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GYROSCOPE DEVIATION FROM GEODESIC MOTION: QUASIRESONANT OSCILLATIONS ON A CIRCULAR ORBIT

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The general relativistic spin–orbit interaction gives rise to a quasiresonant oscillation of the gyroscope mass center along the orbital normal. The oscillation amplitude appears to be measurable by present-day instruments. The influence of oblateness of the field source is investigated.

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1. INTRODUCTION

In general relativity, the motion of a spinning test body (gyroscope) is affected by the spin–orbit interaction in two aspects: 1) the influence of the orbital motion on the orientation of the gyroscope rotation axes, and 2) the influence of the gyroscope intrinsic momentum (spin) on its orbit. The first is comparatively simple when the parallel spin transport is assumed. It is admissible if the deviation from a geodesic motion is small. The Fermi–Walker transport along an appointed world line is also not complicated. In a spherically symmetric field, parallel transport along a geodesic leads to a precession of the gyroscope axes known as the geodetic, or de Sitter precession [1]. In the field of a rotating mass, the gyroscope axes undergo the Schiff precession [2], to be verified in the Gravity Probe B experiment (see [3] for details).

In this work, the second aspect of the spin–orbit interaction is considered. The orbital motion of the gyroscope is a sophisticated problem that has not been fully resolved untill now even in the post-Newtonian approximation. There exist several different approaches with different results in the leading approximation (see, e.g., [4–10]). The only covariant general relativistic equations of motion of the spinning test particles are the well-known Papapetrou equations [5]. This set of equations is incomplete and requires supplementary conditions. It is generally accepted that these conditions single out a representative point as the gyroscope mass center, but there exist diverse other opinions [9–12]. In addition, the Papapetrou equations or alternative ones are very complicated. Their investigation is usually limited by a general analysis; examination of the effects is typically restricted by the motion of the gyroscope with a vertical spin, i.e., with the gyroscope axes orthogonal to the orbital plane [13]. For example, it is known that such a gyroscope moves along a circular orbit with the velocity differing from the one of a body without spin [14]. In [14], the conclusion was drawn that the gyroscope with a horizontal spin leaves the geodesic plane, but an erroneous estimation of this effect was given. The effect is much larger because of a quasiresonant character of the spin–orbit interaction, as was first revealed in [15, 16].

In the present work, the motion of a gyroscope with the horizontal spin is investigated and the general relativistic effect of a quasiresonant beating is proposed. Because of a small denominator, the speed of light is cancelled in the oscillation amplitude, and the effect therefore becomes quite sizeable. The obvious physical interpretation of the effect is given. This effect is independent of supplementary conditions and is the same in the different approaches [4–10]. The description is significantly simplified by expanding the equations of motion up to the linear terms in the displacement from a geodesic. Instead of studying an intricate gyroscope orbit, the small oscillation is investigated. This oscillation gives sufficient information about the gyroscope orbit. It is shown that a Newtonian nonsphericity of the field source causes a specific effacing of the quasires-

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onant beating, retaining the oscillation amplitude measurable.

In what follows, orthonormal bases are used in calculations, Greek indices run from 0 to 3 and Latin indices run from 1 to 3. The signature is $(-+++)$.

2. THE ESSENCE OF THE EFFECT

The general relativistic spin–orbit acceleration a deviating the gyroscope mass center from a geodesic is of the order of

$$a \sim \epsilon \frac{S}{\lambda} g, \qquad (1)$$

where

$$\epsilon = \frac{GM}{c^2 r}$$

is the relativistic small parameter, $g = GM/r^2$ is the Newtonian acceleration due to gravity, S is the spin of the gyroscope, λ is its orbital moment, c is the speed of light, M is the source mass, and G is the gravitational constant. The motion of the rotating body mass center essentially depends (in the leading approximation (1)) on the reference frame in which it is obtained. The general expression for the spin–orbit acceleration in the leading post-Newtonian approximation (1) is [7, 17]

$$\mathbf{a} = 3\frac{GM}{mc^2r^3}[\mathbf{S}\times\mathbf{v} + +(2-\sigma)\hat{\mathbf{r}}\cdot(\mathbf{S}\cdot(\hat{\mathbf{r}}\times\mathbf{v})) - (1+\sigma)(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{S}\times\hat{\mathbf{r}})]. \qquad (2)$$

