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PHYSICAL PROPERTIES OF SCALAR AND SPINOR FIELD STATES WITH THE RINDLER–MILNE (HYPERBOLIC) SYMMETRY

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It is shown that right and left combinations of the positive- and negative-frequency hyperbolically symmetric solutions of the Klein–Fock–Gordon equation possess an everywhere timelike current density vector with a definite Lorentz-invariant sign of the charge density, and similar combinations of solutions to the Dirac equation possess the energy-momentum tensor with everywhere real eigenvalues and a definite Lorentz-invariant sign of the energy density. These right and left modes, just as their $\pm$-frequency components, are eigenfunctions of the Lorentz boost generator with the eigenvalue κ. The sign of the charge (energy) density coincides with the sign of κ for the right scalar (spinor) modes and is opposite to it for the left modes. It is then reasonable to assume that the particles (antiparticles) are precisely described by the right modes with $\kappa > 0$ ($\kappa < 0$) and by the left modes with $\kappa < 0$ ($\kappa > 0$).

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1. INTRODUCTION

Three complete sets of solutions of the Klein–Fock–Gordon (KFG) and Dirac equations are usually considered in relation to the Unruh effect [1]. One of these solution sets is the usual planewave set and the other two are the sets of field modes with a hyperbolic symmetry. The hyperbolically symmetric modes radically differ from the planewave modes by singularities occurring on the light cone. As a result, the corresponding charge and energy densities oscillate with increasing the frequency at Compton distances near the cone and become infinite on the cone. It is not surprising that the charge density of the scalar field and the energy density of the spinor field can have either sign near the singularity. This means that these modes contain both particles and antiparticles near the light cone. It is then difficult to distinguish the hyperbolically symmetric field state created by external sources on the light cone from the state created by the measuring device itself. Nevertheless, there exist right and left states with the hyperbolic symmetry for which the charge density of the scalar field and the energy density of the spinor field possess an everywhere definite Lorentz-invariant sign.

2. PLANE WAVES WITH DEFINITE MOMENTUM AND FREQUENCY

For scalar plane waves

$$\varphi_p^{(\pm)}(x) = \frac{1}{\sqrt{2E}} \exp\left[i(pz \mp Et)\right], \quad E = \sqrt{m^2+p^2}, \quad x^\alpha = (t,z), \tag{1}$$

the current densities $j_p^{(\pm)\alpha}(x) = (\pm 1, p/E)$ are timelike vectors. The signs of the charge densities coinside with the frequency signs. The energy-momentum tensor $t_{\alpha\beta}$ has the components

$$t_{00}^{(\pm)},\ t_{33}^{(\pm)},\ t_{03}^{(\pm)} = E,\ p^2/E,\ \mp p, \tag{2}$$

with $\operatorname{sign} t_{00}^{(\pm)} > 0$.

For spinor plane waves with definite momentum and frequency and with the double spin projection s,

$$\chi_{ps}^{(\pm)}(x) = \varphi_p^{(\pm)}(x)\sqrt{m}u_s^{(\pm)}(\theta), \quad \bar{u}_s^{(\omega)}(\theta)u_{s'}^{(\omega')}(\theta) = 2\omega\delta_{\omega\omega'}\delta_{ss'} \tag{3}$$

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(the bispinors $u_s^{(\pm)}(\theta)$ are given in (41) in the chiral representation in the transposed form), the current densities $j_{ps}^{(\pm)\alpha}(x) = (1,\ p/E)$ are timelike vectors with positive time components. The energy-momentum tensor $t_{\alpha\beta}$ corresponding to (3) has the components

$$t_{00}^{(\pm)},\ t_{33}^{(\pm)},\ t_{03}^{(\pm)} = \pm E,\ \pm p^2/E,\ -p, \tag{4}$$

where $\operatorname{sign} t_{00}^{(\pm)} \gtrless 0$.

A superposition of the scalar positive (negative)-frequency plane waves, unlike the partial waves themselves, does not possess a positive (negative) definite charge density in general. Thus, if

$$\phi(x) = \int dp\, c_p \varphi_p^{(+)}(x), \tag{5}$$

the charge density may not be everywhere positive because of oscillations of the integrand in the representation

$$j^0(x) = i\,\phi^*(x)\,\overset{\leftrightarrow}{\partial_t}\,\phi(x) = \iint \frac{dp\,dp'(E+E')}{\sqrt{2E2E'}} \times \\ \times \exp\{i[(p'-p)z - (E'-E)t]\}\, c_p^* c_{p'}. \tag{6}$$

However, the total charge of the packet is positive and time-independent,

$$Q = \int dz j^0(x) = 2\pi \int dp\, |c_p|^2. \tag{7}$$

Similarly, a superposition of the spinor positive (negative)-frequency plane waves does not possess an everywhere positive (negative) energy density in general. Thus, the positive-frequency wave packet

$$\chi_s(x) = \int dp\, c_p \chi_{ps}^{(+)}(x) \tag{8}$$

has the energy density

$$t_{00}(x) = \frac{1}{2} i \chi_s^+(x)\,\overset{\leftrightarrow}{\partial_t}\,\chi_s(x) = \\ = \frac{1}{2}\iint \frac{dp\,dp'(E+E')}{\sqrt{2E2E'}} \times \\ \times \exp\{i[(p'-p)z - (E'-E)t]\} \times \\ \times m u_s^{(+)+}(\theta) u_s^{(+)}(\theta')\, c_p^* c_{p'} \tag{9}$$

that may not be everywhere positive, but the total energy of the packet is positive and conserved,

$$\mathcal{E} = \int dz\, t_{00}(x) = 2\pi \int dp\, |c_p|^2 E(p). \tag{10}$$

The negative charge (energy) density for a positive-frequency scalar (spinor) wave packet can occur because the packet is nonstationary (cannot be represented as $\exp(-iEt)f(z),\ E>0$). Expressions (6) and (9) imply that the time-averaged values of the charge and energy densities are equal to zero at any point in space. This means that charge and energy come from infinity and go to infinity. In a finite region of space Δz, they can therefore reach perceptible values ΔQ and $\Delta\mathcal{E}$ only for a finite time interval Δt. In addition, each of the quantities

$$\Delta Q(t) = \int\limits_{\Delta z} dz\, j^0(x)$$

and

$$\Delta\mathcal{E}(t) = \int\limits_{\Delta z} dz\, t_{00}(x)$$

can also be negative. This indicates the appearance of the antiparticle in this space-time region.

