] = i\frac{\kappa^2}{4}(\delta_{\alpha\beta}T^{(0)}_{\gamma\delta} + \delta_{\gamma\delta}T^{(0)}_{\alpha\beta}). \tag{14}$$

In the course of these transformations, we omit the terms with extra powers of the graviton momenta because they do not lead to $\ln q^2$ after subsequent loop

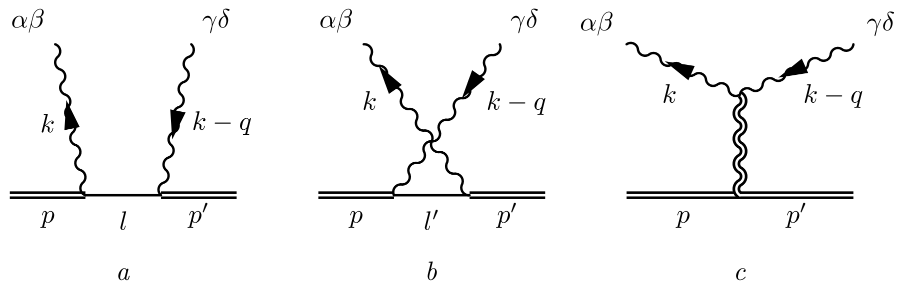


Fig. 3. Pole diagrams

integration. Combining this induced term with (10), we obtain the total effective two-graviton vertex for a scalar particle,

$$V^{(0)eff}_{\varkappa\lambda,\rho\sigma} = i\varkappa^2\, I_{\varkappa\lambda,\alpha\delta} I_{\delta\beta,\rho\sigma} T^{(0)}_{\alpha\beta} = = i\,\frac{\varkappa^2}{2}\, I_{\varkappa\lambda,\alpha\delta} I_{\delta\beta,\rho\sigma} P_\alpha P_\beta\,. \quad (15)$$

For spinor particles, single-graviton vertices (9) also contain terms with the Kronecker δ. Proceeding here with the s- and u-pole diagrams in the same way as in the scalar case, we obtain the following correction to the two-graviton vertex:

$$V^{(1/2)\prime}_{\alpha\beta,\,\gamma\delta} = i\,\frac{\varkappa^2}{4}\,(\delta_{\alpha\beta} T^{(1/2)}_{\gamma\delta} + \delta_{\gamma\delta} T^{(1/2)}_{\alpha\beta}). \quad (16)$$

The total effective two-graviton vertex for a spinor particle is then given by

$$V^{(1/2)eff}_{\varkappa\lambda,\rho\sigma} = i\,\frac{3}{4}\,\varkappa^2\, I_{\varkappa\lambda,\alpha\delta} I_{\delta\beta,\rho\sigma} T^{(1/2)}_{\alpha\beta}\,. \quad (17)$$

If we are interested in spin-independent effects in the graviton scattering off a spinor particle, one more step is possible. The spinor structure of the numerators in the s- and u-pole diagrams can be transformed as follows:

$$\bar u(p')\gamma^\sigma(\hat l + m)\gamma^\omega u(p) = \bar u(p')[l^\sigma\gamma^\omega + l^\omega\gamma^\sigma - - (\hat l - m)\delta^{\sigma\omega} + i\gamma^5\epsilon^{\sigma\xi\omega\eta} l_\xi\gamma_\eta + m\sigma_{\sigma\omega}]\,u(p). \quad (18)$$

The term $\bar u(p')(\hat l - m)u(p)$ in this expression, being averaged over spins, transforms to $l^2 - m^2$ (here, we again omit a term proportional to q^2). After cancelation of the denominators, the sum of these terms in the s- and u-pole diagrams reduces to

$$V^{(1/2)\prime\prime}_{\varkappa\lambda,\,\rho\sigma} = \frac{i\,\varkappa^2}{8} I_{\varkappa\lambda,\,\mu\beta} I_{\rho\sigma,\,\beta\alpha} P_\mu P_\alpha. \quad (19)$$

Because the spin-averaged energy-momentum tensor for spinors coincides with the scalar one, which is equal to $P_\mu P_\alpha/2$, the spin-independent term in the sum of (17) and (19) reduces to (15). In other words, from the fermion diagrams, we can single out the sum of structures that coincides with the effective sea-gull for a scalar particle after averaging over spins.