The parameter σ numbers the different mass centers: $\sigma = 0$ corresponds to the Dixon [6] and Pirani [18] conditions (the intrinsic mass center), $\sigma = 1$ corresponds to the Corinaldesi–Papapetrou conditions [19] (the mass center defined in the «rest» frame in which the gyroscope moves with the velocity $\mathbf{v}$), and $\sigma = 1/2$ leads to the results of Fock [4] and of Refs. [9, 10]. For a circular orbit of the gyroscope ($\mathbf{v}\cdot\mathbf{r} = 0$) with the gyroscope axis lying in the orbital plane ($\mathbf{S}\cdot(\mathbf{r}\times\mathbf{v}) = 0$), spin–orbit acceleration (2) is independent of the parameter σ,

$$\mathbf{a} = 3\frac{GM}{mc^2r^3}\mathbf{S}\times\mathbf{v}. \qquad (3)$$

Parallel transport of the spin vector $\mathbf{S}$ means that in the process of revolution, acceleration (3) is directed along the orbital normal $\mathbf{e}_3$ and is periodic in time τ,

$$\mathbf{a} = \mathbf{e}_3\,\epsilon\,\frac{S}{\lambda}\,g\cos(\omega_s\tau + \beta).$$

The frequency ω_s differs from the orbital frequency ω because of the geodetic precession Ω^G,

$$\Delta\omega = \omega - \omega_s = \Omega^G = \frac{3}{2}\epsilon\,\omega. \qquad (4)$$

On the other hand, the frequency of the free tidal oscillation along the orbital normal is equal to the orbital frequency. This leads to an almost resonant beating with modulation frequency (4) and the maximum amplitude

$$A = \frac{a}{2\omega\,\Delta\omega} = \frac{S}{\lambda}r. \qquad (5)$$

We note the cancellation of the speed of light c in amplitude (5) by the small relativistic denominator $\Delta\omega$ given by Eq. (4). During the time $\tau \ll (\Omega^G)^{-1}$, the quasiresonant oscillation enhances linearly with the rate

$$A\,\Delta\omega = \frac{3}{2}\epsilon\frac{S}{\lambda}v \qquad (6)$$

and reaches the values measurable with present-day instruments. For example, in the case of a gyroscope with the dimension 10^{-1} m and the intrinsic rotation period 10^{-1} s in a near-Earth orbit $r \approx 7\cdot10^3$ km, we obtain the values

$$\epsilon \sim 10^{-10}, \quad \frac{S}{\lambda} \sim 10^{-9}, \quad A\,\Delta\omega \sim 10^{-9}\ \ \mathrm{cm/day}. \qquad (7)$$

Parasitic effects of a nonrelativistic origin are mutually cancelled in the symmetric relative oscillations of two gyroscopes with antiparallel spins.

3. CALCULATION OF THE NET EFFECT

In the post-Newtonian approximation, the static spherically symmetric gravitational field is described by the tetrad

$$\mathring{\mathbf{e}}^\mu = \{(1-\epsilon)\,cdt,\ (1+\epsilon)\,dr,\ r\sin\theta\,d\phi,\ -r\,d\theta\} \qquad (8)$$

that represents the rest observers in the Schwarzschild metric. In this frame, the «electric» part E and the «magnetic» part B of the Riemann tensor $\mathcal{R}$ (see, e.g., [8, 17]),

$$E_{ij} = \mathcal{R}_{i0j0}, \quad 2B_{ij} = \mathcal{R}_{iomn}\varepsilon^{mn}{}_j,$$

are given by

$$\mathring{E}_{ij} = n^2\mathrm{diag}\{-2,\ 1,\ 1\}, \quad \mathring{B}_{ij} = 0, \quad n^2 = GM/r^3. \qquad (9)$$