3. POSITIVE- AND NEGATIVE-FREQUENCY SCALAR WAVES WITH THE HYPERBOLIC SYMMETRY

These scalar waves are defined by the integral representation [2]

$$\phi_\kappa^{(\pm)}(x) = \frac{1}{2}\int\limits_{-\infty}^{\infty} d\theta \exp[i(pz \mp Et) \mp i\kappa\theta], \\ p = m \operatorname{sh}\theta, \quad E = m\operatorname{ch}\theta, \tag{11}$$

where $\theta = \operatorname{Arth}(p/E)$ is the rapidity. In the right and left sectors of the Minkowski plane, these functions can be represented by the Macdonald function of a real argument,

$$\exp(\pm\pi\kappa/2 - i\kappa v)K_{i\kappa}(\zeta), \\ \exp(\mp\pi\kappa/2 - i\kappa v)K_{i\kappa}(\zeta), \\ \zeta = m\sqrt{z^2 - t^2}, \quad v = \operatorname{Arth}(t/z), \tag{12}$$

and in the future and past sectors by the Macdonald function of an imaginary argument,

$$\exp(-i\kappa w)K_{i\kappa}(\pm i\tau), \quad \exp(-i\kappa w)K_{i\kappa}(\mp i\tau), \\ \tau = m\sqrt{t^2 - z^2}, \quad w = \operatorname{Arth}(z/t). \tag{13}$$

Using the Rindler metric

$$ds^2 = dz'^2 - (az')^2 dt'^2$$

in the R and L sectors and the Milne metric

$$ds^2 = (at')^2 dz'^2 - dt'^2$$

in the F and P sectors, we can write

$$\zeta = \pm m z', \quad v = \pm a t',$$

$$\tau = \pm m t', \quad w = \pm a z',$$

where z' and t' are space and time coordinates in the Rindler or Milne spaces, see [3]. It is essential that $\phi_\kappa^{(\pm)}$ includes plane waves with unlimited energy.

The scalar waves have the following properties.

a) $\phi_\kappa^{(\pm)}(x)$ are analytical and finite functions in the lower/upper half-planes of the complex variables $x_+ = t + z$ and $x_- = t - z$.

b) The hyperbolic symmetry implies that $\phi_\kappa^{(\pm)}$ are eigenfunctions of the Lorentz boost operator: under the transformation

$$z,\, t \;\to\; z' = \frac{z - \beta t}{\sqrt{1-\beta^2}}, \quad t' = \frac{t - \beta z}{\sqrt{1-\beta^2}}, \tag{14}$$

the variables ζ and τ remain invariant, while the cyclic variables v and w go to $v' = v - \alpha$ and $w' = w - \alpha$, where $\alpha = \operatorname{Arth}\beta$ is the rapidity corresponding to the Lorentz transformation velocity β. Then

$$\phi_\kappa^{(\pm)}(z,t) \;\to\; \phi_\kappa^{(\pm)}(z',t') = e^{i\alpha\kappa}\phi_\kappa^{(\pm)}(z,t), \tag{15}$$

and therefore, $e^{i\alpha\kappa}$ is an eigenvalue of the Lorentz boost operator $e^{-\alpha\partial_v}$ or $e^{-\alpha\partial_w}$; κ is an eigenvalue of the Lorentz boost generator $i(t\partial_z + z\partial_t) = i\partial_v$ or $i\partial_w$ and is interpreted as the frequency for a Rindler observer or the momentum for a Milne observer.

c) $\phi_\kappa^{(+)}$ and $\phi_\kappa^{(-)}$ are related by complex conjugation accompanied by changing the sign of κ,

$$\phi_\kappa^{(+)*}(x) = \phi_{-\kappa}^{(-)}(x), \quad \phi_{-\kappa}^{(\pm)}(z,t) = \phi_\kappa^{(\pm)}(-z,t). \tag{16}$$

The complex conjugation is equivalent to time reflection. The last property is equivalent to space reflection.

d) As a striking property of $\phi_\kappa^{(\pm)}$, we note that although the current density vectors corresponding to the plane wave components of $\phi_\kappa^{(\pm)}$ are everywhere timelike, the current densities $j_\kappa^{(\pm)\alpha}$ corresponding to $\phi_\kappa^{(\pm)}$ themselves are not timelike vectors in the entire Minkowski space: there are space–time regions inside the light cone where the current densities are spacelike.

The current density j^α for the Minkowski observer is related to the current density J^α for Rindler or Milne observers (more exactly, for local Lorentz observers momentarily comoving to them) by the Lorentz transformation

$$j^0 = \frac{J^0 + \beta J^3}{\sqrt{1-\beta^2}}, \quad j^3 = \frac{J^3 + \beta J^0}{\sqrt{1-\beta^2}}. \tag{17}$$

For the Rindler observer in the R sector with $\beta = t/z$, we have

$$J_\kappa^{(\pm)0} = \frac{2m\kappa e^{\pm\pi\kappa}}{\zeta} K_{i\kappa}^2(\zeta), \quad J^{(\pm)3} = 0. \tag{18}$$

For the L sector, we must replace $e^{\pm\pi\kappa} \to -e^{\mp\pi\kappa}$. The current density vector is timelike.

For the Milne observer with $\beta = z/t$, we have

$$\begin{aligned} J_\kappa^{(\pm)0} &= \pm\frac{\pi}{\sqrt{t^2 - z^2}}, \\ J_\kappa^{(\pm)3} &= -\operatorname{sign}(t)\frac{2m\kappa}{\tau}|K_{i\kappa}(i\tau)|^2. \end{aligned} \tag{19}$$

The current density squared

$$(j_\kappa^{(\pm)})^2 = -\frac{\pi^2}{t^2 - z^2}\left(1 - \frac{4\kappa^2}{\pi^2}|K_{i\kappa}(i\tau)|^4\right) \tag{20}$$

can have either sign when $\tau = m\sqrt{t^2 - z^2} \ll 1$, but is negative for $\tau \gtrsim 1$.

Thus, inside the light cone at invariant distances less than the Compton length from the cone, there are spacetime regions where the current densities $j_\kappa^{(\pm)\alpha}$ are spacelike vectors and the charge density $j_\kappa^{(+)0}$, $\kappa > 0$, is negative, while $j_\kappa^{(-)0}$, $\kappa > 0$, is positive. Because the current densities are timelike vectors for the real particles, we can relate the spacelike current density $j_\kappa^{(+)}$ to antiparticles of the virtual pairs created in regions with a very high energy concentration. The total charge of the $\phi_\kappa^{(+)}$ state on any spacelike surface in Minkowski space is positive and is equal to the charge on this surface entirely situated in the P, $L + R$ or F sector. But the charge density j^0 for this state with $\kappa > 0$ is positive only in R sector, is negative in the L sector, and can have either sign in the P and F sectors.

