ЖЭТФ, 1997, том 111, вып. 2, стр. 514–527

ON THE DYNAMICS OF A CURVED DEFLAGRATION FRONT

V. V. Bychkov, M. A. Liberman***

* *Department of Plasma Physics, Umea University, S–90187, Umea, Sweden*
** *Department of Physics, Uppsala University, Box 530, S–75121, Uppsala, Sweden*
P. Kapitsa Institute for Physical Problems
117334, Moscow, Russia

Submitted 25 June 1996

An equation describing evolution of a curved deflagration front of finite thickness is obtained for the case of an arbitrary equation of state of the «fuel», an arbitrary type of energy release and an arbitrary type of thermal conduction. The equation is complemented by conservation laws for the mass flux and the momentum flux through the deflagration front of finite thickness. As an illustration of the method, the growth rates and the cut-off wavelengths for the linear stage of the flame instability are calculated for the case of a flame in an ideal gaseous fuel and for the case of a thermonuclear deflagration propagating in a strongly degenerate matter of white dwarfs.

1. INTRODUCTION

Dynamics of a deflagration wave is a key problem in many physical phenomena from industrial problems of burning in engines to astrophysical problems of stellar evolution. A deflagration wave (flame) is a front of energy release propagating due to thermal conduction with a subsonic velocity [1]. Usual flame supported by chemical reactions of a gaseous or liquid fuel is a typical example of a deflagration wave. The other examples are waves of laser ablation in inertial fusion [2, 3], fronts of thermonuclear reaction in Supernova events [4, 5], evaporation waves in a superheated fluid [6], etc. A deflagration wave may propagate as a planar stationary front separating cold initial matter (fresh «fuel») from heated products of deflagration. It was shown by Landau and Darrieus [1, 7] that a planar flame front is hydrodynamically unstable against perturbations which bend the front on large length scales compared to the flame front thickness. Similar instability is inherent to all waves of the deflagration type. The instability leads to a cellular structure of a deflagration front [8, 9] or to self-turbulization of the combustion flow [1, 9, 10], which may cause considerable increase of the velocity of the deflagration front. Because of the Landau-Darrieus instability the configuration of a curved deflagration front is much more common than the configuration of a planar front. Besides, some external effects like turbulence may wrinkle a deflagration front as well.

To study propagation of a deflagration wave one has to consider the complete system of hydrodynamic equations, taking into account thermal conduction and kinetic of the energy release. This is a rather complicated problem even for numerical solution and any simplification of the problem is of great interest. The problem may be simplified considerably if the region of thermal conduction and energy release can be replaced by a surface of discontinuity, separating the fuel and the products of deflagration. In this case one must specify the evolution equation for the deflagration front and the boundary conditions at the front. Such a model was employed first by Landau and Darrieus [1, 7] to study stability of a planar flame front of zero thickness. However further investigation of the flame front stability showed that finite thickness of the front has to be taken into account [9, 10]. The influence of thermal conduction and finite flame thickness remains important even on the scales exceeding the flame thickness by several orders of magnitude. At the same time introduction of a discontinuity surface for a flame front does not necessarily imply that the front has zero thickness: the flame thickness may enter the boundary conditions and the evolution equation as an external parameter, depending on the fuel properties. The evolution equation for a flame front of finite thickness and the boundary conditions at the front were obtained first by Matalon and Matkowsky [11] (though misprints in the final results somewhat spoil the excellent work). Matalon and Matkowsky considered the case of a usual laboratory flame propagating in an ideal gas with a large activation energy of the chemical reaction. The same evolution equation for a flame in an ideal gaseous fuel was obtained independently in [12]. However diversity of deflagration waves and their similar properties require development of more general theory of curved deflagration fronts, including the case of an arbitrary «fuel» with an arbitrary type of energy release.

In the present paper we obtain the conservation laws at a deflagration front and the evolution equation for a curved deflagration front of finite thickness propagating in a «fuel» with an arbitrary equation of state, an arbitrary type of thermal conduction and an arbitrary type of energy release. To illustrate the developed method, the growth rates and the cut-off wavelengths for the linear stage of the flame instability are calculated for a flame in an ideal gaseous fuel and for a thermonuclear deflagration propagating in a strongly degenerate matter of white dwarfs. For the particular case of a flame in an ideal gas with large activation energy of the reaction the obtained equations agree with the results of [11].

2. A PLANAR STATIONARY DEFLAGRATION FRONT

As usual the deflagration velocity is essentially subsonic so that the flow caused by a deflagration wave can be treated as incompressible. The equations describing propagation of a deflagration wave are

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0, \tag{1}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + (\rho \mathbf{v} \nabla)\mathbf{v} + \nabla P = \mu_F \Delta \mathbf{v} + \left(\mu_S + \frac{\mu_F}{3}\right) \nabla (\nabla \mathbf{v}), \tag{2}$$

$$\rho \frac{\partial H}{\partial t} + \rho \mathbf{v} \nabla H = \nabla(\kappa \nabla T) + \frac{\rho_- Q}{\tau_R} W(T), \tag{3}$$

where H is the enthalpy per unit mass, κ is the coefficient of thermal conduction, μ_F and μ_S are the coefficients of first and second viscosity respectively, which may be taken constant for majority of physical problems. For incompressible matter the density, the enthalpy and the thermal conduction depend only upon temperature $\rho = \rho(T)$, $H = H(T)$, $\kappa = \kappa(T)$. The last term in Eq. (3) describes the energy release in the deflagration wave, τ_R is a constant of time dimension, Q is the energy release per unit mass. For given enthalpy of the «fuel» (label «$-$») the enthalpy of deflagration products (label «+») is computed through the conservation of energy: $H_+ = H_- + Q$. Then the equation of state gives the final temperature of the deflagration products $T_+ = T(H_+)$, and the final density $\rho_+ = \rho(H_+)$. The dimensionless function $W(T)$ gives the temperature dependence of the reaction rate; this function vanishes for the initial matter and for the deflagration products $W(T_-) = 0$, $W(T_+) = 0$. Strictly speaking, the process of energy release should be described by a separate kinetic equation: the equation of chemical kinetic, the equation of laser light absorption, etc. However in many cases the kinetic equation together with the equation of energy transfer may be reduced to an equation of the form of Eq. (3). This is an exact result for a flame in a usual gaseous fuel with the Lewis number equal to unity, when the coefficients of mass diffusion and thermal diffusivity are equal [9, 10]. This simplification can be made also for the case of negligible mass diffusion, when the electron thermal conduction or thermal conduction of radiation are the dominant processes, as it is for the thermonuclear reaction fronts in Supernovae [13], for laser ablation [3] and for many other physical processes.

