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CONDUCTIVITY IN A TWO-DIMENSIONAL DISORDERED MODEL WITH ANISOTROPIC LONG-RANGE HOPPING

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We consider a two-dimensional system of particles localized on randomly distributed sites of a square lattice with anisotropic transition matrix elements between localized sites. The diagram and replica methods are used. The conductivity of a system in different limits of local sites and particles densities is calculated. The model is relevant to the problem of strong nonmagnetic impurities in superconductors with $d_{x^2-y^2}$ symmetry of the order parameter.

1. INTRODUCTION

We examine a system of randomly distributed impurities at a sites of a two-dimensional square lattice. An impurity potential generates a localized state with a strongly anisotropic (cross-shaped) wave function. The conductivity is produced by due to hopping of particles between local states on the same vertical or horizontal lines. This picture can be realized in two-dimensional $d_{x^2-y^2}$-wave superconductors, where local bound quasiparticle states can arise in the presence of unitary impurities [1]. The wave functions of the local states are strongly anisotropic, with exponential decay in all directions except $\varphi_n = (2n+1)\pi/4$, where the wave function is proportional to r^{-1}.

A similar anisotropy has a wave function of bound states in the vortex core in d-wave superconductors [2]. The wave function in the vicinity of gap nodes at large distances has the form

$$|\Psi|^2 \propto |\varphi - \varphi_n| \exp(-2|\varphi - \varphi_n| r/\xi),$$

with a maximum value $|\Psi|^2 \propto \xi/2r$ in the directions $\varphi - \varphi_n \simeq \xi/2r \to 0$.

We will consider the following tight-binding Hamiltonian:

$$H = \sum_{i,j} t(\mathbf{r}_j - \mathbf{r}_i)\psi^+(\mathbf{r}_i)\psi(\mathbf{r}_j)\rho(\mathbf{r}_i)\rho(\mathbf{r}_j), \tag{1}$$

where $\psi^+(\mathbf{r}_i), \psi^+(\mathbf{r}_j)$ are creation and annihilation operators, $\rho(\mathbf{r}_i)$ is the density of impurities, equal to 1 at the impurity sites, and to 0 otherwise. The transition matrix element has a cross-shaped configuration

$$t(\mathbf{r}) = (\delta_{x,0} + \delta_{y,0}) f(r), \tag{2}$$

with

$$f(r) = J \left(\frac{a}{r}\right)^{\gamma} \exp(-\kappa r),$$

and a is the lattice constant.

The plan of this article is as follows. In Sec. 2 we consider the case of low impurity density. In Sec. 3 we calculate the conductivity in the case of high impurity density. Results are discussed in the Conclusion.

2. LOW DENSITY

We consider now the limit of low impurity concentration ($c \ll 1$). In an external electromagnetic field we substitute in (1)

$$t(\mathbf{r}_i - \mathbf{r}_j) \to t(\mathbf{r}_i - \mathbf{r}_j) \exp\left[ie \int\limits_{\mathbf{r}_i}^{\mathbf{r}_j} \mathbf{A}(\mathbf{r}, t) d\mathbf{r}\right].$$

The electric current is defined as usual with the Hamiltonian (1) by varying over a gauge-invariant vector potential $\mathbf{A}$

$$j_\alpha(t) = -ie \sum_{i,j} (\mathbf{r}_i - \mathbf{r}_j)_\alpha t_{ij} \Psi_i^+(t) \Psi_j(t) \rho_i \rho_j \exp\left(ie\mathbf{A}(t)(\mathbf{r}_i - \mathbf{r}_j)\right), \qquad (3)$$

where $t_{ij} = t(\mathbf{r}_i - \mathbf{r}_j)$. Since we will calculate $j(\omega)$, we assume that the potential $\mathbf{A}$ depends only on time t. Using the equation for the Green function $G(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2)$

$$\frac{\partial}{\partial t_1} G(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) = -i \left\langle T \frac{\partial \Psi(1)}{\partial t_1} \Psi^+(2) \right\rangle - i\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(t_1 - t_2), \qquad (4)$$

we obtain after Fourier transformation in the linear $\mathbf{A}$ approximation

$$j_\alpha(\omega) = \frac{e^2}{c} \sum_{i,j} t_{i,j} (\mathbf{r}_i - \mathbf{r}_j)_\alpha (\mathbf{r}_i - \mathbf{r}_j)_\beta A_\beta(\omega) \int \frac{d\Omega}{2\pi} G(\Omega, \mathbf{r}_j, \mathbf{r}_i) e^{i\Omega\alpha} -$$
$$-\frac{e^2}{c} \sum_{i,j,k,l} t_{i,j} t_{k,l} (\mathbf{r}_i - \mathbf{r}_j)_\alpha (\mathbf{r}_k - \mathbf{r}_l)_\beta A_\beta(\omega) \int \frac{d\Omega}{2\pi} G(\omega + \Omega, \mathbf{r}_j, \mathbf{r}_k) G(\Omega, \mathbf{r}_l, \mathbf{r}_i) e^{i\Omega\alpha}. \qquad (5)$$