Finally, it can be easily demonstrated that after averaging over the spins, all the other terms in the numerators of the s- and u-pole spinor diagrams coincide with the corresponding terms in scalar diagrams with the required accuracy.

For the diagram in Fig. 3c, with the graviton pole in the t-channel, the coincidence between the scalar and spin-averaged spinor cases is obvious.

To summarize, the sum of scalar and spin-averaged spinor tree amplitudes, and hence the sum of the corresponding loop diagrams, coincide with the required accuracy.

4. SPIN-INDEPENDENT EFFECTIVE AMPLITUDES

We start the discussion of loops with the vacuum polarization diagrams, see Fig. 1. The covariant effective Lagrangian corresponding to the sum of these loops was derived in [7] with dimensional regularization. It is given by

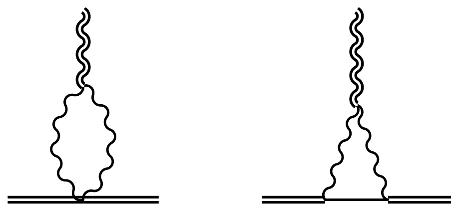
$$L_{RR} = -\frac{1}{960\pi^2(4-d)}\,\sqrt{-g}\,\left(42R_{\mu\nu}R^{\mu\nu} + R^2\right), \quad (20)$$

where, as usual, g is the determinant of the metric tensor, $R_{\mu\nu}$ is the Ricci tensor, and $R = R^\mu_\mu$.

For our purpose, Lagrangian (20) can be conveniently rewritten as [9]

$$L_{RR} = -\frac{1}{1920\pi^2}\ln|q^2|\left(42R_{\mu\nu}R^{\mu\nu} + R^2\right). \quad (21)$$

We are interested, in particular, in the situation where at least one of the particles is considered in the static



a *b*

Fig. 4. Vertex diagrams

limit. In this case, $|q^2| \to \mathbf{q}^2$, and in the coordinate representation we obtain

$$L_{RR} = \frac{1}{3840\pi^3 r^3}\left(42 R_{\mu\nu}R^{\mu\nu} + R^2\right). \qquad (22)$$

The next set of diagrams, Fig. 4, refers to the vertex part. The corresponding effective operator is

$$L_{RT} = -\frac{k}{8\pi^2 r^3}\left(3R_{\mu\nu}T^{\mu\nu} - 2RT\right), \quad T = T^{\mu}_{\mu}. \quad (23)$$

Here and below, $T^{\mu\nu}$ is the spin-independent part of the total energy-momentum tensor of matter.

We finally consider the diagrams in Fig. 5. The first two of them, the diagrams in Fig. 5*a, b*, as well as the diagrams in Figs. 1 and 4, depend only on the momentum transfer $t = q^2$. As regards the box diagrams in Fig. 5*c, d*, their contribution is partly reducible to the same structure as that of diagrams in Fig. 5*a, b*. The sum of all these t-dependent effective operators originating from the diagrams in Fig. 5 is

$$L_{TT} = \frac{k^2}{\pi r^3}T^2. \qquad (24)$$

The irreducible contribution of the s-channel box diagram 5*c* is

$$M_s = \frac{k^2[(s-m_1^2-m_2^2)^2 - 2m_1^2m_2^2]^2}{m_1^2m_2^2|q^2|}\ln\frac{|q^2|}{\lambda^2}\times$$
$$\times\frac{1}{\sqrt{(s-m_-^2)(s-m_+^2)}}\ln\frac{\sqrt{(s-m_-^2)}+\sqrt{(s-m_+^2)}}{\sqrt{(s-m_-^2)}-\sqrt{(s-m_+^2)}}, \qquad (25)$$

where m_1 and m_2 are the particle masses,

$$m_\pm = (m_1 \pm m_2), \quad s = (p_1+p_2)^2,$$

and p_1 and p_2 are the incoming 4-momenta.

The irreducible contribution M_u of the u-channel diagram in Fig. 5*d* is obtained from formula (25) by the substitution

$$s \to u = (p_1 - p_2 - q)^2,$$

with the corresponding analytic continuation.