Eqs. (1)–(3) have a stationary solution which is a planar deflagration front propagating along z-axis. In the frame reference co-moving with the deflagration front the stationary solution consists of the upstream homogeneous flow of the cold matter, the transition region of heating and energy release, and the homogeneous downstream flow of the deflagration products. For the stationary flow we have $\rho u = \rho_- U_f = \mathrm{const}$ and the speed of a stationary deflagration is determined as an eigenvalue of the equation (3) of energy transfer

$$\rho_- U_f \frac{dH}{dz} = \frac{d}{dz}\left(\kappa \frac{dT}{dz}\right) + \frac{\rho_- Q}{\tau_R} W(T) \tag{4}$$

with the boundary conditions $T = T_-$ ahead of the deflagration front and $T = T_+$ behind the front.

It is convenient to introduce the dimensionless variables and parameters $r = \rho/\rho_-$, $\Theta = T/T_-$, $h = H/H_-$, $\Pi = (P - P_-)/\rho_- U_f^2$, $\chi = \kappa/\kappa_-$, $Pr_{F,S} = \mu_{F,S} H_-/T_- \kappa_-$, $\Lambda = QL/(H_- U_f \tau_R)$, and the dimensionless coordinate $\xi = z/L$, where U_f is the velocity and $L = \kappa_- T_-/(H_- \rho_- U_f)$ is the thickness of the planar stationary deflagration front. The parameter Λ is an eigenvalue of the dimensionless equation of heat transfer

$$\frac{d}{d\xi}\left(\chi \frac{d\Theta}{d\xi}\right) - \frac{dh}{d\xi} + \Lambda W(\Theta) = 0, \tag{5}$$

with the boundary conditions $\Theta = 1$ for $\xi \to -\infty$ and $\Theta = \Theta_+$ for $\xi \to \infty$.

Let us consider the approximate analytical solution of Eq. (5) for the case of negligible thickness of the zone of energy release using the method proposed by Zel'dovich and Frank-Kamenetskii [10]. Outside the reaction zone $W \approx 0$ and temperature obeys the equation

$$\chi \frac{d\Theta}{d\xi} = h - 1. \tag{6}$$

Neglecting the second term in Eq. (5) inside the thin zone of energy release and using Eq. (6) as a boundary condition upstream the zone we obtain the following approximate value of Λ

$$\Lambda \approx \frac{(h_+ - 1)^2}{2\chi_+} \left(\int_1^{\Theta_+} W(\Theta)\, d\Theta \right)^{-1}. \tag{7}$$

Eq. (7) makes it possible to obtain the deflagration velocity depending on the particular function of the energy release as follows $U_f = \sqrt{Q\kappa_- T_-/\Lambda H_-^2 \rho_- \tau_R}$.

3. EVOLUTION EQUATION FOR AN INFINITELY THIN DEFLAGRATION FRONT

Let the curved deflagration front be described by a function $z = Rf(\mathbf{x}, t)$, so that evolution of the front is characterized by the length scale $R \gg L$ and by the time scale R/U_f. It is convenient to introduce the dimensionless time $\tau = U_f t/R$, the transverse coordinate $\boldsymbol{\eta} = \mathbf{x}/R$ and the small parameter $\varepsilon = L/R \ll 1$. In a coordinate system co-moving with the flame front

$$\zeta = \xi - f(\boldsymbol{\eta}, \tau)/\varepsilon, \qquad \boldsymbol{\eta} = \boldsymbol{\eta}, \qquad \tau = \tau \tag{8}$$

the hydrodynamic equations take the form

$$\varepsilon \frac{\partial r}{\partial \tau} + \frac{\partial m}{\partial \zeta} + \varepsilon \nabla_\perp (r\boldsymbol{v}) = 0, \tag{9}$$

$$\begin{aligned} \varepsilon r \frac{\partial \boldsymbol{v}}{\partial \tau} + m\frac{\partial \boldsymbol{v}}{\partial \zeta} + \varepsilon r(\boldsymbol{v}\nabla_\perp)\boldsymbol{v} = -\varepsilon \nabla_\perp \Pi + \nabla_\perp f \frac{\partial \Pi}{\partial \zeta} + \\ + Pr_F \Delta_\zeta \boldsymbol{v} + \left(Pr_S + \frac{Pr_F}{3}\right)\left(\varepsilon\nabla_\perp - \nabla_\perp f \frac{\partial}{\partial \zeta}\right)\left(\frac{\partial}{\partial \zeta}\left(\frac{m}{r}\right) + \varepsilon \nabla_\perp \boldsymbol{v}\right), \end{aligned} \tag{10}$$

$$\begin{aligned} \varepsilon r \frac{\partial u}{\partial \tau} + m\frac{\partial u}{\partial \zeta} + \varepsilon r(\boldsymbol{v}\nabla_\perp) u = -\frac{\partial \Pi}{\partial \zeta} + \\ + Pr_F \Delta_\zeta u + \left(Pr_S + \frac{Pr_F}{3}\right)\frac{\partial}{\partial \zeta}\left(\frac{\partial}{\partial \zeta}\left(\frac{m}{r}\right) + \varepsilon \nabla_\perp \boldsymbol{v}\right), \end{aligned} \tag{11}$$

$$\begin{aligned} \varepsilon r \frac{\partial h}{\partial \tau} + m\frac{\partial h}{\partial \zeta} + \varepsilon r \boldsymbol{v}\nabla_\perp h = N^2 \frac{\partial}{\partial \zeta}\left(\chi \frac{\partial \Theta}{\partial \zeta}\right) + \\ + \varepsilon \nabla_\perp \left[\chi\left(\varepsilon \nabla_\perp \Theta - \nabla_\perp f \frac{\partial \Theta}{\partial \zeta}\right)\right] - \varepsilon \nabla_\perp f \frac{\partial}{\partial \zeta}(\chi \nabla_\perp \Theta) + W(\Theta), \end{aligned} \tag{12}$$

where the mass flux m is given by the expression

$$m = ru - r\frac{\partial f}{\partial \tau} - r\boldsymbol{v}\nabla_\perp f, \tag{13}$$