The summation in Eq.(5) is taken over impurity sites, and $\alpha \to +0$. In order to evaluate (5) in the lowest order with respect to the concentration, we examine the case of two randomly located sites. The Green function is found easily:

$$G(\omega, \mathbf{r}_i, \mathbf{r}_i) = \frac{\omega + \mu}{(\omega + \mu)^2 - t_{i,j}^2},$$
$$G(\omega, \mathbf{r}_i, \mathbf{r}_j) = \frac{t_{1,2}}{(\omega + \mu)^2 - t_{i,j}^2}, \qquad (6)$$

where μ is the chemical potential. Substituting (6) into (5), we obtain in the case when the sites are located on the same horizontal chain

$$j_x(\omega) = \frac{e^2}{2c} (x_1 - x_2)^2 t_{1,2} \frac{\omega^2}{(\omega + i\alpha)^2 - 4t_{1,2}^2} A_x(\omega) = Q(\omega) A_x(\omega). \qquad (7)$$

A similar equation can be derived for the case of nonzero temperature. The result differs only by the Fermi filling factor. After averaging over impurity sites we obtain for the conductivity $\sigma(\omega) = iQ(\omega)/\omega$

$$\begin{aligned}\sigma(\omega) &= \frac{\pi e^2}{4} c^2 L \int x^2 t(x)[n_F(\omega-\mu) - n_F(-\omega-\mu)]\delta(\omega - 2t(x))dx = \\ &= \frac{\pi e^2}{8} c^2 L \frac{x_0^2 t(x_0)[n_F(\omega-\mu) - n_F(-\omega-\mu)]}{|t'(x_0)|},\end{aligned} \tag{8}$$

where n_F is the Fermi distribution function, and $2t(x_0) = \omega$.

Substituting $t(x)$ from (2) we get

$$\frac{\omega}{2J} = \frac{\exp(-\kappa x_0)}{x_0^\gamma}, \tag{9}$$

$$\sigma(\omega) = \frac{\pi e^2}{8} c^2 L \frac{x_0^3[n_F(\omega-\mu) - n_F(-\omega-\mu)]}{\gamma + \kappa x_0}. \tag{10}$$

In the limit of low frequency we have the following asymptotic behavior:

1. $\kappa = 0$, $x_0 = (2J/\omega)^{1/\gamma}$:

$$\omega \gg T, \quad \sigma(\omega) \propto \omega^{-3/\gamma}, \tag{11}$$

$$\omega \ll T, \quad \sigma(\omega) \propto \omega^{-3/\gamma+1}; \tag{12}$$

2. $\kappa \neq 0$, $\kappa x_0 \sim \log(2J/\omega)$:

$$\omega \gg T, \quad \sigma(\omega) \propto \log^2(2J/\omega), \tag{13}$$

$$\omega \ll T, \quad \sigma(\omega) \propto \omega \log^2(2J/\omega). \tag{14}$$

3. HIGH DENSITY

3.1. Green function

In case of high density of impurities we assume that the distribution function of impurities is a Gaussian with variance g:

$$\begin{aligned}&\rho(\mathbf{r}_i) = c + \delta\rho(\mathbf{r}_i), \\ &\langle \delta\rho(\mathbf{r}_i)\delta\rho(\mathbf{r}_j)\rangle_\rho = g^2\delta_{ij},\end{aligned} \tag{15}$$

where $\langle ... \rangle_\rho$ denotes the average over possible impurity configurations. We assume that the concentration $c \leq 1$.

The one-particle Green function for the arbitrary impurity distribution is usually defined in terms of a functional integral as

$$G(E,\mathbf{r},\mathbf{r}') = \left[E - t_{ij}\rho(\mathbf{r}_i)\rho(\mathbf{r}_j)\right]^{-1}_{\mathbf{r},\mathbf{r}'} = i\frac{\int D\overline{\psi}D\psi\psi(\mathbf{r})\overline{\psi}(\mathbf{r}')\exp(iS)}{\int D\overline{\psi}D\psi\exp(iS)}, \tag{16}$$

where

$$S = S_0 + S_1, \tag{17}$$

$$iS_0 = i\sum_{\mathbf{r}}\overline{\psi}(\mathbf{r})E\psi(\mathbf{r}), \tag{18}$$

$$iS_1 = -i\sum_{\mathbf{r}_1\mathbf{r}_2}\overline{\psi}(\mathbf{r}_1)t(\mathbf{r}_1-\mathbf{r}_2)\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\psi(\mathbf{r}_2). \tag{19}$$

Introducing an additional integration over new fermion fields χ, $\overline{\chi}$ in order to eliminate the second order terms $\rho\rho$, we get

$$\exp(iS_1) = \int D\overline{\chi}D\chi\exp\left\{i\sum\overline{\chi}(\mathbf{r}_1)t^{-1}(\mathbf{r}_1-\mathbf{r}_2)\chi(\mathbf{r}_2) - ic\sum_{\mathbf{r}}\left[\overline{\chi}(\mathbf{r})\psi(\mathbf{r}) + \overline{\psi}(\mathbf{r})\chi(\mathbf{r})\right] - \right.$$
$$\left. - i\sum_{\mathbf{r}}\delta\rho(\mathbf{r})\left[\overline{\chi}(\mathbf{r})\psi(\mathbf{r}) + \overline{\psi}(\mathbf{r})\chi(\mathbf{r})\right]\right\}\frac{1}{Z}, \tag{20}$$

where

$$Z = \int D\overline{\chi}D\chi\exp\left\{i\sum_{\mathbf{r}_1,\mathbf{r}_2}\overline{\chi}(\mathbf{r}_1)t^{-1}(\mathbf{r}_1-\mathbf{r}_2)\chi(\mathbf{r}_2)\right\}, \tag{21}$$

$$t^{-1}(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}}\varepsilon^{-1}(\mathbf{k})e^{i\mathbf{kr}}, \tag{22}$$

$$\varepsilon(\mathbf{k}) = \sum_{\mathbf{r}}t(\mathbf{r})e^{i\mathbf{kr}} = -J\ln\left[\left(\kappa^2 + 4\sin^2\frac{k_x a}{2}\right)\left(\kappa^2 + 4\sin^2\frac{k_y a}{2}\right)\right]. \tag{23}$$