The expressions for M_s and M_u converge in the ultraviolet sense, but diverge in the infrared limit, depending logarithmically on the «graviton mass» λ. As usual, such behavior is directly related to the necessity to cancel the infrared divergence in the Bremsstrahlung diagrams (evidently, the gravitational Bremsstrahlung in the present case). The box diagrams in Fig. 5*c, d* were considered previously in [23] from a different standpoint.

As regards the three Lagrangians in Eqs. (22), (23), and (24), by virtue of the Einstein equations

$$R_{\mu\nu} = 8\pi k\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \qquad (26)$$

they can be conveniently combined into

$$L_{tot} = -\frac{k^2}{60\pi r^3}\left(138\,T_{\mu\nu}T^{\mu\nu} - 31T^2\right). \qquad (27)$$

The irreducible amplitudes generated by the box diagrams in Fig. 5*c, d* depend nontrivially on s and u, respectively (in line with their simple dependence on $\ln|q^2|/|q^2|$). Therefore, they cannot be reduced to a product of energy–momentum tensors.

5. QUANTUM CORRECTIONS TO METRIC

The effects due to Lagrangian (27) can be conveniently interpreted as generated by quantum corrections to metric. To obtain these corrections, we split the total energy–momentum tensor $T_{\mu\nu}$ into those of a static central body and of a light probe particle, $T^0_{\mu\nu}$ and $t_{\mu\nu}$ respectively. Varying the expression resulting in this way from (27) with respect to $t^{\mu\nu}$, we then obtain a tensor that can be interpreted as a quantum correction $h^{(q)}_{\mu\nu}$ to the metric created by the central body,

$$h^{(q)}_{\mu\nu} = \frac{k^2}{15\pi r^3}\left(138T^0_{\mu\nu} - 31\delta_{\mu\nu}T^0\right). \qquad (28)$$

It follows immediately from this expression that

$$h^{(q)}_{00} = \frac{107}{15}\frac{k^2}{\pi r^3}T^0_{00} = \frac{107}{15}\frac{k^2M}{\pi r^3}, \qquad (29)$$

where M is the mass of the central body.

For the space components $h^{(q)}_{mn}$ of the metric created by a heavy body at rest, one might naively expect from formula (28) that they are given by

$$\frac{31}{15}\frac{k^2}{\pi r^3}\delta_{mn}T^0_{00} = \frac{31}{15}\frac{k^2M}{\pi r^3}\delta_{mn}.$$

But the calculation of $h^{(q)}_{mn}$ actually requires a modification of formula (28). The point is that we work with

a *b* *c* *d*

Fig. 5. Scattering diagrams

gauge condition (4) for the graviton field. It is only natural to require that the resulting effective field $h^{(q)}_{mn}$ should satisfy the same condition, which now simplifies to

$$h^{(q)\mu}{}_{\nu,\mu} - \frac{1}{2} h^{(q)\mu}{}_{\mu,\nu} = 0.$$

The space metric thus obtained is

$$h^{(q)}_{mn} = \frac{k^2 M}{\pi r^3} \left\{ \frac{31}{15} \delta_{mn} - \right.$$
$$\left. - \frac{76}{15} \left[\frac{r_m r_n}{r^2} + \ln\left(\frac{r}{r_0}\right) \left(\delta_{mn} - 3\, \frac{r_m r_n}{r^2} \right) \right] \right\}. \quad (30)$$

Technically, the expression in square brackets in (29) originates from the terms containing structures of the type $\partial_\mu T^{\mu\nu}$. Generally speaking, they arise in calculating Lagrangians (23), (24), and (27), but are omitted there because they vanish on mass shell. These terms are therefore absent in (28). But they can be restored by rewriting the net result (27), by means of Einstein equations (26), as

$$L_{tot} = -\frac{1}{3840 \pi^3 r^3} \left(138 R_{\mu\nu} R^{\mu\nu} - 31 R^2 \right), \quad (31)$$

and then attaching energy-momentum tensors to the double wavy lines using graviton propagators (5). The presence of $\ln(r/r_0)$, where r_0 is some normalization point, is quite natural here if we recall $\ln|q^2|$ in the momentum representation. Fortunately, this term in the square brackets does not influence physical effects.