$u = v_z/U_f$, $\boldsymbol{v} = \mathbf{v}_x/U_f$ are the components of the dimensionless velocity, $\nabla_\perp$ is the part of the operator ∇ related to the coordinate $\boldsymbol{\eta}$ along the front surface. The Laplace operator in the introduced variables takes the form

$$\Delta_\zeta = N^2 \frac{\partial^2}{\partial \zeta^2} - \varepsilon \nabla_\perp^2 f \frac{\partial}{\partial \zeta} - 2\varepsilon \frac{\partial}{\partial \zeta}(\nabla_\perp f \nabla_\perp) + \varepsilon^2 \nabla_\perp^2, \tag{14}$$

where $N = \sqrt{1 + (\nabla_\perp f)^2}$. It is convenient to rewrite Eqs. (10), (11) in another form, extracting the expression for the momentum flux in the left-hand side

$$\begin{aligned} m\frac{\partial}{\partial \zeta}(\boldsymbol{v} + u\nabla_\perp f) = -\varepsilon r\left(\frac{\partial}{\partial \tau} + \boldsymbol{v}\nabla_\perp\right)(\boldsymbol{v} + u\nabla_\perp f) + \varepsilon r u\left(\frac{\partial}{\partial \tau} + \boldsymbol{v}\nabla_\perp\right)\nabla_\perp f - \\ - \varepsilon \nabla_\perp \Pi + Pr_F(\Delta_\zeta \boldsymbol{v} + \nabla_\perp f \Delta_\zeta u) + \varepsilon\left(Pr_S + \frac{Pr_F}{3}\right)\nabla_\perp\left(\frac{\partial}{\partial \zeta}\left(\frac{m}{r}\right) + \varepsilon \nabla_\perp \boldsymbol{v}\right), \end{aligned} \tag{15}$$

$$N^2\frac{\partial}{\partial\zeta}\left(\Pi+\frac{m^2}{N^2r}\right)=-\varepsilon\frac{\partial m}{\partial\tau}-\varepsilon\nabla_\perp(m\boldsymbol{v})-\varepsilon r\frac{\partial^2 f}{\partial\tau^2}-2\varepsilon r\boldsymbol{v}\nabla_\perp\frac{\partial f}{\partial\tau}-$$
$$-\varepsilon r\boldsymbol{v}\left(\boldsymbol{v}\nabla_\perp\right)\nabla_\perp f+\varepsilon\nabla_\perp f\nabla_\perp\Pi+Pr_F\Delta_\zeta\left(\frac{m}{r}\right)+Pr_F\boldsymbol{v}\Delta_\zeta\nabla_\perp f+Pr_F\Delta_\zeta\frac{\partial f}{\partial\tau}+$$
$$+\left(Pr_S+\frac{Pr_F}{3}\right)\left(N^2\frac{\partial}{\partial\zeta}-\varepsilon\nabla_\perp f\nabla_\perp\right)\left(\frac{\partial}{\partial\zeta}\left(\frac{m}{r}\right)+\varepsilon\nabla_\perp\boldsymbol{v}\right). \tag{16}$$

We expand a solution of Eqs. (9)-(12) in the power series of the small parameter ε: $m = m_0+\varepsilon m_1+O\left(\varepsilon^2\right)$, $\Theta=\Theta_0+\varepsilon\Theta_1+O\left(\varepsilon^2\right)$, etc. Then Eqs. (9), (12), (15), (16) with the accuracy $O\left(\varepsilon\right)$ take the form

$$\frac{\partial m}{\partial\zeta}=0, \tag{17}$$

$$m\frac{\partial}{\partial\zeta}\left(\boldsymbol{v}+u\nabla_\perp f\right)=Pr_F N^2\frac{\partial^2}{\partial\zeta^2}\left(\boldsymbol{v}+u\nabla_\perp f\right), \tag{18}$$

$$N^2\frac{\partial}{\partial\zeta}\left(\Pi+\frac{m^2}{N^2\,r}\right)=\left(\frac{4}{3}Pr_F+Pr_S\right)N^2\frac{\partial^2}{\partial\zeta^2}\left(\frac{m}{r}\right), \tag{19}$$

$$m\frac{\partial h}{\partial\zeta}=N^2\frac{\partial}{\partial\zeta}\left(\chi\frac{\partial\Theta}{\partial\zeta}\right)+W\left(\Theta\right). \tag{20}$$

It follows from Eq. (17) that $m=m_0(\boldsymbol{\eta},\tau)+O(\varepsilon)$. Similar to Eq. (5), solution of Eq. (20) can be presented in the form:

$$\Theta_0=\Theta_p\left(\zeta/N\right),\qquad m_0(\boldsymbol{\eta},\tau)=N\equiv\sqrt{1+(\nabla_\perp f)^2}, \tag{21}$$

where $\Theta=\Theta_p(\xi)$ is the temperature profile for a planar deflagration front, which is solution of the ordinary differential equation (5). The equation of state of the fuel gives us immediately

$$r_0=r\left(\Theta_0\right)=r_p\left(\zeta/N\right),\qquad h_0=h\left(\Theta_0\right)=h_p\left(\zeta/N\right). \tag{22}$$

Eqs. (18), (19) being integrated across the deflagration front supply the conservation laws for the momentum flux at the infinitely thin surface of discontinuity. Particularly, we find from Eq. (18)

$$\boldsymbol{v}+u\nabla_\perp f=\boldsymbol{v}_-+u_-\nabla_\perp f+O(\varepsilon). \tag{23}$$

Eqs. (13), (23) give the solution for the velocity components

$$u=u_--\frac{1}{N}\,\frac{r-1}{r}+O(\varepsilon), \tag{24}$$

$$\boldsymbol{v}=\boldsymbol{v}_-+\frac{\nabla_\perp f}{N}\,\frac{r-1}{r}+O(\varepsilon). \tag{25}$$

The expression for pressure follows from Eq. (19)

$$\Pi=\Pi_-+\frac{r-1}{r}+\left(\frac{4}{3}Pr_F+Pr_S\right)N\frac{\partial}{\partial\zeta}\left(\frac{1}{r}\right)+O(\varepsilon). \tag{26}$$

Eqs. (13), (21) give the evolution equation for an infinitely thin deflagration front

$$\frac{\partial f}{\partial\tau}+\boldsymbol{v}_-\nabla_\perp f+\sqrt{1+(\nabla_\perp f)^2}-u_-=O(\varepsilon). \tag{27}$$

Evolution equation Eq. (27) and the relations (24)–(26) determine the dynamics of an infinitely thin deflagration front, which is, in fact, the Landau-Darrieus model of a deflagration front.