Thus, unlike the sign of the total charge, the sign of the charge density is not well defined by the frequency sign of the $\phi_\kappa^{(\pm)}$ states. This situation occurs in external field problems due to a possible pair creation by the external field, or in problems of forming wave packets with a high energy density. The appearance of a negative charge density in the P, F, and L sectors for the positive-frequency state $\phi_\kappa^{(+)}$ is a consequence of the hyperbolic symmetry of the state. The hyperbolic symmetry divides Minkowski space into spacelike and timelike subspaces with the Rindler and Milne metrics. These metrics have singularities on the light cone (which is their common boundary) and can be considered as a limiting case of a global nonsingular smooth metric of the space with a nonzero external field near the light cone. The pair creation by this field is then possible and the appearance of a negative charge density in the positive-frequency state $\phi_\kappa^{(+)}$ after switching the field off can be understood.

The states $\phi_\kappa^{(+)}$ and $\phi_\kappa^{(-)}$ possess respectively the positive and negative total charge but do not possess an everywhere positive and negative charge density. This means that both the particle and the antiparticle can be detected in any of these states.

4. RIGHT AND LEFT SCALAR MODES

In each of the R and L sectors, $\phi_\kappa^{(+)}$ and $\phi_\kappa^{(-)}$ differ only by factors. According to Unruh [1], one can find remarkable right and left combinations

$$\phi_\kappa^R = \alpha_\kappa \phi_\kappa^{(+)} + \beta_\kappa \phi_\kappa^{(-)}, \quad \phi_\kappa^L = \beta_\kappa \phi_\kappa^{(+)} + \alpha_\kappa \phi_\kappa^{(-)}, \quad (21)$$

such that $\phi_\kappa^R = 0$ in the L sector and $\phi_\kappa^L = 0$ in the R sector. In these combinations,

$$\beta_\kappa = -\alpha_\kappa e^{-\pi\kappa}, \quad \alpha_\kappa = \frac{e^{\pi\kappa/2}}{\sqrt{2\,\mathrm{sh}\,\pi\kappa}}, \quad |\alpha_\kappa|^2 - |\beta_\kappa|^2 = \varepsilon_\kappa \equiv \mathrm{sgn}\,\kappa. \quad (22)$$

For $\kappa < 0$, we have $\sqrt{\mathrm{sh}\,\pi\kappa} = i\sqrt{\mathrm{sh}\,\pi|\kappa|}$. The set $\phi_\kappa^{R,L}$ possesses the same hyperbolic symmetry as the set $\phi_\kappa^{(\pm)}$, but the striking property of these functions is that the corresponding current densities $j_\kappa^{R\,\alpha}$ and $j_\kappa^{L\,\alpha}$ are timelike vectors in the entire spacetime region where they are nonzero. Lorentz transformation (17) again relates the current density j^α for the Minkowski observer to the current density J^α for the Rindler or Milne observers.

For the Rindler observer with $\beta = t/z$, we have

$$J_\kappa^{R0} = \frac{4m\kappa\,\mathrm{sh}\,\pi|\kappa|}{\zeta}|K_{i\kappa}(\zeta)|^2, \quad J_\kappa^{R3} = 0. \quad (23)$$

The current density vector is then timelike.

For the Milne observer with $\beta = z/t$, we have

$$J_\kappa^{R,L0} = \pm\frac{\mathrm{sgn}(\kappa t)\,\pi}{\sqrt{t^2 - z^2}}, \quad J_\kappa^{R,L3} = -\frac{\mathrm{sgn}(\kappa t)\,\pi^2\kappa}{\sqrt{t^2 - z^2}\,\mathrm{sh}\,\pi\kappa}|J_{i\kappa}(\tau)|^2. \quad (24)$$

The Lorentz invariant current density squared is nonpositive,

$$(j_\kappa^{R,L})^2 = -\frac{\pi^2}{t^2 - z^2} \times \left[1 - \left(\frac{\pi\kappa}{\mathrm{sh}\,\pi\kappa}\right)^2 |J_{i\kappa}(\tau)|^4\right] \le 0, \quad (25)$$

for all real κ and $\tau \geqslant 0$ [4]. The current density vector is timelike.

It is interesting to note that in the R sector, the current density squared $(j_\kappa^R)^2$ tends to infinity as $\zeta \to 0$, but in the P or F sectors, it is finite at $\tau = 0$:

$$(j_\kappa^R)^2|_{\tau\to 0} = -\frac{\pi^2 m^2}{1 + \kappa^2} + \ldots \quad (26)$$

The state ϕ_κ^R (ϕ_κ^L) describes a wave with the hyperbolic symmetry and the charge density that is only positive for $\kappa > 0$ ($\kappa < 0$) or only negative for $\kappa < 0$ ($\kappa > 0$). We can then say that the respective state describes the particle or the antiparticle. In other words, the state ϕ_κ^L describes the particle or the antiparticle with the sign of κ that is opposite to the sign used in describing the ϕ_κ^R state [4].

We note that complex conjugation (time reflection) of the functions $\phi_\kappa^{R,L}$ is equivalent to changing the sign of κ, while the space reflection is equivalent to changing the sign of κ and replacing $R \rightleftarrows L$:

$$\phi_\kappa^{R*}(x) = \phi_{-\kappa}^R(x), \quad \phi_\kappa^{L*}(x) = \phi_{-\kappa}^L(x); \quad \phi_\kappa^R(-z,t) = i\phi_{-\kappa}^L(z,t). \quad (27)$$

In the R sector, where κ is interpreted as energy by the Rindler observer and $\phi_\kappa^L = 0$, particles are described by the functions $\phi_\kappa^R, \kappa > 0$, and antiparticles by the complex conjugate functions, i.e., by $\phi_\kappa^R, \kappa < 0$. In the F or P sectors, where κ is interpreted as momentum by the Milne observer, particles with the momentum κ are described by the functions $\phi_\kappa^R, \kappa > 0$, and $\phi_\kappa^L, \kappa < 0$, while antiparticles with same momenta are described by the complex conjugate functions $\phi_{-\kappa}^R$ and $\phi_{-\kappa}^L$.