The obtained quantum corrections $h^{(q)}_{00}$ and $h^{(q)}_{mn}$ to the metric are universal, i.e., are the same when created by a spinless or spinning heavy point-like particle.

Our results (29) and (30) differ from the corresponding ones in [18]. The main reason is that the contribution of operator (24) to the metric is absent in [18]. This omission does not look logical to us: on mass shell, one cannot distinguish this operator from other ones (see (27), (31)). One more disagreement is perhaps due to the same inconsistency: the contribution of operator (23) to the metric, as given in [18], is two times smaller than ours.

In addition, the Fourier transformation of $(q_m q_n / \mathbf{q}^2) \ln \mathbf{q}^2$ is performed in [18] incorrectly, which gives a wrong result ($r_m r_n / r^2$ only) for the term in the square brackets in (30).

In conclusion of this section, we consider the $0n$ component of tensor (28). It is given by

$$h^{(q)}_{0n} = \frac{46}{5} \frac{k^2}{\pi r^3} T^0_{0n} = -\frac{46}{5} \frac{k^2 M \mathbf{v}}{\pi r^3}, \quad (32)$$

where $\mathbf{v}$ is the velocity of the source.

We are interested in the situation corresponding to a compound central body rotating with the angular velocity $\boldsymbol{\omega}$, but with its centre of mass being at rest. The velocity of a separate element of the body is then given by $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the coordinate of this element. In addition, we must shift $\mathbf{r} \to \mathbf{r} + \boldsymbol{\rho}$ in formula (32). Then, following [22, § 106, Problem 4], we obtain a quantum correction to the Kerr metric,

$$h^{(q)}_{0n} = \frac{69}{5} \frac{k^2}{\pi r^5} [\mathbf{S} \times \mathbf{r}]. \quad (33)$$

We emphasize that spin $\mathbf{S}$ involved here is in fact the internal angular momentum of a rotating compound central body with spinless constituents. We cannot see any reason why this last quantum correction (33) should be universal (as distinct from $h^{(q)}_{00}$ and $h^{(q)}_{mn}$). If instead of a compound body discussed here, we deal with a particle of spin 1/2, the general structure of $h^{(q)}_{0n}$ is of course the same, but the numerical coefficient can be quite different.

The last problem, that of a quantum correction to the Kerr metric created by a particle of spin 1/2, was addressed in [18]. However, the treatment of this correction there raises the same objections: the contribution of operator (24) to $h^{(q)}_{0n}$ is missed at all, and the corresponding effect of operator (23) is not taken into account properly.

6. QUANTUM CORRECTIONS TO GRAVITATIONAL EFFECTS. I

We start with the correction to the Newton law. As usual, it is generated by the 00 component of metric. Here, expression (29) gives

$$U^{qr}(r) = \frac{107}{30}\,\frac{k^2 Mm}{\pi r^3}. \tag{34}$$

However, in line with (29), we must now take the irreducible contribution of the box diagrams in Fig. 5*c, d* into account, which cannot be reduced to metric. Having other applications in mind, we write the sum of the two amplitudes, retaining in it the terms of not only the zeroth order in c^{-2}, but also the first order,

$$M_s + M_u = -k^2 m_1 m_2 \ln(\mathbf{q}^2 - \omega^2) \times \\ \times \frac{2}{3}\left(23 + \frac{524}{5}\,\frac{p_1 p_2 - m_1 m_2}{m_1 m_2}\right). \tag{35}$$

In the static limit, $\omega \to 0$, $p_1 p_2 \to m_1 m_2$, expression (35) reduces to

$$M_s + M_u \to -\frac{46}{3}\,k^2 m_1 m_2 \ln \mathbf{q}^2. \tag{36}$$

Changing the sign (in passing from the amplitude to the potential) and performing the Fourier transformation, we obtain [16, 17]

$$U^{qi}(r) = -\frac{23}{3}\,\frac{k^2 Mm}{\pi r^3}. \tag{37}$$

Thus, the net correction to the Newton law is

$$U^{q}(r) = -\frac{41}{10}\,\frac{k^2 Mm}{\pi r^3}. \tag{38}$$

This result was also cross-checked and confirmed by the independent calculation in the standard harmonic gauge, with the field variables