Transition to the orbital frame $\mathbf{e}^\nu$ is performed by the boost

$$\mathring{\mathbf{e}}^\mu = L^\mu_\nu\,\mathbf{e}^\nu$$

in the $\mathring{\mathbf{e}}_2$ direction. The Lorentz matrix L has the standard form. Namely, the components of the 4-velocity of the fiducial orbital motion $\phi = nt = \omega\tau$ are

$$u^\mu = L_0^\mu = \gamma\{1,\ 0,\ \beta,\ 0\}, \tag{10}$$

where

$$\begin{array}{rclrcl} \gamma &=& (1-\beta^2)^{-1/2}, & \beta &=& v/c, \\ v &=& (1+\epsilon)nr, & \omega &=& \gamma v/r = n(1+3\epsilon/2), \end{array}$$

and τ is the proper time. The $\mathbf{e}_1$ axis is directed along the current radius vector, the $\mathbf{e}_2$ axis is along the orbital motion velocity, and $\mathbf{e}_3$ is orthogonal to the orbital plane,

$$L_1^1 = 1, \quad L_2^2 = \gamma, \quad L_3^3 = 1.$$

The angular velocity vector $\mathbf{\Omega}$ of rotation of the orbital triad

$$\nabla_u \mathbf{e}^i = \Omega^i{}_k \mathbf{e}^k$$

has the only component

$$\Omega_3 = \Omega_{12} \stackrel{\text{def}}{=} \omega_s = n. \tag{11}$$

The transformation of the «magnetic» matrix [17]

$$\begin{aligned} B_{ij} = 4\mathring{B}_{kl} L^{[k}{}_i u^{0]} L^{[l}{}_j u^{0]} - \mathring{B}_{pq}\varepsilon^p{}_{km}\varepsilon^q{}_{ln} \times \\ \times L^k{}_i u^m L^l{}_j u^n - 4\mathring{E}_{km}\varepsilon^m{}_{ln} L^{[k}{}_{(i} u^{0]} L^l{}_{j)} u^n \end{aligned} \tag{12}$$

leads to the appearance of the component

$$B_{31} = \beta(\mathring{E}_{33} - \mathring{E}_{11}) = 3n^2\beta \tag{13}$$

in the orbital frame. The transformation of the «electric» matrix is analogous to (12) with the substitution

$$B \to E, \quad E \to -B$$

(see [17]). The result is

$$\begin{aligned} E_{11} &= -2n^2(1+\epsilon/2), \quad E_{22} = n^2, \\ E_{33} &= n^2(1+3\epsilon) = \omega^2. \end{aligned} \tag{14}$$

We note that the component E_{22} parallel to the boost is invariant and the equality $E_{33} = \omega^2$ is exact.

The equation of motion of the gyroscope mass center in the orbital frame is the equation of geodesic deviation with spin–orbit acceleration (2) in its right-hand side[1)],

$$\nabla_u \nabla_u \xi^i + E_k^i \xi^k = a^i, \tag{15}$$

where

$$\nabla_u \nabla_u \boldsymbol{\xi} = \ddot{\boldsymbol{\xi}} + 2\mathbf{\Omega} \times \dot{\boldsymbol{\xi}} + \dot{\mathbf{\Omega}} \times \boldsymbol{\xi} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \boldsymbol{\xi}).$$

The dot denotes the derivative with respect to the proper time τ. In the post-Newtonian approximation, the spin–orbit force applied to the intrinsic mass center of the rotating body is

$$ma^i = -c^{-1} B_k^i S^k. \tag{16}$$

This formula can be obtained, for example, by the matched asymptotic expansions method [8] or directly from the Papapetrou equations with the supplementary conditions of Pirani or Dixon (see [17]; distinctions between the exact conditions of Pirani and Dixon are also discussed there).