Completeness of the sets $\phi_\kappa^{(\pm)}$ and $\phi_\kappa^{R,L}$ is expressed by

$$\Delta^{(\pm)}(x - x') = \pm i \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi^2}\phi_\kappa^{(\pm)}(x)\phi_\kappa^{(\pm)*}(x') = = \frac{\pm i}{2\pi}K_0(m\sqrt{y^2}) \quad \text{if } y^2 > 0, \\ = \frac{1}{4}\left[\varepsilon(y^0)J_0(m\sqrt{|y^2|}) \mp iN_0(m\sqrt{|y^2|})\right] \quad \text{if } y^2 < 0,\ y = x - x', \quad (28)$$

$$\Delta(y) = \sum_\pm \Delta^{(\pm)}(y) = = i\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi^2}\varepsilon_\kappa\left[\phi_\kappa^R(x)\phi_\kappa^{R*}(x') - \phi_\kappa^L(x)\phi_\kappa^{L*}(x')\right] = = \frac{1}{2}\varepsilon(y^0)\,\theta(-y^2)\,J_0(m\sqrt{|y^2|}). \quad (29)$$

It is interesting to note that analytical properties of the functions ϕ_κ^R and ϕ_κ^L in each of the variables u and v are similar to the properties of the Pauli–Jordan function $\Delta(x)$ in x^2. Indeed, $\Delta(x)$ is also equal to the sum of the positive-frequency and negative-frequency functions $\Delta^{(\pm)}(x)$, which are boundary values of some function $F(x^2)$ that is analytical in the complex plane of x^2 cut along the real negative semi-axis $x^2 < 0$:

$$\Delta^{(\pm)}(x) = \pm F(x^2 \pm i\varepsilon\,\mathrm{sgn}\,x^0), \quad \varepsilon \to +0.$$

It follows that $\Delta(x)$ differs from zero only for $x^2 < 0$ and is equal to the jump of $F(x^2)$ on the cut.

The solution of the Cauchy problem and the normalization condition are given by

$$\begin{gathered}\phi(y) = \int\limits_S d\sigma^\alpha \Delta(y-x) \overset{\leftrightarrow}{\partial}_\alpha \phi(x), \\ i\int\limits_S d\sigma^\alpha \phi_\kappa^{(\omega)*} \overset{\leftrightarrow}{\partial}_\alpha \phi_{\kappa'}^{(\omega')} = 2\pi^2\omega\delta(\kappa-\kappa')\delta_{\omega\omega'},\end{gathered} \tag{30}$$

where S is a spacelike surface in Minkowski space or in any of the P, $L+R$, F sectors. For the functions ϕ_κ^a and $\phi_{\kappa'}^{a'}$, $a, a' \in R, L$, the right-hand side of the normalization condition is $2\pi^2\varepsilon_\kappa\varepsilon_a\delta(\kappa-\kappa')\delta_{aa'}$, where $\varepsilon_R = -\varepsilon_L = 1$. In accordance with the normalization condition, all the states have the same magnitude of the conserved total charge; the sign of the charge coincides with the frequency sign for the $\phi_\kappa^{(\pm)}$ states and with the sign of the product $\varepsilon_\kappa\varepsilon_a$ for the ϕ_κ^a states, $a \in R, L$.

An arbitrary solution of the KFG equation can be represented by the expansions

$$\phi(x) = \int\limits_{-\infty}^{\infty} \frac{dp}{\sqrt{2E}} \times \left[c_p \exp\left[i(pz-Et)\right] + d_p^* \exp\left[i(pz+Et)\right]\right] = \tag{31}$$

$$= \int\limits_{-\infty}^{\infty} \frac{d\kappa}{2\pi^2}[a_\kappa\phi_\kappa^{(+)}(x) + b_\kappa^*\phi_\kappa^{(-)}(x)] = \int\limits_{-\infty}^{\infty} \frac{d\kappa}{2\pi^2}[r_\kappa\phi_\kappa^R(x) + l_\kappa^*\phi_\kappa^L(x)]. \tag{32}$$

As an example, we consider

$$\phi(x) = \frac{1}{\sqrt{2E_1}}\exp\left[i(p_1z - E_1t)\right],$$

$$c_p = \delta(p-p_1), \quad d_p = 0.$$

It then follows that

$$a_\kappa = \frac{2\pi}{\sqrt{2E_1}}e^{i\kappa\theta_1}, \quad b_\kappa = 0, \quad \theta_1 = \mathrm{Arth}(p_1/E_1), \tag{33}$$

$$\begin{gathered}r_\kappa = \varepsilon_\kappa\frac{\pi\exp(\pi\kappa/2 + i\kappa\theta_1)}{(\sqrt{E_1}\,\mathrm{sh}\,\pi\kappa)^*}, \\ l_\kappa^* = \varepsilon_\kappa\frac{\pi\exp(-\pi\kappa/2 + i\kappa\theta_1)}{(\sqrt{E_1}\,\mathrm{sh}\,\pi\kappa)^*}.\end{gathered} \tag{34}$$

The spectra are given by (with $g_1 = 2\pi^2/E_1$)

$$|r_\kappa|^2 = \begin{cases}\dfrac{g_1e^{2\pi\kappa}}{e^{2\pi\kappa}-1}, & \kappa>0, \quad j^0>0; \\ \dfrac{g_1}{e^{2\pi|\kappa|}-1}, & \kappa<0, \quad j^0<0,\end{cases} \tag{35}$$

$$|l_\kappa|^2 = \begin{cases}\dfrac{g_1}{e^{2\pi\kappa}-1}, & \kappa>0, \quad j^0<0; \\ \dfrac{g_1e^{2\pi|\kappa|}}{e^{2\pi|\kappa|}-1}, & \kappa<0, \quad j^0>0.\end{cases} \tag{36}$$

There are no reasons to associate these spectra with thermodynamical ones, especially for a uniformly moving Milne observer, for whom κ is not the energy but the momentum, and all the more so for a Minkowski observer, for whom κ is an eigenvalue of the Lorentz boost generator and is odd under space and time reflections. We have

$$\begin{gathered}\phi_\kappa^{(+)} = \varepsilon_\kappa(\alpha_\kappa^*\phi_\kappa^R - \beta_\kappa^*\phi_\kappa^L), \\ \phi_\kappa^{(-)} = \varepsilon_\kappa(\alpha_\kappa^*\phi_\kappa^L - \beta_\kappa^*\phi_\kappa^R),\end{gathered} \tag{37}$$

$$|\alpha_\kappa|^2 = \frac{e^{2\pi\kappa}}{e^{2\pi\kappa}-1}, \quad |\beta_\kappa|^2 = \frac{1}{e^{2\pi\kappa}-1}, \quad \kappa>0, \tag{38}$$

$$|\alpha_\kappa|^2 = \frac{1}{e^{2\pi|\kappa|}-1}, \quad |\beta_\kappa|^2 = \frac{e^{2\pi|\kappa|}}{e^{2\pi|\kappa|}-1}, \quad \kappa<0, \tag{39}$$

where $|\beta_\kappa/\alpha_\kappa|^2$ is the probability to find any nonzero number of pairs and $|\alpha_\kappa|^{-2}$ is the probability to find no pairs in the state $\phi_\kappa^{(+)}$, $\kappa>0$, etc, cf. [5]. This interpretation follows from the none-one-particle consideration of the wave equation solutions and does not require transition to the secondary quantization, although is confirmed by it [6].