The Green function in terms of the new two-component field φ,

$$\begin{aligned}\varphi_1(\mathbf{r}) &= \psi(\mathbf{r}),\\ \varphi_2(\mathbf{r}) &= \chi(\mathbf{r}),\\ \overline{\varphi}_1(\mathbf{r}) &= \overline{\psi}(\mathbf{r}),\\ \overline{\varphi}_2(\mathbf{r}) &= \overline{\chi}(\mathbf{r}),\end{aligned} \tag{24}$$

reads

$$\hat{G}(\mathbf{r}_1,\mathbf{r}_2) = -i\langle\hat{\varphi}(\mathbf{r}_1)\otimes\hat{\overline{\varphi}}(\mathbf{r}_2)\rangle, \tag{25}$$

where $\hat{\overline{\varphi}}(\mathbf{r}) = (\overline{\varphi}_1(\mathbf{r}), \overline{\varphi}_2(\mathbf{r}))$, and angle brackets are defined as

$$\langle\ldots\rangle = \frac{\int D\overline{\psi}D\psi D\overline{\chi}D\chi(\ldots)\exp(iS_{eff})}{\int D\overline{\psi}D\psi D\overline{\chi}D\chi\exp(iS_{eff})} \tag{26}$$

with

$$iS_{eff} = i\sum_{\mathbf{r}} \hat{\bar{\varphi}}(\mathbf{r}) \begin{pmatrix} E & -c-\delta\rho(\mathbf{r}) \\ -c-\delta\rho(\mathbf{r}) & t^{-1}(\mathbf{r}_1-\mathbf{r}_2) \end{pmatrix} \hat{\varphi}(\mathbf{r}). \tag{27}$$

The equation for the Green function after averaging over impurities in the Born approximation reads

$$\hat{G}(\mathbf{k}) = \hat{G}^0(\mathbf{k}) + \hat{G}^0(\mathbf{k})\hat{\Sigma}(\mathbf{k})\hat{G}(\mathbf{k}), \tag{28}$$

with the bare Green function

$$[\hat{G}^0(\mathbf{k})]^{-1} = \begin{bmatrix} E & -c \\ -c & \varepsilon^{-1}(\mathbf{k}) \end{bmatrix} \tag{29}$$

and the self-energy $\hat{\Sigma}$ obtained by summing diagrams without intersections

$$\hat{\Sigma}(\mathbf{k}) = g^2a^2 \int \frac{d\mathbf{k}_1}{(2\pi)^2} \sigma^x \hat{G}(\mathbf{k}_1)\sigma^x. \tag{30}$$

The solution of Eqs. (28), (30) is

$$\Sigma(\mathbf{k}) = \begin{pmatrix} A & C \\ C & B \end{pmatrix}, \tag{31}$$

$$\hat{G}(\mathbf{k}) = \frac{1}{[1+\varepsilon(\mathbf{k})R](E-A)-(c+iC)^2\varepsilon(\mathbf{k})} \begin{bmatrix} 1+\varepsilon(\mathbf{k})B & -i(c+iC)\varepsilon(\mathbf{k}) \\ -i(c+iC)\varepsilon(\mathbf{k}) & \varepsilon(\mathbf{k})(E-A) \end{bmatrix}, \tag{32}$$

where

$$A = g^2a^2 \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{(E-A)\varepsilon(\mathbf{k})}{[1+\varepsilon(\mathbf{k})B](E-A)-(c+iC)^2\varepsilon(\mathbf{k})},$$
$$B = -g^2a^2 \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{1-\varepsilon(\mathbf{k})B}{[1+\varepsilon(\mathbf{k})B](E-A)-(c+iC)^2\varepsilon(\mathbf{k})},$$
$$iC = g^2a^2 \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{(c+iC)\varepsilon(\mathbf{k})}{[1+\varepsilon(\mathbf{k})B](E-A)-(c+iC)^2\varepsilon(\mathbf{k})}.$$

In the limit $g^2 \ll 1$, we obtain

$$A^{R,A} = \pm i\frac{\gamma}{2}, \quad B^{R,A} = \mp i\frac{c^2}{E^2}\frac{\gamma}{2}, \quad C^{R,A} = \pm i\frac{c}{E}\frac{\gamma}{2}, \tag{33}$$

$$\gamma = 2\pi g^2a^2\frac{E^2}{c^4}\nu\left(\frac{E}{c^2}\right), \tag{34}$$

where $\nu(\varepsilon)$ is the density of states of the pure model ($c=1$, $g=0$):

$$\nu(\varepsilon) = \int \frac{d\mathbf{k}}{(2\pi)^2}\delta(\varepsilon-\varepsilon(\mathbf{k})). \tag{35}$$

Taking into account that $A \ll E$, $BE \ll 1$, $C \ll c$, we find for the Green function in the limit $g^2 \ll 1$

$$\hat{G}^{R,A}(\mathbf{k}) = \frac{1}{E-c^2\varepsilon(\mathbf{k}) \pm i\gamma\left[1+3(c^2/E)\varepsilon(\mathbf{k})\right]/2} \begin{pmatrix} 1 & c\varepsilon(\mathbf{k}) \\ c\varepsilon(\mathbf{k}) & \varepsilon(\mathbf{k})E \end{pmatrix}, \tag{36}$$

where $G^R$and G^A are the retarded and advanced Green functions.