4. INFLUENCE OF FINITE THICKNESS OF A DEFLAGRATION FRONT ON THE FRONT EVOLUTION

Evolution of a deflagration front depends essentially upon the finite thickness of the front even on the length scales which exceed the front thickness by several orders of magnitude [9, 10]. Next terms of the expansion in ε describe the influence of the finite thickness of the deflagration front. The equation for the mass flux, Eq. (9), with the accuracy ε^2 may be written in the form

$$\frac{\partial m_1}{\partial \zeta} = N^{-1}\left(\frac{\partial N}{\partial \tau} + \boldsymbol{v}_-\nabla_\perp N + \frac{\nabla_\perp f \nabla_\perp N}{N}\right)\frac{\partial}{\partial \zeta}((r_0-1)\zeta) - \frac{r_0-1}{N}\left(\frac{\partial N}{\partial \tau} + \nabla_\perp(N\boldsymbol{v}_-) + \nabla_\perp^2 f\right) - \nabla_\perp \boldsymbol{v}_-. \tag{28}$$

The solution of Eq. (28) is

$$m_1 = m_{1-} - \zeta\nabla_\perp \boldsymbol{v}_- + \left(\frac{\partial N}{\partial \tau} + \boldsymbol{v}_-\nabla_\perp N + \frac{\nabla_\perp f \nabla_\perp N}{N}\right)\frac{\zeta}{N}(r_0-1) - \left(\frac{\partial N}{\partial \tau} + \nabla_\perp(N\boldsymbol{v}_-) + \nabla_\perp^2 f\right) J\left(\frac{\zeta}{N}\right), \tag{29}$$

where

$$J\left(\frac{\zeta}{N}\right) = \int\limits_{-\infty}^{\zeta/N} \left(r_p(\nu) - 1\right) d\nu. \tag{30}$$

Substituting the expression for the mass flux Eq. (29) into the equation of thermal conduction Eq. (12) and extracting the terms of order of $O(\varepsilon)$ we have

$$m_{1-}\frac{\partial h_0}{\partial \zeta} - \left(\frac{\partial N}{\partial \tau} + \nabla_\perp(N\boldsymbol{v}_-)\right)\frac{\zeta}{N}\frac{\partial h_0}{\partial \zeta} - \left(\frac{\partial N}{\partial \tau} + \nabla_\perp(N\boldsymbol{v}_-) + \nabla_\perp^2 f\right) J\left(\frac{\zeta}{N}\right)\frac{\partial h_0}{\partial \zeta} + $$
$$+\nabla_\perp^2 f\chi\frac{\partial \Theta_0}{\partial \zeta} - 2\nabla_\perp f\frac{\nabla_\perp N}{N}\frac{\partial}{\partial \zeta}\left(\zeta\chi\frac{\partial \Theta_0}{\partial \zeta}\right) = $$
$$= N^2\frac{\partial^2}{\partial \zeta^2}(\chi\Theta_1) - N\frac{\partial}{\partial \zeta}\left(\frac{dh}{d\Theta}\Theta_1\right) + \frac{dW}{d\Theta}\Theta_1. \tag{31}$$

One can check by simple substitution and comparison with Eq. (5) that the right-hand side of Eq. (31) is zero for $\Theta_1 = \partial\Theta_0/\partial\zeta$:

$$\hat{F}\left(\frac{\partial \Theta_0}{\partial \zeta}\right) \equiv N^2\frac{\partial^2}{\partial \zeta^2}\left(\chi\frac{\partial \Theta_0}{\partial \zeta}\right) - N\frac{\partial}{\partial \zeta}\left(\frac{dh}{d\Theta}\frac{\partial \Theta_0}{\partial \zeta}\right) + \frac{dW}{d\Theta}\frac{\partial \Theta_0}{\partial \zeta} = 0. \tag{32}$$

The operator $\hat{F}(\Theta_1)$ can be reduced to a Hermitian form $\hat{F}_H(\psi)$ by the substitution

$$\Theta_1 = \frac{\psi}{\chi}\exp\left(\frac{1}{2}\int\limits_{\zeta_-/N}^{\zeta/N} \chi_p^{-1}\frac{dh_p}{d\Theta_p}d\nu\right) \tag{33}$$

where ζ_- is an arbitrary position since the operator $\hat{F}(\Theta_1)$ is linear. The expression $\hat{F}_H(\psi)$ becomes zero for

$$\psi_0 = \chi \frac{\partial \Theta_0}{\partial \zeta} \exp\left(-\frac{1}{2} \int\limits_{\zeta_-/N}^{\zeta/N} \chi_p^{-1} \frac{dh_p}{d\Theta_p} d\nu \right). \tag{34}$$

If we write the right-hand side of Eq. (31) in the Hermitian form, multiply it by ψ_0 and integrate, we obtain the relation

$$m_{1-} = \left(\frac{\partial N}{\partial \tau} + \nabla_\perp (N \boldsymbol{v}_-) + \nabla_\perp^2 f \right) \langle (\xi + J(\xi)) \frac{dh_p}{d\xi} \rangle - \\ - \nabla_\perp^2 f \left\langle \xi \frac{dh_p}{d\xi} + \chi \frac{d\Theta_p}{d\xi} \right\rangle + 2 \frac{\nabla_\perp f \nabla_\perp N}{N} \left\langle \frac{d}{d\xi} \left(\xi \chi \frac{d\Theta_p}{d\xi} \right) \right\rangle, \tag{35}$$

where

$$\langle \Phi \rangle = \left(\int\limits_{-\infty}^{\infty} \Phi \Psi^2 d\xi \right) \left(\int\limits_{-\infty}^{\infty} \frac{dh_p}{d\xi} \Psi^2 d\xi \right)^{-1} \tag{36}$$

and

$$\Psi^2 = \chi \frac{d\Theta_p}{\partial \xi} \exp\left(- \int\limits_{\xi_-}^{\xi} \chi_p^{-1} \frac{dh_p}{d\Theta_p} d\nu \right). \tag{37}$$