$$\psi^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \delta^{\mu\nu}$$

and the gauge condition

$$\partial_\mu \psi^{\mu\nu} = 0.$$

We now consider the quantum correction to the interaction of the orbital momentum $\mathbf{l}$ of a light particle with its own spin $\mathbf{s}$, i.e., to the gravitational spin–orbit interaction. It is most easily obtained with the general expression for the frequency $\boldsymbol{\omega}$ of the spin precession in a gravitational field derived in [24]. For a nonrelativistic particle in a weak static centrally symmetric field, this expression simplifies to

$$\omega_i = \frac{1}{2}\,\varepsilon_{imn}(\gamma_{mnk} v_k + \gamma_{0n0} v_m), \tag{39}$$

where

$$\gamma_{mnk} = \frac{1}{2}(\partial_m h_{nk} - \partial_n h_{mk}), \quad \gamma_{0n0} = -\frac{1}{2}\partial_n h_{00}$$

are the Ricci rotation coefficients and $\mathbf{v}$ is the particle velocity (the present sign convention for $\boldsymbol{\omega}$ is opposite to that in [24]). A simple calculation results in

$$U^{q}_{ls}(r) = -\frac{169}{20}\,\frac{k^2}{\pi r^5}\,\frac{M}{m}(\mathbf{l}\cdot\mathbf{s}). \tag{40}$$

Finally, with formula (33), we easily derive the quantum correction to the interaction of the orbital momentum $\mathbf{l}$ of a light particle with the internal angular momentum (spin) $\mathbf{S}$ of a compound central body, i.e., to the Lense–Thirring effect,

$$U^{q,r}_{LT}(r) = -\frac{69}{5}\,\frac{k^2}{\pi r^5}(\mathbf{l}\cdot\mathbf{S}). \tag{41}$$

7. ASIDE ON CLASSICAL RELATIVISTIC CORRECTIONS

In this section, we first consider the classical velocity-dependent correction to the Newton law. On one hand, this is an introduction to the derivation of quantum velocity-dependent corrections in the next section. On the other hand, this is necessary for the discussion of another, velocity-independent relativistic correction to the Newton law. The derivation of the classical velocity-independent correction via the diagram technique served in [16, 17] as a check of calculations of quantum corrections to the Newton law.

We consider the Born scattering amplitude with the graviton exchange in the harmonic gauge,

$$M_B = 8\pi k\,\frac{T^1_{\mu\nu}\,T^2_{\mu\nu} - (1/2)\,T^1_{\mu\mu}\,T^2_{\nu\nu}}{\mathbf{q}^2 - \omega^2}, \tag{42}$$

where $T^{1,2}_{\mu\nu}$ are the energy-momentum tensors of particles with the respective masses $m_{1,2}$ and velocities $\mathbf{v}_{1,2}$. To the adopted accuracy, the numerator simplifies to

$$\frac{1}{2}\,T^1_{00}\,T^2_{00} - 2T^1_{0n}\,T^2_{0n} = \frac{m_1 m_2}{2}(1 - 4\,\mathbf{v}_1\cdot\mathbf{v}_2).$$

We then expand the denominator to the first order in $\omega^2/\mathbf{q}^2$, and thus arrive at the expression

$$\frac{4\pi k m_1 m_2}{\mathbf{q}^2}\left(1 - 4\mathbf{v}_1\cdot\mathbf{v}_2 + \frac{\omega^2}{\mathbf{q}^2}\right).$$

The term of the zeroth order in c^{-2} in this formula, $4\pi k m_1 m_2/\mathbf{q}^2$, is obviously (after the necessary sign reversal) the Fourier transform of the Newton potential.

However, we are interested here in the terms of the first order in c^{-2}. To transform $\omega^2/\mathbf{q}^2$, we note that ω is in fact the energy difference between the initial and final energies of a particle. The particles can now be considered nonrelativistic, and this difference therefore transforms (to the first order in $\mathbf{p}' - \mathbf{p}$) as follows:

$$\varepsilon' - \varepsilon = (\mathbf{p}' - \mathbf{p}) \cdot \mathbf{v}.$$

Therefore, the terms of the first order in c^{-2} are rewritten as

$$\frac{4\pi k m_1 m_2}{\mathbf{q}^2}\left[-4\mathbf{v}_1 \cdot \mathbf{v}_2 + \frac{(\mathbf{q}\cdot\mathbf{v}_1)(\mathbf{q}\cdot\mathbf{v}_2)}{\mathbf{q}^2}\right].$$