In Eq. (15), E_{ik} is measured on the fiducial geodesic u, but B_{ik} in (16) must be calculated in the frame comoving with the gyroscope mass center. This «mixing» is admissible in the approximation linear in S (Eq. (1)) and linear in ξ (Eq. (15)) if the displacement ξ is induced by the spin–orbit interaction,

$$\xi \sim S, \quad \xi S \sim S^2 \sim \xi^2 = 0.$$

On the same ground, we transport the spin vector along the fiducial geodesic according to Fermi–Walker,

$$\nabla_u \mathbf{S} = \dot{\mathbf{S}} + \mathbf{\Omega} \times \mathbf{S} = 0, \tag{17}$$

$$\dot{S}_1 = \omega_s S_2, \quad \dot{S}_2 = -\omega_s S_1, \quad \dot{S}_3 = 0.$$

Parallel transport equation (17) describes the known geodetic precession (4),

$$S_1 = S\cos(\omega_s\tau + \beta), \quad S_2 = -S\sin(\omega_s\tau + \beta). \tag{18}$$

For the spin in the fiducial plane ($S_3 = 0$), equations (15) and (16) of the mass center motion become

$$\left.\begin{array}{rcl} \ddot{\xi}_1 - 2\omega_s\dot{\xi}_2 &+& (E_{11} - \omega_s^2)\xi_1 = 0, \\ \ddot{\xi}_2 + 2\omega_s\dot{\xi}_1 &=& 0, \end{array}\right\} \tag{19}$$

$$\ddot{\xi}_3 + E_{33}\xi_3 = 3g\epsilon\frac{S_1}{\lambda}. \tag{20}$$

Equations (19) describe the free oscillation with the frequency

$$\omega' = \sqrt{E_{11} - 3\omega_s^2} = n(1 - 3\epsilon/2),$$

induced by the initial perturbation in the fiducial plane. The difference between ω' and the orbital frequency ω

1) Equation (15) can be obtained by expanding the Papapetrou equations up to linear terms in the displacement ξ in the leading approximation (1) of the spin–orbit interaction. At $\mathbf{S} = 0 \to \mathbf{a} = 0$, Eq. (15) is reduced to the geodesic deviation equation.

is caused by the general relativistic pericenter drift of the perturbed quasielliptic orbit,

$$\omega - \omega' = 3\epsilon n.$$

If the initial perturbation in the fiducial plane is zero, the trajectory of the gyroscope projection onto the plane coincides with the circular geodesic.

The equation of forced oscillations (20) along the orbital normal,

$$\ddot{\xi}_3 + \omega^2 \xi_3 = 3\epsilon \frac{S}{\lambda} g \cos(\omega_s + \beta), \tag{21}$$

proves to be quasiresonant due to proximity of the frequencies of the natural tidal oscillation $\sqrt{E_{33}} = \omega$ and of the compelling force ω_s. The difference of the frequencies $\Delta\omega$ in Eq. (4), which prevents the oscillation from becoming resonance, is equal to the geodetic precession Ω^G. The general solution of Eq. (21),

$$\begin{aligned} \xi_3 &= A\cos\zeta - C\cos\eta, \\ \zeta &= \omega_s \tau + \beta, \quad \eta = \omega\tau + \alpha, \end{aligned} \tag{22}$$

contains the amplitude A given by (5) and two integration constants, C and α. If $C = 0$, oscillation (22) describes the precession of the gyroscope orbit tilted by the angle

$$A/r = S/\lambda$$

relative to the fiducial plane, with the angular velocity of the geodetic precession given by (4) (Figure *a*). The evolution of the gyroscope orbital moment with arbitrary C is presented in Figure *b*. If $C = A$, pure beating occurs,

$$\xi_3 = 2A \sin\frac{\eta - \zeta}{2} \sin\frac{\eta + \zeta}{2}. \tag{23}$$

The mass center oscillates along the orbital normal with a variable amplitude modulated by geodetic precession (4). The initial condition