3.2. Drude formula

The conductivity in our case is defined as in Sec. 1 in terms of the four-particle correlation function

$$\sigma_E(\omega) = \frac{e^2}{2\pi}\int \frac{d\mathbf{k}_1}{(2\pi)^2}\frac{d\mathbf{k}_2}{(2\pi)^2} v_\alpha(\mathbf{k}_1) v_\alpha(\mathbf{k}_2) K_{E\omega}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_2,\mathbf{k}_1), \tag{37}$$

where E is taken at the Fermi level,

$$K_{E\omega}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_2,\mathbf{k}_1) = \frac{1}{V}\times$$
$$\times\sum_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}} \exp(i\mathbf{k}_1(\mathbf{x}-\mathbf{y}))\exp(i\mathbf{k}_2(\mathbf{z}-\mathbf{t}))\left\langle \rho_\mathbf{x}\rho_\mathbf{y}\rho_\mathbf{z}\rho_\mathbf{t} G_{11}^R\left(\mathbf{y},\mathbf{z},E+\frac{\omega}{2}\right) G_{11}^A\left(\mathbf{t},\mathbf{x},E-\frac{\omega}{2}\right)\right\rangle_\rho \tag{38}$$

and

$$v_\alpha(\mathbf{k}) = \frac{\partial\varepsilon(\mathbf{k})}{\partial k_\alpha}. \tag{39}$$

Inserting the solution (36) into (37), we find in the lowest approximation

$$\sigma_E(\omega) = \frac{e^2}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^2} v_\alpha^2(\mathbf{k}) G_{11}^R\left(\mathbf{k},E+\frac{\omega}{2}\right) G_{11}^A(\mathbf{k},E-\omega/2) = \frac{e^2c^6}{2\pi}\left(\frac{J}{E}\right)^2\frac{A(E)}{B(E)}, \tag{40}$$

where

$$A(E) = \int\limits_{\varepsilon(\mathbf{k})=E/c^2}\frac{dl_\mathbf{k}}{v(\mathbf{k})}v_x^2(\mathbf{k}), \qquad B(E) = \int\limits_{\varepsilon(\mathbf{k})=E/c^2}\frac{dl_\mathbf{k}}{v(\mathbf{k})}, \tag{41}$$

$v(\mathbf{k}) = \sqrt{v_x^2(\mathbf{k}) + v_y^2(\mathbf{k})}$, and $dl_\mathbf{k}$ is the element of length of the Fermi surface.

The conductivity can be expressed in terms of a particle density defined by

$$n_0(E) = a^2\int\limits_{-\pi/a}^{\pi/a} dk_x \int\limits_{-\pi/a}^{\pi/a} dk_y\theta\left(\frac{E}{c^2} - \varepsilon(\mathbf{k})\right). \tag{42}$$

We obtain in the low and high density limits

$$\sigma \doteq \begin{cases} \dfrac{e^2c^6}{g^2}\dfrac{n_0}{32\pi\log^2 2}, & \text{for} \quad n_0 \ll 1, \\ \dfrac{e^2c^6}{g^2}\dfrac{1-n_0}{4\kappa^4\log^2\kappa}, & \text{for} \quad 1-n_0 \ll 1. \end{cases} \tag{43}$$

The asymptotic behavior in the intermediate range $0 < n_0 < 1$ is

$$\sigma = 2\pi^4\frac{e^2c^6}{g^2}\frac{1}{(1-n_0)^2\log[1/(1-n_0)]}, \tag{44}$$

with maximum value

$$\sigma_{max} \sim \frac{1}{\kappa^{8/3}\log\kappa} \tag{45}$$

reached at $1 - n_0 \sim \kappa^{4/3}$.

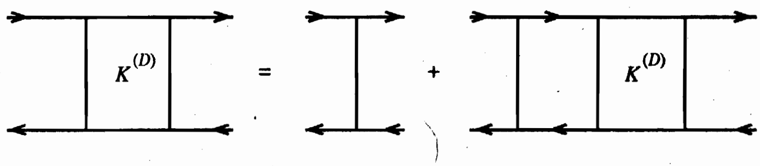

Fig. 1. Diffusion vertex

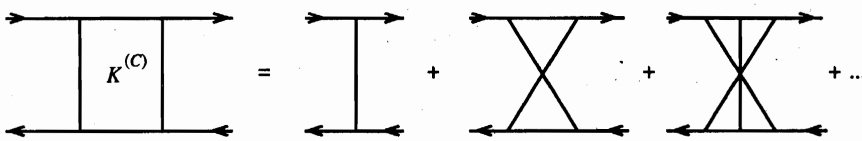

Fig. 2. Crossed-ladder vertex

3.3. Perturbation corrections

To go beyond the quasiclassical approximation we include contributions to the conductivity from «diffusion-ladder» and «crossed-ladder» or «cooperon» vertices [3, 4]. The additional term is

$$\delta\sigma_E(\omega) = \frac{e^2c^4}{2\pi}\int \frac{d\mathbf{k}_1}{(2\pi)^2}\frac{d\mathbf{k}_2}{(2\pi)^2} v_\alpha(\mathbf{k}_1)v_\alpha(\mathbf{k}_2)G^R_{1a}(\mathbf{k}_1)G^R_{c1}(\mathbf{k}_2)G^A_{1b}(\mathbf{k}_2)G^A_{d1}(\mathbf{k}_1)K_{ac;bd}(\mathbf{k}_1,\mathbf{k}_2), \tag{46}$$

where for the «diffusion» vertex contribution we obtain (see Fig. 1)

$$K^{(D)}_{ac;bd} = g^2\sigma^x_{ac}\sigma^x_{bd} + g^2\int\frac{d\mathbf{l}}{(2\pi)^2}\sigma^x_{aa_1}G^R_{a_1c_1}(\mathbf{l})K^{(D)}_{c_1c;bb_1}G^A_{b_1d_1}(\mathbf{l})\sigma^x_{d_1d}. \tag{47}$$