The Fourier transform of this expression, taken with the opposite sign, is the well-known relativistic velocity-dependent correction to the Newton potential [19, 20, 22]

$$U^{cl}_{vv} = \frac{k m_1 m_2}{2r}[7\mathbf{v}_1\cdot\mathbf{v}_2 + (\mathbf{n}\cdot\mathbf{v}_1)(\mathbf{n}\cdot\mathbf{v}_2)], \qquad \mathbf{n} = \frac{\mathbf{r}}{r}. \tag{43}$$

We here essentially follow the derivation by Iwasaki [21].

At least equally simple is the derivation of the relativistic velocity-independent correction to the Newton potential. In the harmonic gauge, the metric created by a point-like mass m_1 is

$$ds^2 = \frac{r - km_1}{r + km_1}\,dt^2 - \frac{r + km_1}{r - km_1}\,dr^2 - (r + km_1)^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{44}$$

In the expansion in r_g of the classical action $-m_2\int ds$ for a probe particle of mass m_2, the second-order term is $-k^2 m_1^2 m_2/2r^2$. Now, reversing the sign (to pass from a Lagrangian to a potential) and restoring the symmetry between m_1 and m_2, we arrive at the discussed correction

$$U^{cl} = \frac{k^2 m_1 m_2 (m_1 + m_2)}{2r^2}. \tag{45}$$

The classical correction (45) was found long ago in [19, 20] (see also the textbook [22, § 106]), and was derived later in [21] by calculating the corresponding parts of the diagrams in Fig. 4*b*, and 5*b, c, d* in the harmonic gauge. A subtle point of the last calculation [21] refers to the box diagrams in Fig. 5*c, d*. Obviously, the classical c^{-2} contribution of these diagrams, in particular, contains the result of iteration of the usual Newton interaction and the velocity-dependent interaction (43). Therefore, the result of this iteration should be subtracted from the sum of the contributions of the diagrams in Figs. 4*b*, and Figs. 5*b, c, d*. This has been done properly by Iwasaki [21]).

However, Bjerrum-Bohr, Donoghue, and Holstein argue (see sec. 2.1 in [17]) that in the scattering problem, as distinct from the bound state one, this subtraction is unnecessary. They claim that there is a difference between what they call «the lowest-order scattering potential» without this subtraction, and the classical correction U^{cl}, which they call the bound state potential. For our part, we do not see any difference of principle between the bound state problem and the scattering one[2)], and therefore believe that it is just (45) which should be considered as the relativistic correction to the Newton law, both in the scattering and bound state problems.

8. QUANTUM CORRECTIONS TO GRAVITATIONAL EFFECTS. II

We now address the quantum correction to the classical velocity-dependent gravitational interaction (43). We start with the amplitude (27) written in the momentum representation,

$$L_{tot} = \frac{k^2}{30}\ln|q^2|\left(138\,T_{\mu\nu}T^{\mu\nu} - 31T^2\right). \tag{46}$$

Unlike with the previous quantum corrections, we here go beyond the static approximation, and in the spirit of the previous section, expand

$$\ln|q^2| = \ln(\mathbf{q}^2 - \omega^2)$$

to the first order in ω^2. Following the same lines of reasoning further, we easily obtain the quantum velocity-dependent correction

$$U^{q,r}_{vv}(\mathbf{r}) = -\frac{k^2 m_1 m_2}{60\pi r^3}\times \times[445(\mathbf{v}_1\cdot\mathbf{v}_2) + 321(\mathbf{n}\cdot\mathbf{v}_1)(\mathbf{n}\cdot\mathbf{v}_2)], \quad \mathbf{n} = \frac{\mathbf{r}}{r}. \tag{47}$$

With formula (47), we can derive (in the spirit of [22, § 106, Problem 4]) the quantum correction to the spin–spin interaction of compound bodies 1 and 2 rotating with the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, but with their centres of masses at rest. The velocity of a separate element of the body i is then given by $\mathbf{v}_i = \boldsymbol{\omega}_i \times \boldsymbol{\rho}_i$, where $\boldsymbol{\rho}_i$ is the coordinate of this element counted off the center of mass of this body. In formula (47), where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we then shift