$$\xi_3(\tau = 0) = 0$$

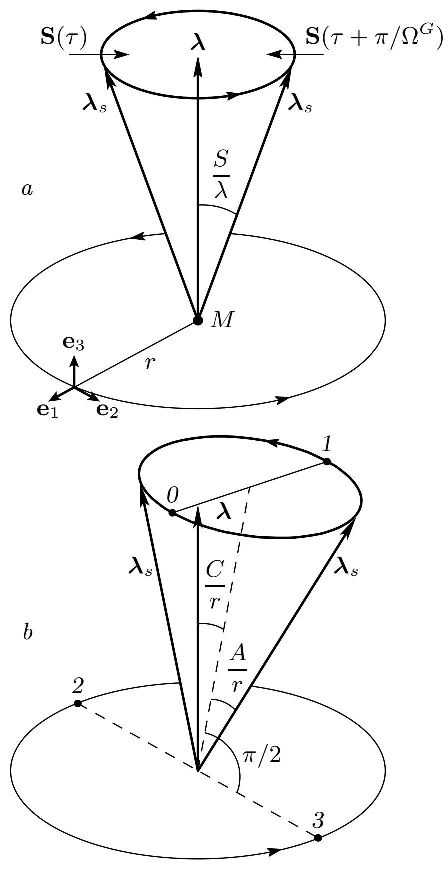
is provided by choosing the constant $\alpha = -\beta$,

$$\xi_3 = 2A \sin\frac{\Delta\omega}{2}\tau \, \sin\left(\frac{\omega + \omega_s}{2}\tau + \beta\right). \tag{24}$$

Within a time $\tau \ll (\Delta\omega)^{-1}$, the oscillation amplitude grows at the rate $A\,\Delta\omega$ given by (6) and (7). The condition

$$\dot{\xi}_3(\tau = 0) = 0$$

fixes the initial spin orientation $\sin\beta = 0$ along the radial direction (see (18)).



Orbit of the gyroscope. Orbital moments of the fiducial geodesic and the gyroscope are $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_s$ respectively. *a* — Precession of the gyroscope orbit at $C = 0$. *b* — Variable inclination of the gyroscope orbit, the constant C is arbitrary. The orbital moment $\boldsymbol{\lambda}_s$ points at the positions marked 0 and 1 when $\sin((\eta - \zeta)/2)$ equals 0 and 1 respectively. At points *2* and *3*, it turns out that $\cos\eta = 0$

The problem of measuring oscillation (24) is complicated by the circumstance that initial perturbations lead to the natural tidal oscillation with the orbital frequency $\omega^2 = E_{33}$ (see (21)). Therefore, gyroscopes with antiparallel spins must be manufactured to be coaxial. In order that the Newtonian harmonic oscillation due to instrumental error be smaller than the relativistic oscillation induced by the spin–orbit interaction, strong restrictions on the initial perturbations $\xi_3(0)$ and $\dot{\xi}_3(0)$ are required,

$$\xi_3(0) \ll \xi \sim A\,\Delta\omega\,\tau_f, \quad \dot{\xi}_3(0) \ll \omega\xi, \tag{25}$$

where τ_f is the formation time of the amplitude measured.

4. THE EFFECT OF FIELD OBLATENESS

The Newtonian oblateness of the source does not lead to forced gyroscope oscillations. The oblateness affects the natural tidal oscillation frequency $(\tilde{E}_{33})^{1/2}$, the orbital frequency $\tilde{\omega}$, and consequently, the angular velocity $\tilde{\omega}_s$ of the spin rotation relative to the orbital triad. The two frequencies, $(\tilde{E}_{33})^{1/2}$ and $\tilde{\omega}_s$, enter the equation of motion of the gyroscope mass center,

$$\ddot{\tilde{\xi}}_3 + \tilde{E}_{33}\tilde{\xi}_3 = 3\epsilon\frac{S}{\lambda}g\cos(\tilde{\omega}_s\tau + \beta). \qquad (26)$$