The solution of this equation does not depend on $\mathbf{k}_1$ and $\mathbf{k}_2$. Therefore the contribution of the «diffusion» vertex to the conductivity is equal to zero because

$$\int\frac{d\mathbf{k}_1}{(2\pi)^2}v_\alpha(\mathbf{k}_1)G^R_{1a}(\mathbf{k}_1)G^A_{d1}(\mathbf{k}_1) = 0. \tag{48}$$

Now we consider the «cooperon» vertex contribution. The vertex $K^{(C)}$ obeys the equation (see Fig. 2)

$$K^{(C)}_{ac;bd}(\mathbf{q}) = g^2\sigma^x_{ac}\sigma^x_{bd} + g^2\int\frac{d\mathbf{l}}{(2\pi)^2}\sigma^x_{ae}\sigma^x_{bg}G^R_{ea_1}(\mathbf{l})G^A_{gb_1}(\mathbf{q}-\mathbf{l})K^{(C)}_{a_1c;b_1d}(\mathbf{q}), \tag{49}$$

where $\mathbf{q} = \mathbf{k}_1 + \mathbf{k}_2$.

Using the Green function from (36) we can rewrite this equation:

$$K^{(C)}_{ac;bd} - \frac{1}{4}b(\omega,k)\sigma^x_{ae}\sigma^x_{bg}\tau_{ea_1}\tau_{gb_1}K^{(C)}_{a_1c;b_1d} = g^2\sigma^x_{ac}\sigma^x_{bd}, \tag{50}$$

where

$$b = 1 + \frac{1}{4\gamma}(i\omega - Dk^2), \qquad D = \frac{c^4\overline{v}_x^2}{4\gamma}, \tag{51}$$

and

$$\hat{\tau} = \begin{pmatrix} \exp(\theta) & 1 \\ 1 & \exp(-\theta) \end{pmatrix}, \tag{52}$$

where $\exp(\theta) = c/E$.

We seek a solution of (50) in the form

$$K^{(C)}_{ac;bd} = g^2 \sum_{\mu\nu} \sigma^{\mu}_{ab} \sigma^{\nu}_{cd} K^{\mu\nu}, \tag{53}$$

where $\mu, \nu \in \{0, x, y, z\}$, and $K^{\mu\nu}$ satisfies

$$K^{\mu\nu} - b(\omega, k) \sum_{\lambda=0,x,y,z} \Lambda_{\mu\lambda} K^{\lambda\nu} = \frac{1}{2} g^{\mu\nu}_c, \tag{54}$$

with

$$\hat{\Lambda} = \frac{1}{2} \begin{pmatrix} \cosh^2\theta & \cosh\theta & 0 & \sinh\theta\cosh\theta \\ \cosh\theta & 1 & 0 & \sinh\theta \\ 0 & 0 & 0 & 0 \\ -\sinh\theta\cosh\theta & -\sinh\theta & 0 & -\sinh^2\theta \end{pmatrix} \tag{55}$$

and

$$\hat{g}_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{56}$$

A solution of (54) can be found in terms of matrices U, B:

$$K^{\mu\nu} = U_{\mu M} B_{LN} (U^{-1})_{N\nu}, \tag{57}$$

where

$$(U^{-1})_{M\mu} \Lambda_{\mu\nu} U_{\nu N} = z_M \delta_{MN}, \tag{58}$$

$$B_{MN} = \frac{1}{1 - b(\omega, k) z_M} (U^{-1})_{M\mu} g^{\mu\nu} U_{\nu N}. \tag{59}$$

The eigenvalues z_M and matrix U are

$$z_0 = 1, \qquad z_1 = 0, \qquad z_2 = 0, \qquad z_3 = 0, \tag{60}$$

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{2}}\cosh\theta & \frac{1}{\sqrt{2}}\cosh\theta & -i\sinh\theta & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}}\sinh\theta & -\frac{1}{\sqrt{2}}\sinh\theta & i\cosh\theta & 0 \end{pmatrix}. \tag{61}$$

We see from (51) and (60) that the eigenmode with $N = 1$ only has singular behavior for ω, $k \to 0$. Substituting solutions (60), (59), (57), (54), and (53) into (46) we obtain a logarithmically divergent correction to the conductivity:

$$\delta\sigma = -\frac{e^2}{2\pi^2}\int\frac{dq}{q}. \tag{62}$$

We consider the problem in detail in the next section using field-theoretic methods for diffusion mode interactions.

3.4. Field-theoretic description

Quantum corrections to the conductivity can be described in terms of a diffusion modes interactions. To derive an effective Lagrangian we make use of a replica method. Conductivity properties are determined by the density–density correlation function

$$K(\omega) = \langle G^R_{E+\omega/2}(\mathbf{r}_1,\mathbf{r}_2)G^A_{E-\omega/2}(\mathbf{r}_1,\mathbf{r}_2)\rangle, \tag{63}$$

where

$$\hat{G}^{(R,A)}(E) = \left[E \pm i\delta - t_{ij}\rho(\mathbf{r}_i)\rho(\mathbf{r}_j)\right]^{-1}, \tag{64}$$

and angle brackets denote impurity averaging.