Profiles of temperature, enthalpy and density for a planar stationary deflagration front $\Theta_p = \Theta_p(\xi)$, $h_p = h_p(\xi)$, $\rho_p = \rho_p(\xi)$ follow from Eq. (5) and the equation of state. Taking into account Eq. (35) we obtain the evolution equation for a deflagration front of finite thickness with the accuracy $O(\varepsilon^2)$

$$\frac{\partial f}{\partial \tau} + \boldsymbol{v}_- \nabla_\perp f - u_- + N = -\varepsilon \left(\frac{\partial N}{\partial \tau} + \nabla_\perp (N \boldsymbol{v}_-) + \nabla_\perp^2 f \right) \left\langle (\xi + J) \frac{dh_p}{d\xi} \right\rangle + \\ + \varepsilon \nabla_\perp^2 f \left\langle \xi \frac{dh_p}{d\xi} + \chi \frac{d\Theta_p}{d\xi} \right\rangle - 2\varepsilon \frac{\nabla_\perp f \nabla_\perp N}{N} \left\langle \frac{d}{d\xi} \left(\xi \chi \frac{d\Theta_p}{d\xi} \right) \right\rangle, \tag{38}$$

where $N = \sqrt{1 + (\nabla_\perp f)^2}$. Eq. (38) is obtained for the case of an arbitrary equation of state of the «fuel», an arbitrary type of thermal conduction and an arbitrary type of energy release.

Let us consider a deflagration wave, for which energy is released in a zone of negligible thickness compared to the total thickness of the deflagration front. For a usual flame this case corresponds to a reaction with large activation energy. In this case we have from Eq. (6) $\Psi^2 = 1$ for $\xi < 0$ and $\Psi^2 = 0$ for $\xi > 0$, so that the averaging Eq. (36) becomes

$$\langle \Phi \rangle = \frac{1}{h_+ - 1} \int\limits_{-\infty}^{0} \Phi d\xi. \tag{39}$$

Calculating the coefficients in Eq. (38) we can write the evolution equation for the deflagration front in the form

$$\frac{\partial f}{\partial \tau} + \boldsymbol{v}_- \nabla_\perp f - u_- + N = \varepsilon\alpha \left(\frac{\partial N}{\partial \tau} + \nabla_\perp (N\boldsymbol{v}_-) + \nabla_\perp^2 f \right). \tag{40}$$

Eq. (40) describes the influence of the front stretch and the finite thickness of the deflagration wave on the velocity of the front. The numerical factor depends on the «fuel» properties

$$\alpha = \frac{h_+}{h_+ - 1} \int_1^{\Theta_+} r\chi d\Theta + \int_1^{\Theta_+} \frac{1 - rh}{h - 1} \chi d\Theta. \tag{41}$$

For many physical problems the dimensionless equation of state for an isobaric flow have the form $rh = 1$. It takes place when pressure is determined by an ideal gas, by a non-relativistic or ultra-relativistic fermi-gas, by radiation pressure, etc. In this case the numerical factor becomes

$$\alpha = \frac{h_+}{h_+ - 1} \int_1^{\Theta_+} r\chi d\Theta. \tag{42}$$

For the particular case of an ideal gas ($r\Theta = 1$) with a constant coefficient of thermal conduction ($\chi = 1$) the evolution equation obtained by Matalon and Matkowsky [11] is recovered from Eq. (40).

5. CONSERVATION LAWS AT A DEFLAGRATION FRONT OF FINITE THICKNESS

The evolution equation Eq. (40) should be complemented by the conservation laws at the deflagration front which take into account the finite thickness of the front. These conservation laws follow from Eqs. (9), (15), (16) integrated with the accuracy $O\left(\varepsilon^2\right)$. The conservation law for the mass flux follows immediately from Eq. (29)

$$\left(ru - r\boldsymbol{v}\nabla_\perp f - r\frac{\partial f}{\partial \tau} \right)\Big|_-^+ = \varepsilon J_1 \left(\frac{\partial N}{\partial \tau} + \nabla_\perp (N\boldsymbol{v}) + \nabla_\perp^2 f \right), \tag{43}$$

where the numerical factor is given by the formula

$$J_1 = \int_{-\infty}^{\infty} \xi \frac{dr_p}{d\xi} d\xi. \tag{44}$$

To obtain the conservation laws for the momentum flux we need to integrate Eqs. (15), (16), which have the form with the accuracy $O\left(\varepsilon^2\right)$

$$\begin{aligned} m\frac{\partial}{\partial \zeta}(\boldsymbol{v} + u\nabla_\perp f) = & -\varepsilon r \left(\frac{\partial}{\partial \tau} + \boldsymbol{v}_- \nabla_\perp \right) \boldsymbol{v}_- - \varepsilon r \nabla_\perp f \left(\frac{\partial}{\partial \tau} + \boldsymbol{v}_- \nabla_\perp \right) u_- - \\ & -\varepsilon \frac{r-1}{N} (\nabla_\perp f \nabla_\perp) \boldsymbol{v}_- - \varepsilon \frac{r-1}{N} \nabla_\perp f (\nabla_\perp f \nabla_\perp) u_- - \varepsilon \nabla_\perp \Pi_- - \\ & -\varepsilon \frac{r-1}{N} \left(\frac{\partial}{\partial \tau} + \boldsymbol{v}_- \nabla_\perp + \frac{\nabla_\perp f}{N} \nabla_\perp \right) \nabla_\perp f + \varepsilon \frac{\nabla_\perp N}{N} \frac{\partial}{\partial \zeta} \left(\zeta \frac{r-1}{r} \right) + \\ & + Pr_F N^2 \frac{\partial^2}{\partial \zeta^2} (\boldsymbol{v} + u\nabla_\perp f) - \varepsilon Pr_F \nabla_\perp N \frac{\partial}{\partial \zeta} \left(\frac{r-1}{r} \right), \end{aligned} \tag{45}$$

$$N^2\frac{\partial}{\partial\zeta}\left(\Pi+\frac{m^2}{N^2r}\right)=\left(\frac{4}{3}Pr_F+Pr_S\right)N^2\frac{\partial^2}{\partial\zeta^2}\left(\frac{m}{r}\right)-\varepsilon\frac{\partial N}{\partial\tau}-\varepsilon\nabla_\perp\left(N\boldsymbol{v}_-\right)-$$
$$-\varepsilon r\frac{\partial^2 f}{\partial\tau^2}-2\varepsilon r\boldsymbol{v}_-\nabla_\perp\frac{\partial f}{\partial\tau}-2\varepsilon\left(r-1\right)\left(\frac{\partial}{\partial\tau}-\boldsymbol{v}_-\nabla_\perp\right)N-\varepsilon\frac{(r-1)^2}{r}\,\frac{\nabla_\perp f\nabla_\perp N}{N}-$$
$$-\varepsilon r v_-\left(\boldsymbol{v}_-\nabla_\perp\right)\nabla_\perp f+\varepsilon\nabla_\perp f\nabla_\perp\Pi_--\varepsilon\frac{r-1}{r}\,\nabla_\perp^2 f+$$
$$+\varepsilon\left(\frac{4}{3}Pr_F+Pr_S\right)N^2\nabla_\perp\left(\frac{\nabla_\perp f}{N}\right)\frac{\partial}{\partial\zeta}\left(\frac{r-1}{r}\right). \tag{46}$$