Considering only the quadrupole moment J_2 (which is given by $J_2 \approx 1\cdot 10^{-3}$ for the Earth), we obtain for an equatorial orbit that

$$\sqrt{\tilde{E}_{33}} = \omega\left(1 + \frac{9}{4}J_2\frac{R^2}{r^2}\right), \qquad (27)$$

$$\tilde{\omega} = \omega\left(1 + \frac{3}{4}J_2\frac{R^2}{r^2}\right), \qquad (28)$$

$$\tilde{\omega}_s = \omega_s\left(1 + \frac{3}{4}J_2\frac{R^2}{r^2}\right), \qquad (29)$$

where R is the equatorial radius of the source. The frequency $(\tilde{E}_{33})^{1/2}$ differs from the orbital frequency $\tilde{\omega}$ because of the Newtonian quadrupole precession Ω^J of the orbital plane,

$$\tilde{\omega} - \sqrt{\tilde{E}_{33}} = -\frac{3}{2}\omega J_2\frac{R^2}{r^2} = \Omega^J. \qquad (30)$$

The gyroscope axis does not undergo the additional Newtonian precession,

$$\tilde{\omega} - \tilde{\omega}_s = \Omega^G.$$

As a result of the difference in Eq. (30), small denominator (4) is changed as

$$\Delta\tilde{\omega} = \sqrt{\tilde{E}_{33}} - \tilde{\omega}_s = \Omega^G - \Omega^J \approx -\Omega^J \approx \approx \Delta\omega\frac{J_2}{\epsilon}\frac{R^2}{r^2}. \qquad (31)$$

The oscillation modulation period is then given by

$$\tilde{T} = \frac{2\pi}{\Delta\tilde{\omega}}$$

and amplitude (5) becomes

$$\tilde{A} = -A\,\frac{\Omega^G}{\Omega^J} = \frac{S}{\lambda}\,\frac{\epsilon}{J_2}\,\frac{r^2}{R^2}. \qquad (32)$$

The gyroscope orbital moment vector describes a conic surface with the apex angle $2\tilde{A}/r$ and the time period $\tilde{T}$. The quadrupole precession period $\tilde{T}$ of a near-Earth orbit is 2 months. For the pure beating

$$\tilde{\xi}_3 = 2\tilde{A}\,\sin\frac{\Delta\tilde{\omega}}{2}\tau\,\sin\left[\left(\tilde{\omega}_s + \frac{\Delta\tilde{\omega}}{2}\right)\tau + \beta\right] \qquad (33)$$

within the timescale $\tau \ll \tilde{T}$, the oscillation increases precisely as in the case of a spherically symmetric field (see Eq. (6)),

$$\tilde{A}\,\Delta\tilde{\omega} = A\,\Delta\omega = \frac{3}{2}\,\epsilon\,\frac{S}{\lambda}\,v. \qquad (34)$$

The maximum amplitude formed in time $\tilde{T}/2$ on a near-Earth orbit for a gyroscope with $S/\lambda \sim 10^{-9}$ (see Eq. (7)) is

$$\tilde{A} \sim 10^{-7}\ \text{cm}, \qquad (35)$$

which is several orders as good as the present-day limit of measuring small oscillations.

5. CONCLUSIONS

The general relativistic quasiresonant spin–orbit interaction leads to oscillation of the gyroscope mass center relative to the fiducial geodesic along the orbital normal. The beating amplitude does not include the speed of light and equals the ratio of the intrinsic moment of the gyroscope to its orbital moment. The modulation frequency equals the angular velocity of the geodetic precession. The oscillation represents the precession of the gyroscope orbital moment. Within an acceptable time, the oscillation amplitude reaches the values that are amenable to experimental analysis.

Taking the source oblateness into account decreases the beating amplitude and increases the modulation frequency by the factor that is equal to the ratio of the quadrupole precession velocity to the geodetic precession velocity. The period of the quadrupole precession turns out to be a quite sufficient time to form a measurable amplitude of the oscillation. The tidal acceleration, providing the quasiresonant character of the oscillation, leads to strong restrictions that must be imposed on the initial perturbations in order to distinguish the relativistic spin–orbit oscillation in the background of the Newtonian tidal oscillation.

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