Integrating over anticommuting Grassmann variables κ^*, κ we can write

$$G^R_{E+\omega/2}(\mathbf{r}_1,\mathbf{r}_2)G^A_{E-\omega/2}(\mathbf{r}_1,\mathbf{r}_2) = -\frac{\int D\kappa^* D\kappa\,\kappa_1(\mathbf{r}_1)\kappa_1^*(\mathbf{r}_2)\kappa_{N+1}(\mathbf{r}_1)\kappa^*_{N+1}(\mathbf{r}_2)\exp(iS)}{\int D\kappa^* D\kappa\exp(iS)}, \tag{65}$$

where

$$iS = i\sum_{\mathbf{r}_1,\mathbf{r}_2}\sum_{n=1}^{2N}\kappa_n^*(\mathbf{r}_1)\left\{[E + (\omega + i\delta)\lambda_n]\,I_{\mathbf{r}_1,\mathbf{r}_2} - t_{\mathbf{r}_1,\mathbf{r}_2}\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\right\}\kappa_n(\mathbf{r}_2), \tag{66}$$

$$\lambda_n = \begin{cases} 1, & n \le N, \\ -1, & n > N, \end{cases} \tag{67}$$

and N is the number of replicas.

The quadratic term $\rho\rho$ can be transformed with the help of the additional Grassmann fields ν^*, ν

$$\exp\left\{-i\sum_{\mathbf{r}_1,\mathbf{r}_2}\sum_{n=1}^{2N} t(\mathbf{r}_1 - \mathbf{r}_2)\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\kappa_n^*(\mathbf{r}_1)\kappa_n(\mathbf{r}_2)\right\} = \int D\nu^* D\nu \times$$
$$\times \exp\left\{i\sum_{\mathbf{r}_1,\mathbf{r}_2}\sum_{n=1}^{2N} t^{-1}(\mathbf{r}_1 - \mathbf{r}_2)\nu^*(\mathbf{r}_1)\nu(\mathbf{r}_2) - i\sum_{\mathbf{r}}\sum_{n=1}^{2N}\rho(\mathbf{r})\left[\nu_n^*(\mathbf{r})\kappa_n(\mathbf{r}) + \kappa_n^*(\mathbf{r})\nu_n(\mathbf{r})\right]\right\} \times$$
$$\times\left[\int D\nu^* D\nu\exp\left\{i\sum_{\mathbf{r}_1,\mathbf{r}_2}\sum_{n=1}^{2N} t^{-1}(\mathbf{r}_1 - \mathbf{r}_2)\nu^*(\mathbf{r}_1)\nu(\mathbf{r}_2)\right\}\right]^{-1}, \tag{68}$$

It is convenient to define spinors Ψ, χ:

$$\sqrt{2}\Psi_n = \begin{pmatrix} \Psi_{n1} \\ \Psi_{n2} \end{pmatrix} = \begin{pmatrix} \kappa_n^* \\ \kappa_n \end{pmatrix}, \qquad \sqrt{2}\overline{\Psi}_n = \begin{pmatrix} -\kappa_n \\ \kappa_n^* \end{pmatrix}, \tag{69}$$

$$\sqrt{2}\chi_n = \begin{pmatrix} \chi_{n1} \\ \chi_{n2} \end{pmatrix} = \begin{pmatrix} \nu_n^* \\ \nu_n \end{pmatrix}, \qquad \sqrt{2}\overline{\chi}_n = \begin{pmatrix} -\nu_n \\ \nu_n^* \end{pmatrix}, \tag{70}$$

or

$$\overline{\Psi} = (C\Psi)^T, \quad \overline{\chi} = (C\chi)^T, \qquad C = \delta_{mn} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{71}$$

where C is the charge conjugation matrix. The same relations hold between the original and Fourier components

$$\overline{\Psi}_{ni}(p) = C_{ij}\Psi_{ni}(-p), \qquad \Psi_{ni}(p) = C_{ji}\overline{\Psi}_{ni}(-p), \tag{72}$$

$$\overline{\chi}_{ni}(p) = C_{ij}\chi_{ni}(-p), \qquad \chi_{ni}(p) = C_{ji}\overline{\chi}_{ni}(-p), \tag{73}$$

where

$$\Psi_{ni}(\mathbf{r}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{N}} \Psi_{ni}(\mathbf{p}) e^{i\mathbf{pr}}, \qquad \chi_{ni}(\mathbf{r}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{N}} \chi_{ni}(\mathbf{p}) e^{i\mathbf{pr}}. \tag{74}$$

The action (66) takes the form

$$iS = i\left\{ \sum_{\mathbf{r}} \overline{\Psi}(\mathbf{r}) \left[E + \left(\frac{\omega}{2} + i\delta\right)\Lambda \right] \Psi(\mathbf{r}) + \sum_{\mathbf{r}_1,\mathbf{r}_2} \overline{\chi}(\mathbf{r}_1) t^{-1}(\mathbf{r}_1 - \mathbf{r}_2)\chi(\mathbf{r}_2) - \right.$$

$$\left. - \sum_{\mathbf{r}} \left[\overline{\chi}(\mathbf{r})\Psi(\mathbf{r}) + \overline{\Psi}(\mathbf{r})\chi(\mathbf{r}) \right] \right\}, \tag{75}$$

where Λ is a diagonal matrix consisting of elements λ_n. Introducing new bispinors