Integrating Eqs. (45), (46) and taking into account asymptotic behavior outside the deflagration front we obtain the jump of the components of the momentum flux across the deflagration front of finite thickness

$$\left(\boldsymbol{v}+u\nabla_\perp f\right)\Big|_-^+=Pr_FN\left(\frac{\partial\boldsymbol{v}}{\partial\zeta}+\nabla_\perp f\frac{\partial u}{\partial\zeta}\right)\Big|_-^+-\varepsilon Pr_F\frac{r_+-1}{r_+}\frac{\nabla_\perp N}{N}+$$
$$+\varepsilon J_1\left(\hat{D}\boldsymbol{v}_-+\nabla_\perp f\hat{D}u_-+N^{-1}\hat{D}\nabla_\perp f\right), \tag{47}$$

$$\left(\Pi+\frac{m^2}{N^2r}\right)\Big|_-^+=\varepsilon J_2\nabla_\perp\left(\frac{\nabla_\perp f}{N}\right)+$$
$$+\frac{\varepsilon J_1}{N}\left(\frac{\partial^2 f}{\partial\tau^2}+2\boldsymbol{v}_-\frac{\partial}{\partial\tau}\nabla_\perp f+\boldsymbol{v}_-\left(\boldsymbol{v}_-\nabla_\perp\right)\nabla_\perp f+2\hat{D}N-\frac{\nabla_\perp f\nabla_\perp N}{N}\right)+$$
$$+\left(\frac{4}{3}Pr_F+Pr_S\right)\left(\frac{\partial}{\partial\zeta}\left(\frac{m}{r}\right)\Big|_-^++\varepsilon\frac{r_+-1}{r_+}\nabla_\perp\left(\frac{\nabla_\perp f}{N}\right)\right) \tag{48}$$

where the coefficient J_2 is

$$J_2=\int\limits_{-\infty}^{\infty}\frac{\xi}{r_p^2}\frac{dr_p}{d\xi}d\xi \tag{49}$$

and the differential operator $\hat{D}$ is introduced for convenience

$$\hat{D}=\frac{\partial}{\partial\tau}+\boldsymbol{v}_-\nabla_\perp+N^{-1}\nabla_\perp f\nabla_\perp. \tag{50}$$

It can be obtained using the hydrodynamic equations Eqs. (9)–(11) outside the deflagration front [11] that

$$\left(\frac{\partial\boldsymbol{v}}{\partial\zeta}+\nabla_\perp f\frac{\partial u}{\partial\zeta}\right)\Big|_-^+=-\varepsilon\frac{r_+-1}{N}\left(\hat{D}\boldsymbol{v}_-+\nabla_\perp f\hat{D}u_-+\frac{1}{N}\hat{D}\nabla_\perp f\right)+\varepsilon\frac{r_+-1}{r_+}\frac{\nabla_\perp N}{N^2}, \tag{51}$$

$$\frac{\partial}{\partial\zeta}\left(\frac{m}{r}\right)\Big|_-^+=-\varepsilon\frac{r_+-1}{r_+}\nabla_\perp\left(\frac{\nabla_\perp f}{N}\right). \tag{52}$$

Taking into account Eqs. (51)-(52), we can rewrite the conservation laws Eqs. (47), (48) in the form

$$(\boldsymbol{v} + u\nabla_\perp f)|_-^+ = \varepsilon \left(J_1 - Pr_F\left(r_+ - 1\right)\right)\left(\hat{D}\boldsymbol{v}_- + \nabla_\perp f \hat{D} u_- + N^{-1}\hat{D}\nabla_\perp f\right), \tag{53}$$

$$\left(\Pi + \frac{m^2}{N^2 r}\right)\Big|_-^+ = \varepsilon J_2 \nabla_\perp\left(\frac{\nabla_\perp f}{N}\right) +$$
$$+\frac{\varepsilon J_1}{N}\left(\frac{\partial^2 f}{\partial \tau^2} + 2\boldsymbol{v}_-\nabla_\perp\frac{\partial f}{\partial \tau} + \boldsymbol{v}_-(\boldsymbol{v}_-\nabla_\perp)\nabla_\perp f + 2\hat{D}N - \frac{\nabla_\perp f \nabla_\perp N}{N}\right). \tag{54}$$

Equations (43), (53), (54) represent the conservation laws for the mass flux, tangential and normal components of the momentum flux for a deflagration front of finite thickness for the case of an arbitrary fuel. The left-hand sides of these equations represent the conservation laws for an infinitely thin deflagration front, the right-hand sides describe the influence of the transport processes and the finite thickness of the deflagration front. The dependence upon the equation of state of the «fuel» and the particular type of the energy release enters the conservation laws through the coefficients J_1, J_2. For the common case of a thin zone of energy release the coefficients take the form

$$J_1 = \int_1^{\Theta_+} r\chi d\Theta + \int_1^{\Theta_+} \frac{1 - rh}{h - 1}\chi d\Theta, \tag{55}$$

$$J_2 = \int_1^{\Theta_+} \chi d\Theta + \int_1^{\Theta_+} \frac{1 - rh}{h - 1}\frac{\chi}{r} d\Theta. \tag{56}$$

The evolution equation Eq. (38) and the conservation laws Eqs. (43), (53), (54) determine dynamics of a curved deflagration front in a «fuel» with an arbitrary equation of state and an arbitrary type of energy release. It should be mentioned that second viscosity does not enter either the evolution equation or the conservation laws, though this process has been taken into account in calculations.