We note that the modes $\phi_\kappa^{R,L}(x)$ with $\kappa = 0$ are not defined by Eq. (21) because the coefficients α_κ and β_κ are infinite at $\kappa = 0$. The term with $\kappa = 0$ in expansions (32) of an arbitrary solution of the KFG equation is nevertheless finite and can be defined as the $\kappa \to 0$ limit of

$$r_\kappa\phi_\kappa^R + l_\kappa^*\phi_\kappa^L \equiv a_\kappa\phi_\kappa^{(+)} + b_\kappa^*\phi_\kappa^{(-)}|_{\kappa\to0} = a_0\phi_0^{(+)} + b_0^*\phi_0^{(-)}.$$

A similar remark applies to the term with $\kappa = 0$ in expansion (29).

5. DIRAC EQUATION SOLUTIONS WITH THE HYPERBOLIC SYMMETRY

Solutions $\psi_{\kappa s}^{(\pm)}$ of the Dirac equation in the Rindler or Milne space are related to solutions $\chi_{\kappa s}^{(\pm)}$ of this equation in Minkowski space by the Lorentz transformation

$$\psi_{\kappa s}^{(\pm)}(x) = e^{-\alpha\alpha_3/2}\chi_{\kappa s}^{(\pm)}(x), \quad \alpha = \mathrm{Arth}\beta, \quad \alpha_3 = \mathrm{diag}(\sigma_3, -\sigma_3), \tag{40}$$

where $\beta = t/z$ or z/t for the Rindler or the Milne space respectively. We use the chiral representation

$$\chi_{\kappa s}^{(\pm)}(x) = \frac{1}{2}\int_{-\infty}^{\infty} d\theta \exp\left[i(pz \mp Et) \mp i\kappa\theta\right] u_s^{(\pm)}(\theta),$$
$$p = m\,\mathrm{sh}\,\theta, \quad E = m\,\mathrm{ch}\,\theta,$$
$$\tilde{u}_1^{(\pm)}(\theta) = (e^{\pm\theta/2}, 0, \pm e^{\mp\theta/2}, 0),$$
$$\tilde{u}_{-1}^{(\pm)}(\theta) = (0, \pm e^{\mp\theta/2}, 0, e^{\pm\theta/2}), \tag{41}$$

where $s = \pm 1$ are the eigenvalues of the matrix

$$\Sigma_3 = \mathrm{diag}(\sigma_3, \sigma_3).$$

This representation defines the bispinor $\chi_{\kappa s}^{(+)}(x)$ ($\chi_{\kappa s}^{(-)}(x)$) as an analytical function in the lower (upper) half-plane of the respective complex variable $x_+ = t + z$ and $x_- = t - z$.

Bispinor components of $\psi_{\kappa s}$ and $\chi_{\kappa s}$ can be expressed through the Macdonald functions with the indices $i\kappa \pm 1/2$. For example, in the R and F sectors, $\psi_{\kappa 1}^{(\pm)}$ can be represented by the respective expression

$$\exp(\pm\pi\kappa/2 \pm i\pi/4 - i\kappa v)\begin{pmatrix} K_{i\kappa-1/2}(\zeta) \\ 0 \\ -iK_{i\kappa+1/2}(\zeta) \\ 0 \end{pmatrix} \quad \text{and}$$
$$\exp(-i\kappa w)\begin{pmatrix} K_{i\kappa-1/2}(\pm i\tau) \\ 0 \\ \pm K_{i\kappa+1/2}(\pm i\tau) \\ 0 \end{pmatrix}. \tag{42}$$

In other sectors, these functions can be obtained using the symmetry relations

$$\psi_{\kappa s}^{(\pm)}(t,z) = \alpha_3\psi_{\kappa s}^{(\mp)}(-t,-z) = \pm\beta\psi_{-\kappa s}^{(\pm)}(t,-z) = \pm\beta\psi_{\kappa s}^{(\pm)*}(-t,z), \tag{43}$$

where

$$\alpha_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{44}$$

The functions $\psi_{\kappa -1}^{(\pm)}$ with the opposite spin direction can be obtained from (42) by transposing the first row elements with the fourth row and the third row elements with the second row.

The orthogonality and normalization condition for $\psi_{\kappa s}^{(\pm)}$ is

$$\int_S d\sigma_\alpha \bar{\psi}_{\kappa s}^{(\omega)}(x)\gamma^\alpha(x)\psi_{\kappa' s'}^{(\omega')}(x) = \frac{2\pi^2}{m}\delta_{\omega\omega'}\delta_{ss'}\delta(\kappa - \kappa'). \tag{45}$$

This involves an oriented surface element $d\sigma_\alpha = n_\alpha d\sigma$, where $d\sigma$ is the invariant surface measure and n_α is the timelike normal to the surface. Because $\psi_{\kappa s}^{(\pm)}$ are solutions of the covariant Dirac equation with the coordinate-dependent metric $g_{\alpha\beta}$ and the matrices γ^α (see, e.g., § 3.8 in [3]), the normalization condition for these functions also contains $\gamma^\alpha(x)$ and it is convenient to choose the spacelike integration surface S entirely in one of the P, $L + R$ or F subspaces with either the Milne or the Rindler metric. For a constant-t' surface S, the surface element reduces to

$$d\sigma_0 = dz'\sqrt{\gamma}\,n_0, \quad n_0 = \sqrt{-g_{00}}$$

and $\gamma = |g_{33}|$ is the determinant of the space metric.

Because the Rindler and Milne spaces and the corresponding metrics only represent nonstandard coordinate forms of the flat space-time, the solutions $\psi_{\kappa s}^{(\pm)}$ must be related to the solutions $\chi_{\kappa s}^{(\pm)}$ of the usual Dirac equation in Minkowski space by a Lorentz transformation. These solutions satisfy the same symmetry relations (43) and orthogonality and normalization condition (45) with the standard γ matrices. For a constant-t surface S, the surface element becomes $d\sigma_0 = dz$ and the right-hand side of (45) immediately follows when one uses integral representation (41) for $\chi_{\kappa s}^{(\pm)}$ and performs the integration over z first.

In representation (42), the functions $\chi_{\kappa s}^{(\pm)}$ differ from $\psi_{\kappa s}^{(\pm)}$ by the factors $e^{v/2}$ and $e^{-v/2}$ of the first and the third bispinor elements in the R sector and by $e^{w/2}$ and $e^{-w/2}$ in the F sector.