$$\varphi(\mathbf{r}) = \begin{pmatrix} \Psi(\mathbf{r}) \\ \chi(\mathbf{r}) \end{pmatrix}, \qquad \overline{\varphi}(\mathbf{r}) = \left(\overline{\Psi}(\mathbf{r}) \quad \overline{\chi}(\mathbf{r}) \right)$$

we can rewrite the action as

$$iS = i \sum_{\mathbf{r}_1,\mathbf{r}_2} \overline{\varphi}(\mathbf{r}_1) G_0^{-1}(\mathbf{r}_1, \mathbf{r}_2) \varphi(\mathbf{r}_2) - i \sum_{\mathbf{r}} \delta\rho(\mathbf{r}) \overline{\varphi}(\mathbf{r}) \sigma^x \varphi(\mathbf{r}), \tag{76}$$

where

$$G_0^{-1}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} E + \left(\frac{\omega}{2} + i\delta\right)\Lambda & -c \\ -c & t^{-1}(\mathbf{r}_1 - \mathbf{r}_2) \end{pmatrix} \tag{77}$$

is the bare Green function.

Performing the average over a Gaussian distribution for the randomness we obtain

$$\langle \exp(iS_{int}) \rangle = \left\langle \exp\left[-i \sum_{\mathbf{r}} \delta\rho(\mathbf{r})\overline{\varphi}(\mathbf{r})\sigma^x\varphi(\mathbf{r}) \right] \right\rangle = \exp\left\{ -\frac{g^2}{2} \sum_{\mathbf{r}} [\overline{\varphi}(\mathbf{r})\sigma^x\varphi(\mathbf{r})]^2 \right\}. \quad (78)$$

The order parameter in the localization theory is a traceless tensor proportional to $\langle \Psi\overline{\Psi} \rangle$. Taking into account only long-wavelength fluctuations we can rewrite (78) as

$$\langle \exp(iS_{int}) \rangle = \exp\left[g^2 \sum_{\mathbf{r}} \sigma^x_{ab}\sigma^x_{cd} P^{\nu\mu}_{da}(\mathbf{r}) P^{\mu\nu}_{bc}(\mathbf{r}) \right], \quad (79)$$

where

$$P^{\mu\nu}_{bc}(\mathbf{r}) = \frac{1}{L} \sum_{\mathbf{p},\mathbf{q}} \varphi^{\mu}_{b}(\mathbf{p})\overline{\varphi}^{\nu}_{c}(\mathbf{p}+\mathbf{q}) \exp(-i\mathbf{q}\mathbf{r}), \qquad \nu = \{n, i\}.$$

Spinors φ and $\overline{\varphi}$ are related by charge conjugation (71). This imposes symmetry conditions

$$(P^{\mu\nu}_{bc}(\mathbf{r}))^* = P^{\nu\mu}_{cb}(\mathbf{r}), \quad \text{or} \quad P^+ = P, \quad (80)$$

$$P = CP^TC^T. \quad (81)$$

Introducing a Gaussian integration over the c-number matrix field Q, we obtain

$$\exp(iS_{eff}) = \int DQ \exp\left\{ \sum_{\mathbf{r}_1,\mathbf{r}_2} \overline{\varphi}(\mathbf{r}_1) \left[iG_0^{-1}(\mathbf{r}_1,\mathbf{r}_2) + I_{\mathbf{r}_1\mathbf{r}_2}\frac{\beta}{4}Q(\mathbf{r}_1) \right] \varphi(\mathbf{r}_2) - \right.$$
$$\left. - \sum_{\mathbf{r}} \frac{\alpha}{4} \mathrm{Tr}\,(Q\sigma^x Q\sigma^x) \right\} \left\{ \int DQ \exp\left[-\sum_{\mathbf{r}} \frac{\alpha}{4} \mathrm{Tr}\,(Q\sigma^x Q\sigma^x) \right] \right\}^{-1}. \quad (82)$$

After integration over the bispinor field φ we obtain

$$iS_{eff} = \mathrm{Tr}\,\ln\left[iG_0^{-1} + \frac{\beta}{4}Q \right] - \sum_{\mathbf{r}} \frac{\alpha}{4} \mathrm{Tr}\,(Q\sigma^x Q\sigma^x). \quad (83)$$

In the saddle-point approximation we use the equation

$$\delta S/\delta Q = 0,$$

or

$$\left(iG_0^{-1} + \frac{\gamma}{2}Q \right)^{-1} - \frac{\gamma}{2g^2}\sigma^x Q\sigma^x = 0. \quad (84)$$

The solution of this equation to lowest order in g^2 is

$$Q_{sp} = \Lambda(\cosh\theta - \sigma^z\sinh\theta + \sigma^x)\exp\theta, \quad (85)$$

$$E/c = \exp\theta, \quad (86)$$

with the Green function

$$G = \frac{1}{E + \Lambda\omega/2 - c^2\varepsilon(\mathbf{p}) - 2i\gamma\Lambda} \begin{pmatrix} 1 & E/c \\ E/c & E^2/c^2 \end{pmatrix}. \tag{87}$$

Expanding near the saddle point and using the symmetry of the tensor Q, we find

$$iS_{eff} = \mathrm{Tr}\ \ln\left(iG_0^{-1} + \frac{\gamma}{2}\delta Q\right) - \frac{\gamma^2}{16g^2}\sum_{\mathbf{r}} \mathrm{Tr}\ \{[Q + \delta Q(\mathbf{r})]\,\sigma^x\,[Q + \delta Q(\mathbf{r})]\,\sigma^x\}. \tag{88}$$