6. SUMMARY OF THE RESULTS AND DISCUSSION

In the present paper we have obtained the evolution equation for a curved deflagration front of finite thickness propagating in a «fuel» with an arbitrary equation of state and an arbitrary type of energy release. The evolution equation is complemented by the boundary conditions: the conservation laws for the mass flux and for the momentum flux through the deflagration front. Let us write the obtained results in dimensional units in a coordinate free form. For a deflagration front described by a function $F(z, \mathbf{x}, t) = 0$ (for example, $F = z - Rf(\mathbf{x}, t) = 0$) we have $N = |\nabla F|$ and therefore the evolution of a deflagration front is described by the equation

$$\mathbf{n}\mathbf{v}_- + \frac{1}{N}\frac{\partial F}{\partial t} = U_f - \alpha L K, \tag{57}$$

where $\mathbf{v}_-$ is the flow velocity at the cold boundary of the deflagration front, $\mathbf{n} = \nabla F / |\nabla F|$ is the unit normal vector directed towards the products of deflagration, K is the front stretch, which may be written in a coordinate free form as [11]

$$K = \frac{1}{N}\frac{\partial N}{\partial t} + \frac{1}{N}\nabla\left(N\left(\mathbf{v}_- - U_f\mathbf{n}\right)\right), \tag{58}$$

and the numerical factor for the case of a narrow zone of energy release is given by

$$\alpha = \frac{Q+H_-}{Q}\int\limits_{T_-}^{T_+}\frac{\rho\kappa dT}{\rho_-\kappa_- T_-} + \int\limits_{T_-}^{T_+}\frac{\rho_- H_- - \rho H}{H - H_-}\frac{\kappa dT}{\rho_-\kappa_- T_-}. \tag{59}$$

Taking into account Eqs. (57), (58) we can write the conservation laws in the following coordinate free form

$$\rho\left(\mathbf{nv} + \frac{1}{N}\frac{\partial F}{\partial t}\right)\Big|_-^+ = \rho_- L J_1 K, \tag{60}$$

$$[\mathbf{nv}]|_-^+ = \frac{L}{U_f}\left(J_1 + Pr_F\frac{\rho_- - \rho_+}{\rho_-}\right)\mathbf{n}\times\hat{D}\left(\mathbf{v}_- - U_f\mathbf{n}\right), \tag{61}$$

$$\begin{gathered}\left(P + \rho(\mathbf{vn})^2\right)\big|_-^+ = -L\rho_- U_f^2 J_2\nabla\mathbf{n} + \\ +L\rho_- J_1\left(\frac{U_f}{N}\hat{D}N + \mathbf{n}\frac{\partial\mathbf{v}_-}{\partial t} + \mathbf{n}\left(\mathbf{v}_-\nabla\right)\mathbf{v}_-\right)\end{gathered} \tag{62}$$

where the dimensional form of the operator $\hat{D}$ is

$$\hat{D} = \frac{\partial}{\partial t} + \mathbf{v}_-\nabla - U_f\mathbf{n}\nabla \tag{63}$$

and the coefficients J_1, J_2 for the common case of a deflagration wave with a thin zone of energy release are given by the formulas

$$J_1 = \int\limits_{T_-}^{T_+}\frac{\rho\kappa}{\rho_-\kappa_- T_-}dT + \int\limits_{T_-}^{T_+}\frac{\rho_- H_- - \rho H}{H - H_-}\frac{\kappa dT}{\rho_-\kappa_- T_-}, \tag{64}$$

$$J_2 = \int\limits_{T_-}^{T_+}\frac{\kappa}{\kappa_- T_-}dT + \int\limits_{T_-}^{T_+}\frac{\rho_- H_- - \rho H}{\rho(H - H_-)}\frac{\kappa dT}{\kappa_- T_-}. \tag{65}$$

The evolution equation Eq. (57) and the conservation laws Eqs. (60)–(62) determine the dynamics of a deflagration front in a flow of a «fuel» with an arbitrary equation of state, arbitrary type of thermal conduction and energy release. An essential advantage of the present model is possibility to apply it to physical problems of different nature, i. e. flame propagation in a chemical or thermonuclear fuel, propagation of an ablation front in a laser target, evaporation front in a superheated fluid, etc.

Let us apply the obtained results to some particular physical situations. In the simplest case of a flame in an ideal gaseous fuel with a large activation energy and equal coefficients of thermal conductivity and fuel diffusion the evolution equation (57) and the conservation laws (60)–(62) coincide with the equations derived in [11]. The well known expression [14–16] for the instability growth rate σ of perturbations at a planar flame front of finite thickness may be also derived directly from the obtained equations. The linearized equations (57), (60)–(62) couple the amplitudes of the perturbation modes generated by an unstable flame front in the upstream and the downstream flows. Taking into account the explicit expressions for the perturbation modes [16] one finds the instability growth rate from the consistency condition of the linearized equations (57), (60)–(62):

$$\sigma = SU_f k\left(1 - k\lambda_c/2\pi\right), \tag{66}$$

where $S = (1+r_+)^{-1}\left(\sqrt{1+1/r_+ - r_+} - 1\right)$ is the coefficient obtained in the Landau-Darrieus theory of the flame front instability. The instability is suppressed by thermal conduction for perturbations of a wavelength shorter than the cut-off wavelength

$$\lambda_c = \frac{\pi L r_+}{2S\left(S\left(r_+ + 1\right) + r_+\right)}\left(1 - r_+ - \frac{\ln r_+}{1 - r_+}\left(2Sr_+ + 1 + r_+\right)\right). \tag{67}$$

Because of the large numerical coefficient in Eq. (67) the cut-off wavelength λ_c for typical laboratory flames ($r_+ = 0.1 - 0.2$) exceeds the flame thickness L essentially, $\lambda_c \approx 20L$. By this reason smooth planar flames are observed much more often, than it was expected from the Landau-Darrieus theory [1, 7, 10].