Under Lorentz transformation (14), the functions $\chi_{\kappa s}^{(\pm)}$ go to

$$\chi_{\kappa s}^{(\pm)}(x') = \exp(i\alpha\kappa - \alpha\alpha_3/2)\chi_{\kappa s}^{(\pm)}(x), \quad \alpha = \mathrm{Arth}\,\beta, \quad \alpha_3 = \mathrm{diag}(\sigma_3, -\sigma_3). \tag{46}$$

The eigenvalues are again independent of the frequency sign. The current densities j^α and J^α for the Minkowski and Rindler or Milne observers are again related by (17).

For the Rindler observer with $\beta = t/z$, we have in the R sector:

$$J_\kappa^{(\pm)0} = 2e^{\pm\pi\kappa}|K_{i\kappa-1/2}(\zeta)|^2, \quad J_\kappa^{(\pm)3} = 0. \tag{47}$$

For the L sector, we must replace $e^{\pm\pi\kappa} \to e^{\mp\pi\kappa}$.

For the Milne observer with $\beta = z/t$, we have

$$\begin{aligned} J_\kappa^{(\pm)0} &= |K_{i\kappa-1/2}(i\tau)|^2 + |K_{i\kappa+1/2}(i\tau)|^2, \\ J_\kappa^{(\pm)3} &= \pm|K_{i\kappa-1/2}(i\tau)|^2 \mp |K_{i\kappa+1/2}(i\tau)|^2. \end{aligned} \tag{48}$$

The current density is a timelike vector and its time component is positive (a well known fact for the spinor field). But the striking feature of $\chi_{\kappa s}^{(\pm)}$ is that the eigenvalues of the corresponding energy-momentum tensor $t_{\alpha\beta}^{(\pm)}$ are not everywhere real. There are some places inside the light cone where these eigenvalues are complex conjugate.

The energy-momentum tensors $t_{\alpha\beta}$ and $T_{\alpha\beta}$ for the Minkowski and Rindler or Milne observers are related by the Lorentz transformation

$$\begin{aligned} t_{00} &= \gamma^2(T_{00} - 2\beta T_{03} + \beta^2 T_{33}), \\ t_{33} &= \gamma^2(T_{33} - 2\beta T_{03} + \beta^2 T_{00}), \\ t_{03} &= \gamma^2[T_{03}(1+\beta^2) - \beta T_{00} - \beta T_{33}], \\ \gamma &= (1-\beta^2)^{-1/2}. \end{aligned} \tag{49}$$

For the Rindler observer with $\beta = t/z$ in the R sector, we have

$$T_{00}^{(\pm)},\ T_{33}^{(\pm)},\ T_{03}^{(\pm)} = \frac{2m\kappa e^{\pm\pi\kappa}}{\zeta} \times \\ \times \left(|K_{i\kappa-1/2}(\zeta)|^2,\ \int_\zeta^\infty \frac{d\zeta}{\zeta}|K_{i\kappa-1/2}(\zeta)|^2,\ 0 \right). \tag{50}$$

For the L sector, we replace $e^{\pm\pi\kappa} \to -e^{\mp\pi\kappa}$.

For the Milne observer with $\beta = z/t$, we have

$$\begin{aligned} T_{00}^{(\pm)} &= \pm\frac{m\kappa}{\tau}\left(\int_\tau^\infty \frac{d\tau}{\tau}A(\tau) + \frac{\pi}{\kappa}\right), \\ T_{33}^{(\pm)} &= \pm\frac{m\kappa}{\tau}A(\tau), \quad T_{03}^{(\pm)} = \frac{m\pi\kappa}{\tau^2}, \\ A(\tau) &= |K_{i\kappa+1/2}(i\tau)|^2 - |K_{i\kappa-1/2}(i\tau)|^2. \end{aligned} \tag{51}$$

The eigenvalues (invariants) of the energy-momentum tensor,

$$\lambda_{1,2} = \frac{1}{2}(T_{33} - T_{00}) \pm \sqrt{\frac{1}{4}(T_{00}+T_{33})^2 - T_{03}^2}, \tag{52}$$

are real and have opposite signs in the Rindler space, while in the Milne space, they are complex conjugate for $\tau \ll 1$, when the momentum density (energy flux) is greater than half the sum of the energy density and the pressure:

$$\lambda_{1,2}(\tau) \approx \frac{R(\tau)}{\tau} \pm i\frac{\pi m\kappa}{\tau^2 \operatorname{ch}\pi\kappa} + \ldots, \quad \tau \ll 1. \tag{53}$$

As $\tau \to 0$, $R(\tau)$ oscillates with a finite amplitude and an increasing frequency.

6. RIGHT AND LEFT SPINOR MODES

In the spinor case, the right and left superpositions of the positive- and negative-frequency modes are defined as in the scalar case, but the Dirac scalar product leads to different Bogoliubov coefficients,

$$\begin{aligned} \chi_{\kappa s}^R &= \alpha_\kappa \chi_{\kappa s}^{(+)} + \beta_\kappa \chi_{\kappa s}^{(-)}, \\ \chi_{\kappa s}^L &= \beta_\kappa \chi_{\kappa s}^{(+)} + \alpha_\kappa \chi_{\kappa s}^{(-)}, \\ \beta_\kappa &= i\alpha_\kappa e^{-\pi\kappa}, \quad \alpha_\kappa = \frac{e^{\pi\kappa/2}}{\sqrt{2\operatorname{ch}\pi\kappa}}, \\ &|\alpha_\kappa|^2 + |\beta_\kappa|^2 = 1. \end{aligned} \tag{54}$$

Evidently, the right and left modes satisfy the orthogonality and normalization condition

$$\int_S d\sigma_\alpha \bar\chi_{\kappa s}^a(x)\gamma^\alpha \chi_{\kappa' s'}^{a'}(x) = \frac{2\pi^2}{m}\delta_{aa'}\delta_{ss'}\delta(\kappa-\kappa'), \tag{55}$$

where $a, a' \in L, R$ and S is a spacelike surface as in (30) or (45).

The modes $\chi_{\kappa s}^{(\pm)}$ and $\chi_{\kappa s}^{L,R}$ form two complete sets of Dirac equation solutions and any other solution $\chi(x)$ can be decomposed into the corresponding integrals

$$\chi(x) = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi^2}\left[a_{\kappa s}\chi_{\kappa s}^{(+)}(x) + b_{\kappa s}^*\chi_{\kappa s}^{(-)}(x)\right] = \\ = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi^2}\left[r_{\kappa s}\chi_{\kappa s}^R(x) + l_{\kappa s}^*\chi_{\kappa s}^L(x)\right], \tag{56}$$

where summation over s is assumed.