Taking Fourier transforms, Eq.(88) reads

$$i\delta S_{eff} = -\frac{\gamma^2}{16g^2}\frac{1}{L}\sum_{\mathbf{q}}\Big\{\mathrm{Tr}[\delta Q(\mathbf{q})\sigma^x\delta Q(-\mathbf{q})\sigma^x] - \\ - g^2\int\frac{d\mathbf{p}}{(2\pi)^2}\mathrm{Tr}[G(\mathbf{p})\delta Q(\mathbf{q})G(\mathbf{p}+\mathbf{q})\delta Q(-\mathbf{q})] + \mathrm{Tr}\left[\frac{i\omega\gamma}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\delta Q(\mathbf{q})\right]\Big\}. \tag{89}$$

Due to the symmetry of the Q we can write the variation δq as

$$\delta Q_{bc} = Q_\alpha \sigma^\alpha_{bc_1}\sigma^x_{c_1c}, \tag{90}$$

where all matrices Q_α are real. Inserting the relation (90) into (89), we find, taking into account only low-energy transverse ($Q\delta Q + \delta Q Q = 0$) modes,

$$\delta S_{eff} = -\frac{\gamma^2}{8g^2}\sum_{\alpha,\beta}\left[\delta_{\alpha\beta} - \frac{b_0}{8}\mathrm{Tr}(\sigma^\beta\sigma^x\tau\sigma^\alpha\sigma^x\tau)\right]Q_\alpha Q_\beta + Q_\alpha T_\alpha, \tag{91}$$

where

$$b_0 = 1 - \frac{Dk^2}{4\gamma}, \quad D = \frac{c^4\overline{v}_x^2}{4\gamma}, \quad \tau = \cosh\theta + \sigma^z\sinh\theta + \sigma^x, \\ T_\alpha = \mathrm{Tr}\left[\frac{i\omega}{2}\Lambda\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\sigma^\alpha\sigma^x\right]. \tag{92}$$

After diagonalization we obtain

$$iS_{eff} = \frac{1}{L}\sum_{\mathbf{k}}\mathrm{Tr}\left\{-\frac{\gamma Dk^2}{32g^2}q_1(\mathbf{k})q_1(-\mathbf{k}) - \frac{\gamma}{8g^2}\sum_{l=2,3,4}\mathrm{Tr}\,[q_l(\mathbf{k})q_l(-\mathbf{k})] + \frac{i\omega\gamma}{16g^2}\Lambda U_{\alpha l}q_l(\mathbf{k})\right\}, \tag{93}$$

where

$$Q_\alpha = U_{\alpha l}q_l, \tag{94}$$

$$U_{\alpha l} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}}\cosh\theta & \frac{1}{\sqrt{2}}\cosh\theta & -i\sinh\theta & 0 \\ -\frac{i}{\sqrt{2}}\sinh\theta & -\frac{i}{\sqrt{2}}\sinh\theta & -\cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{95}$$

We see that the action (93) consists of only one diffusion type mode ($l = 1$). Taking into account only these modes and neglecting interactions with other higher-energy modes, we obtain the well-known action of the nonlinear σ model,

$$S = \frac{1}{t} \int d\mathbf{r} \left[\mathrm{Tr}(\nabla Q)^2 - \tilde{\omega} \mathrm{Tr}(\Lambda Q) \right], \tag{96}$$

where

$$\tilde{\omega} = \frac{\gamma}{16g^2} \frac{\omega}{D}, \qquad \frac{1}{t} = \frac{\gamma}{16g^2} D.$$

The renormalization group equations for this case were studied in Ref. [5]. In the limit $N \to 0$, we see that in lowest order

$$\frac{dt}{d\ln k} = \frac{t^2}{8}, \tag{97}$$

$$\frac{d\ln\tilde{\omega}}{d\ln k} = 0. \tag{98}$$

Since the conductivity σ is proportional to the diffusion constant D, equations (97) and (98) determine the frequency and system length dependence of the conductivity. In particular, we have from (98) that $\sigma(\omega) \propto D \propto \omega$ for small ω.

4. CONCLUSIONS

We have investigated a two-dimensional model with a new type of disorder due to a random distribution of local states with strongly anisotropic overlaps of wave functions. This type of disorder, described by a quadratic impurity density Hamiltonian (1), was not considered previously. The conductivity of the system was calculated in the limits of low (Sec. 2) and high densities (Sec. 3) of local states. Since perturbation theory leads to divergent terms («Cooperon» vertices) we used the field theoretic description in terms of a diffusion mode interaction. Introducing an additional integration over fermion field and performing the average over impurities with the help of a replica trick, we obtained the action of the nonlinear σ model. Renormalization group equations for this model determine the behavior of the diffusion constant and the conductivity. We have shown that this type of disorder leads to weak localization phenomena in the high density limit, as in the usual two-dimensional case [6].

As mentioned in the Introduction, a similar picture can be realized in nontrivial superconductors. A strong scattering impurity potential produces a resonant or marginally bound state inside the gap in a d-wave superconductor. The wave function of the impurity bound state is highly anisotropic, with $1/r$ decay along the nodes of the gap, and, an exponential, angle-dependent decay range otherwise. A finite concentration of impurities leads to an formation of the narrow quasiparticle band. If we simplify a picture and take into account scattering processes of quasiparticles inside this band only, we obtain our model. We note that our consideration is applicable only in the case of strong unitary impurities producing local bound states. Opposite cases of weak impurity scattering with different types of disorder were studied in Refs. [7, 8].

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