Another interesting application of the obtained nonlinear model is dynamics of a thermonuclear deflagration in Supernova events. It is generally believed that the observed spectrum of a Supernova of Type Ia is determined mostly by the first deflagration stage of white dwarf burning [4, 17]. At the same time investigation of the deflagration stage is strongly complicated by the huge range of the length scales which have to be taken into account. The hydrodynamic length scale of the deflagration dynamics in Supernovae is about the white dwarf radius $\sim$ 100km, which exceeds the flame thickness ($\sim 10^{-5}$cm) by 12 orders of magnitude. By this reason a direct numerical simulation of white dwarf burning is impossible and a simplified model is required. The most probable scenario of the deflagration dynamics in white dwarfs is development of a fractal structure, which implies many cascades of cells of different sizes growing because of the Landau-Darrieus instability [18]. The velocity of a fractal flame increases with increase of the flame radius R as [18, 19]

$$U = r_+^{-1} U_f\left(\frac{2\pi R}{n_c\lambda_c}\right)^d, \tag{68}$$

where $2+d$ is the fractal dimension, n_c is the critical number of a spherical function for a fractal accelerating flame found in [19], which determines the upper limit of the fractal structure. The cut-off wavelength λ_c gives the low limit of the fractal structure of the deflagration in white dwarfs. This value may be calculated on the basis of the model Eqs. (57), (60)–(62), taking into account the properties of the white dwarf fuel. In the dense central layers of a white dwarf with $\rho = 3\cdot 10^9\text{g/cm}^3$ the equation of state is determined by degenerate ultrarelativistic electrons, so that [20]

$$H = \frac{4P}{\rho}, \tag{69}$$

$$P = 0.77\hbar c\left(\frac{\rho Z}{M_i}\right)^{4/3}\left(1 + 2.06\left(\frac{M_i}{\rho Z}\right)^{2/3}\frac{T^2}{\hbar^2 c^2}\right), \tag{70}$$

where eZ and M_i are the average electrical charge and the average mass of the ions. Because of the relatively low temperature the thermal conduction of the degenerate electrons dominates in a white dwarf over the radiation thermal conduction, therefore [21]

$$\kappa = \frac{0.81\hbar c^2}{e^4 Z \Lambda_{ei}} \left(\frac{\rho Z}{M_i}\right)^{1/3} T, \tag{71}$$

where Λ_{ei} is the Coulomb logarithm. Using the equation of state and the expression for the thermal conduction we calculate the coefficients of the nonlinear model Eqs. (59), (64), (65). Then similar to the case of a flame in an ideal gaseous fuel, we find the cut-off wavelength of the Landau-Darrieus instability of a deflagration front in white dwarfs

$$\lambda_c = \frac{2\pi L}{S r_+} \frac{b_1 + S b_2}{(1 + r_+) S + 1} \tag{72}$$

with the coefficients depending upon the ratio of densities r_+ of the fuel and the products of the thermonuclear reaction

$$b_1 = \frac{3 - 2r_+^{1/3} - r_+^{4/3}}{6\left(1 - r_+^{4/3}\right)} + \frac{r_+ + 1}{12 r_+^{4/3} (1 - r_+)\left(1 + r_+^{4/3}\right)} \left(3 - 3r_+^{2/3} + 4r_+^{4/3} + r_+^{6/3} + r_+^{8/3}\right), \tag{73}$$

$$b_2 = \frac{1}{2r_+^{4/3}(1 - r_+)} - \frac{1 - r_+^{2/3}}{6r_+^{2/3}\left(1 + r_+^{4/3}\right)} \left(3 + 2r_+^{2/3} + r_+^{4/3}\right). \tag{74}$$

The fuel close to the white dwarf center is strongly degenerate so that energy release in the reaction causes only slight decrease of density of the burning matter, $r_+ = 0.85 - 0.9$. For such small expansion the Landau-Darrieus instability is rather weak, while the thermal stabilization is strong and the cut-off wavelength exceeds the flame thickness by almost three orders of magnitude, $\lambda_c = 410L$. This value differs essentially from the crude estimate of the cut-off wavelength made in [22], where the stabilization of the hydrodynamic instability was expected on the length scales of about $10L$.

The developed model may be useful as well for investigation of the fractal dimension of the deflagration fronts in white dwarfs, which up to now has been estimated with rather low accuracy ($\sim 50\%$) [18] not acceptable in the theory of thermonuclear Supernovae.

This work was supported in part by the Swedish National Board for Industrial and Technical Development (NUTEK), grant P2204-1 and in part by the Swedish Natural Science Research Council (NFR), grant F-AA/Fu 10297-307.

References

1. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford (1987).
2. W. H. Manheimer, D. G. Colombant, and G. H. Gardner, Phys. Fluids **25**, 1644 (1982).
3. V. V. Bychkov, S. M. Golberg, and M. A. Liberman, Phys. Plasmas **1**, 2976 (1994).
4. S. E. Woosley and T. A. Weaver, Ann. Rev. Astron. Astroph. **24**, 20 (1986).
5. V. V. Bychkov and M. A. Liberman, Astron. Astroph. **302**, 727 (1995).

6. J. E. Shepherd and B. Sturtevant, J. Fluid Mech. **121**, 379 (1982).
7. L. D. Landau, Zh. Exp. Teor. Fiz. **14**, 240 (1944).
8. G. I. Sivashinsky, Ann. Rev. Fluid. Mech. **15**, 179 (1983).
9. F. A. Williams, *Combustion Theory*, 2nd Ed., Benjamin, Reading, MA (1985).
10. Ya. B. Zel'dovich, G. I. Barenblatt, V. B. Librovich, and G. M. Makhviladze, *The Mathematical Theory of Combustion and Explosion*, Consultants Bureau, New York (1985).
11. M. Matalon and B. J. Matkowsky, J. Fluid Mech. **124**, 239 (1982).
12. P. Clavin and G. Joulin, J. Physique Lettres **44**, L1 (1983).
13. V. V. Bychkov and M. A. Liberman, Astroph. J. **451**, 711 (1995).
14. P. Pelce and P. Clavin, J. Fluid Mech. **124**, 219 (1982).
15. M. L. Frankel and G. I. Sivashinsky, Comb. Sci. Tech. **29**, 207 (1982).
16. M. A. Liberman, V. V. Bychkov, S. M. Golberg, and D. L. Book, Phys. Rev. **E49**, 445 (1994).
17. A. M. Khokhlov, Astron. Astroph. **245**, 114 (1991).
18. S. I. Blinnikov and P. V. Sasorov, Phys. Rev. E **53**, 4827 (1996).
19. V. V. Bychkov and M. A. Liberman, Phys. Rev. Lett. **76**, 2814 (1996).
20. L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon, Oxford (1959).
21. F. X. Timmes and S. E. Woosley, Astroph. J. **396**, 649 (1992).
22. A. M. Khokhlov, Astrophys. J. **449**, 695 (1995).