For example, for the positive-frequency plane wave solution with $s = 1$,

$$\begin{aligned} \chi_{p_1 1}^{(+)}(x) &= \frac{1}{\sqrt{2E_1}}\exp[i(p_1 z - E_1 t)]\, u_1^{(+)}(\theta_1), \\ \theta_1 &= \operatorname{Arth}\frac{p_1}{E_1}, \end{aligned} \tag{57}$$

we have

$$a_{\kappa 1} = \frac{2\pi}{\sqrt{2E_1}}e^{i\kappa\theta_1}, \quad b_{\kappa 1}^* = 0, \tag{58}$$

$$r_{\kappa 1} = \frac{\pi \exp[\pi\kappa/2 + i\kappa\theta_1]}{\sqrt{E_1 \operatorname{ch} \pi\kappa}},$$
$$l^*_{\kappa 1} = \frac{-i\pi \exp[-\pi\kappa/2 + i\kappa\theta_1]}{\sqrt{E_1 \operatorname{ch} \pi\kappa}}. \quad (59)$$

For the spectra of the right and left modes, we then obtain (with $g_1 = 2\pi^2/E_1$)

$$|r_{\kappa 1}|^2 = \frac{g_1 e^{2\pi\kappa}}{e^{2\pi\kappa}+1}, \quad |l_{\kappa 1}|^2 = \frac{g_1}{e^{2\pi\kappa}+1}. \quad (60)$$

For the negative-frequency plane wave solution, the coefficients in expansions (56) are

$$a_{\kappa 1} = 0, \quad b^*_{\kappa 1} = \frac{2\pi}{\sqrt{2E_1}} e^{-i\kappa\theta_1}, \quad (61)$$

$$r_{\kappa 1} = \frac{-i\pi \exp[-\pi\kappa/2 - i\kappa\theta_1]}{\sqrt{E_1 \operatorname{ch} \pi\kappa}},$$
$$l^*_{\kappa 1} = \frac{\pi \exp[\pi\kappa/2 - i\kappa\theta_1]}{\sqrt{E_1 \operatorname{ch} \pi\kappa}}. \quad (62)$$

The spectra for the left and right modes then coincide with the respective expressions in (60).

Although these spectra resemble the thermal distribution of the Fermi-particle gas, this similarity seems to be artificial for the same reasons as in the scalar case. Moreover, decompositions (56) of the plane wave in the hyperbolic modes $\chi^{(\pm)}_{\kappa s}$ or $\chi^{R,L}_{\kappa s}$ and the inverse expansions of these modes in plane waves in Eqs. (41) and (54) confirm the completeness of these three sets and the absence of the loss of information or purity of states. We see that the hyperbolic symmetry and a definite frequency sign preserve good analytical properties of the modes but lead to an indefinite sign of their charge density or energy density.

The «thermal» spectra appear when one preserves the hyperbolic symmetry of modes and requires the definiteness of the charge density or energy density signs in the entire Minkowski space. This can only be achieved at the expense of loosing good analytical properties of the modes and essentially consists in the transition from the boundary value of an analytical function on the cut to its jump on this cut. We have

$$\chi^{(+)}_{\kappa s} = \alpha^*_\kappa \chi^R_{\kappa s} + \beta^*_\kappa \chi^L_{\kappa s}, \quad \chi^{(-)}_{\kappa s} = \beta^*_\kappa \chi^R_{\kappa s} + \alpha^*_\kappa \chi^L_{\kappa s}, \quad (63)$$

$$|\alpha_\kappa|^2 = \frac{e^{2\pi\kappa}}{e^{2\pi\kappa}+1}, \quad |\beta_\kappa|^2 = \frac{1}{e^{2\pi\kappa}+1}, \quad (64)$$

where $|\alpha_\kappa|^2$ and $|\beta_\kappa|^2$ are the respective probabilities to find no pairs (one pair) and one pair (no pairs) in the state $\chi^{(+)}_{\kappa s}$, $\kappa > 0$ ($\kappa < 0$). This interpretation follows from the none-one-particle analysis of wave equation solutions and does not require the transition to the secondary quantization, although is confirmed by it [5, 6].

For the Rindler observer with $\beta = t/z$, we have

$$T^R_{00},\ T^R_{33},\ T^R_{03} = \frac{4m\kappa \operatorname{ch} \pi\kappa}{\zeta} \times$$
$$\times \left(|K_{i\kappa-1/2}(\zeta)|^2,\ \int\limits_\zeta^\infty \frac{d\zeta}{\zeta} |K_{i\kappa-1/2}(\zeta)|^2,\ 0 \right) \quad (65)$$

and for the Milne observer with $\beta = z/t$,

$$T^R_{00} = \frac{\pi m\kappa}{\tau^2} \left(1 + \frac{\pi\tau}{\operatorname{ch}\pi\kappa} \int\limits_0^\tau \frac{d\tau}{\tau} |J_{i\kappa+1/2}(\tau)|^2 \right),$$
$$T^R_{33} = \frac{\pi m\kappa}{\tau^2} \left(1 - \frac{\pi\tau}{\operatorname{ch}\pi\kappa} |J_{i\kappa+1/2}(\tau)|^2 \right),$$
$$T^R_{03} = \frac{\pi m\kappa}{\tau^2}. \quad (66)$$

The energy density is greater than the pressure. As $\tau \to 0$, we have

$$T^R_{00} \approx T^R_{33} \approx T^R_{03} = \pi m\kappa/\tau^2$$

similarly to the energy-momentum tensor of electromagnetic waves.

It is interesting to note that in the R sector, the eigenvalues $\lambda^R_{1,2}$ tend to infinity as $\zeta \to 0$, while in the P or F sectors, they are finite at $\tau = 0$,

$$\lambda^R_{1,2}|_{\tau\to 0} = -\frac{2\pi m\kappa}{1+4\kappa^2} \pm \frac{2\pi m\kappa}{1+4\kappa^2} \sqrt{\frac{1+4\kappa^2}{9+4\kappa^2}} + \ldots \quad (67)$$

The sign of t_{00} is relativistically invariant in only two cases:

1) the eigenvalues λ_1 and λ_2 are real and have opposite signs,

$$\lambda_1\lambda_2 = T^2_{03} - T_{00}T_{33} < 0, \quad (68)$$

2) the eigenvalues are real, have the same sign, and the energy density is greater than the pressure in magnitude:

$$(\lambda_1 - \lambda_2)^2 = (T_{00} + T_{33})^2 - 4T^2_{03} > 0,$$
$$\lambda_1\lambda_2 > 0, \quad \operatorname{sign}(T^2_{00} - T^2_{33}) > 0. \quad (69)$$

We note that the sign of $(t^2_{00} - t^2_{33})$ is relativistically invariant only if λ_1 and λ_2 are real, i.e., if $(\lambda_1 - \lambda_2)^2 > 0$. Then, if $\lambda_{1,2}$ are complex or if they are real and have the same sign, but $\operatorname{sgn}(T^2_{00} - T^2_{33}) < 0$, the sign of t_{00} can be changed by a Lorentz transformation.