[2)] For instance, the second Born approximation to a scattering amplitude is as legitimate a notion as the second-order correction to a bound state energy.

$$\mathbf{r} \to \mathbf{r} + \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2.$$

Again following [22], we thus obtain

$$U_{ss}^{q,r}(\mathbf{r}) = \frac{69}{10}\frac{k^2}{\pi r^5}[3(\mathbf{S}_1\cdot\mathbf{S}_2) - 5(\mathbf{n}\cdot\mathbf{S}_1)(\mathbf{n}\cdot\mathbf{S}_2)], \quad \mathbf{n} = \frac{\mathbf{r}}{r}, \tag{48}$$

where $\mathbf{S}_i$ are the internal angular momenta (spins) of the rotating compound central bodies.

We note that quantum correction (41) to the Lense–Thirring effect can also be derived in the same way.

We finally consider the corresponding corrections induced by irreducible amplitude (35), which is now conveniently rewritten as

$$M_s + M_u = -k^2 m_1 m_2 \ln(\mathbf{q}^2 - \omega^2) \times \times \frac{2}{3}\left(23 - \frac{524}{5}\mathbf{v}_1\cdot\mathbf{v}_2\right). \tag{49}$$

This amplitude also generates quantum corrections to the velocity-dependent, Lense–Thirring, and spin–spin interactions. The calculations are practically identical with the previous ones, and give the respective corrections

$$U_{vv}^{q,\,irr}(\mathbf{r}) = = \frac{k^2 m_1 m_2}{10\pi r^3}[311(\mathbf{v}_1\cdot\mathbf{v}_2) + 115(\mathbf{n}\cdot\mathbf{v}_1)(\mathbf{n}\cdot\mathbf{v}_2)], \tag{50}$$

$$U_{LT}^{q,\,irr}(r) = \frac{262}{5}\frac{k^2}{\pi r^5}(\mathbf{l}\cdot\mathbf{S}), \tag{51}$$

$$U_{ss}^{q,\,irr}(\mathbf{r}) = = -\frac{131}{5}\frac{k^2}{\pi r^5}[3(\mathbf{S}_1\cdot\mathbf{S}_2) - 5(\mathbf{n}\cdot\mathbf{S}_1)(\mathbf{n}\cdot\mathbf{S}_2)]. \tag{52}$$

Now, combining these contributions with those originating from quantum corrections to the metric, we finally obtain

$$U_{vv}^{q}(\mathbf{r}) = U_{vv}^{q,r}(\mathbf{r}) + U_{vv}^{q,\,irr}(\mathbf{r}) = = \frac{k^2 m_1 m_2}{60\pi r^3}[1421(\mathbf{v}_1\cdot\mathbf{v}_2) + 369(\mathbf{n}\cdot\mathbf{v}_1)(\mathbf{n}\cdot\mathbf{v}_2)], \tag{53}$$

$$U_{LT}^{q}(r) = U_{LT}^{q,r}(r) + U_{LT}^{q,\,irr}(r) = \frac{193}{5}\frac{k^2}{\pi r^5}(\mathbf{l}\cdot\mathbf{S}), \tag{54}$$

$$U_{ss}^{q}(\mathbf{r}) = U_{ss}^{q,r}(\mathbf{r}) + U_{ss}^{q,\,irr}(\mathbf{r}) = = -\frac{193}{10}\frac{k^2}{\pi r^5}[3(\mathbf{S}_1\cdot\mathbf{S}_2) - 5(\mathbf{n}\cdot\mathbf{S}_1)(\mathbf{n}\cdot\mathbf{S}_2)]. \tag{55}$$

We are grateful to N. G. Ural'tsev and A. I. Vainshtein for useful discussions. This paper was supported by the RFBR (grant № 03-02-17612).

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Note added in proofs. After submitting our manuscript to the journal, we became aware that the problem of long-range quantum corrections in gravity was also addressed by D. Dalvit and F. D. Mazzitelli (Phys. Rev. D **56**, 7779 (1997); E-print archives hep-th/9708102). In particular, they found the contribution of the vacuum polarization diagrams 1*a, b* to the metric and to the Newton law.