The tensor $t^R_{\alpha\beta}$ possesses the first property in the R sector and either the first or the second property depending on the value of τ in the F and P sectors. In the F and P sectors, the eigenvalues λ_1^R and λ_2^R are real because of the inequality

$$\int_0^\tau \frac{d\tau}{\tau}|J_{i\kappa+1/2}(\tau)|^2 - |J_{i\kappa+1/2}(\tau)|^2 > 0. \qquad (70)$$

Inequalities (70) and (25) that are essential in this paper were not found in the mathematical literature.

7. CONCLUSION

Hyperbolic symmetry of scalar and spinor field states requires plane waves with unlimited frequencies to participate in the corresponding superpositions. For scalar field, field states with the quantum number κ that are formed as superpositions and are analytic in the coordinates $x_\pm = t \pm z$ do not possess an everywhere timelike current density, while for the spinor field, they do not possess the energy-momentum tensor with everywhere real eigenvalues. This means that these states describe both particles and antiparticles. Nevertheless, it is possible to construct hyperbolically symmetric right and left states that are not analytic in $x_\pm$ but possess an everywhere timelike current density and the energy-momentum tensor with everywhere real eigenvalues. Precisely these states describe the particle or the antiparticle.

This implies that the charge densities j_κ^{R0} and J_κ^{R0} for the scalar particle (antiparticle) states ϕ_κ^R and the energy densities $t_{\kappa 00}^R$ and $T_{\kappa 00}^R$ for the spinor particle (antiparticle) states $\chi_{\kappa s}^R$ are everywhere positive (negative) for $\kappa > 0$ ($\kappa < 0$) and are equal to zero in the L sector. This assertion remains valid after replacing $R \rightleftarrows L$ and changing the sign of κ.

It is known [7] that if a wave packet is formed from plane waves and is localized in a region of the order of or less than the Compton wave length, it must contain both positive and negative frequencies. The superpositions $\phi_\kappa^{(+)}$ and $\chi_{\kappa s}^{(+)}$ do not contradict this assertion because each of them is localized in a region of the order of the Compton length only for $|t| \lesssim m^{-1}$, while for $|t| \gg m^{-1}$, each superposition consists of two waves that propagate along the light cone boundaries $z = \pm t$, exponentially decaying outside the cone for $\zeta = m\sqrt{z^2 - t^2} \gg 1$ and oscillating and falling off only as τ^{-1} inside the cone for $\tau = m\sqrt{t^2 - z^2} \gg 1$. Therefore, these two waves remain coherently connected in a single wave packet with the width $\approx 2|t|$.

In the well-known review [8], Pauli made the following remark about energy density in the Dirac electron field theory: «The concept of the energy density seems to be more problematic in this theory than that of the volume integrated total energy. The energy density is no longer positive definite for the theory of holes, in contradistinction to the case for the theories discussed in §§ 1 and 2. This is also shown in the c number theory; even if limitation is made to wave packets in which the partial waves all have the same sign of the frequency in the phase exp $i(\mathbf{k}\cdot\mathbf{x} - k_0 x_0)$ the energy density (as distinguished from the total energy) cannot be made positive definite.» I do not know whether Pauli had some example of such a wave packet. In any case, each of the modes $\chi_{\kappa s}^{(\pm)}$ can serve as a specific illustration to his remark. The energy density for each of these modes can accept both signs near the light cone owing to singularities on the cone related to the hyperbolic symmetry of the modes. On the other hand, each of the modes $\chi_{\kappa s}^{R,L}$ is an example of such a superposition of positive- and negative-frequency spinor plane waves with a sign-definite energy density in the entire Minkowski space.

It is interesting that the scalar eigenfunctions of the Lorentz boost operator appear in the analysis of the photon wave function in localized near the photon propagation plane 3 + 1-space [9]. However, a scalar product different from (30) is used in this analysis.

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APPENDIX

The integral $J_{\kappa\kappa'}^R$ defined in [4] by Eq. (14), being the integral of a total differential, does not actually depend on the form of the spacelike surface over which it extends, but depends only on the parameters mt and ζ fixing the coordinates of the left boundary of this surface. Namely, the z coordinate of the left boundary is equal to $\sqrt{t^2 + \zeta^2/m^2}$, while the right boundary is at infinity. When the left boundary tends to zero at a fixed ratio mt/ζ, we obtain the result (20) from [4] without any uncertainties related to the factor exp $[i(\kappa - \kappa')\mathrm{Arsh}(mt/\zeta)]$, which eventually turns into 1 at fixed mt/ζ and $\kappa = \kappa'$. Thus, the normalization integral (20) in [4] is correct for any spacelike surface lying in the R sector with the left boundary at zero, rather than at $z = |t|$ as was assumed in [4].

Similarly, expression (28) for the normalization integral $J_{\kappa\kappa'}^L$ in [4] is correct for any spacelike surface lying in the L sector with the right boundary at zero, rather than at $z = -|t|$ as was assumed in [4].

The integral $J_{\kappa\kappa'}^F$ defined by Eqs. (22) and (23) in [4] is justified for any spacelike surface lying inside the F sector with the boundaries at the points defined by fixed values of mt and $\tau = m\sqrt{t^2 - z^2}$. The z coordinates of the left and right boundaries of this surface are then given by $z_{1,2} = \mp\sqrt{t^2 - \tau^2/m^2}$. As t tends to infinity at fixed τ, we obtain the result (25) from [4] without any ambiguity related to the factor inside the parentheses in Eq. (23) in [4], which turns into π at fixed τ and $\kappa = \kappa'$. Thus, normalization integral (25) in [4] is correct for any spacelike integration surface lying in the F sector and having the boundaries at $z_{1,2} = \mp\infty$ but not at $z_{1,2} = \mp|t|$, as was understood in [4]. A similar comment applies to the integral $J^P_{\kappa\kappa'}$.

On any spacelike surface entirely lying in the P, $L + R$ or F sectors with the left and right boundaries at infinities, each of the states $\phi^{(\pm)}_\kappa$ has the same conserved total charge

$$Q^{(\pm)}_\kappa = Q^{(\pm)}_{\kappa P} = Q^{(\pm)}_{\kappa L} + Q^{(\pm)}_{\kappa R} = Q^{(\pm)}_{\kappa F} \gtrless 0. \qquad (71)$$

Therefore, the factor 1/2 in the right-hand sides of Eqs. (34) and (35) in [4] must be replaced by 1.

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