Dynamics of soliton-like wave signals propagating in smoothly inhomogeneous and weakly nonstationary nonlinear media

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By employing the nonlinear Klein–Gordon equations with small nonlocal corrections we develop a self-consistent description of one-dimensional soliton-like wave signals in smoothly inhomogeneous nonstationary nonlinear media. For the quasisoliton velocity and the field's integral characteristics we develop a closed system of ordinary differential equations written in a form common in relativistic mechanics. We study the interaction, accompanied by frequency conversion, of quasisolitons with the waves of the medium parameters; in particular, the laws that govern the penetration of these waves into the bulk of an advancing dense-plasma barrier and their reflection from the barrier. Finally, we demonstrate the possibility of the signal frequency growing considerably without temporal-spectrum broadening by employing the example of relativistic quasisolitons propagating in an initially homogeneous and stationary plasma with small additional ionization in the medium. © *1996 American Institute of Physics.* [S1063-7761(96)028609-0]

1. INTRODUCTION

Lately there has been an upsurge of interest in studies of electromagnetic propagation in inhomogeneous nonstationary dispersive media. In the three decades of active work in this field, the emphasis has shifted from purely linear problems[1–5] to nonlinear.[6–12] Presently of great interest are frequency self-conversion and the structure of quasimonochromatic pulsed signals in media with a nonlocal nonlinearity due, in particular, to electric breakdown or additional ionization[6–8] and excitation of low-frequency (electronic and acoustic) wake waves.[9,10]

Two main approaches (often inconsistent) are employed to solve these problems: the truncated parabolic equation method, which allows for diffraction of the wave fields but makes it possible to describe only relatively small variations of the angular and frequency spectra of these fields,[7,10] and the space–time geometrical optics approximation, which ignores diffraction completely but makes it possible, within its range of applicability, to study cases in which the frequency of the wave signal is converted significantly.[4,6]

Below we examine the dynamics of one-dimensional soliton-like pulses ("quasisolitons") in inhomogeneous nonstationary media. Neither the truncated parabolic equation method[1)] nor the nonlinear space–time geometrical optics approximation can be used to solve this problem. Nonlinearity in quasisolitons almost perfectly balances diffraction stemming from the medium's dispersion, and the frequency can be converted over a wide range. In studying quasisoliton behavior one must almost always remain within the framework of a nonlinear wave equation, and here we take a nonlinear Klein–Gordon equation, often used in various areas of physics, which contains small nonlocal terms in addition to a local nonlinearity.

The plan of the paper is as follows. Section 2 deals with the asymptotic procedure that makes it possible to derive a closed system of ordinary differential equations for the quasisoliton velocity and the field's integral characteristics, and to express the signal's frequency and duration in terms of its "energy" and "dynamic" parameters. In Sec. 3 we analyze the case of smoothly inhomogeneous nonstationary media with a purely local nonlinearity and, in particular, study the laws governing the interaction of quasisolitons with medium parameter waves. In Sec. 4 we examine the effect of a nonlocal nonlinearity on the propagation of soliton-like signals in initially homogeneous and stationary media. Finally, Sec. 5 summarizes the main results of the work.

2. BASIC EQUATIONS AND APPROXIMATIONS

We examine the propagation of one-dimensional pulsed signals in an inhomogeneous nonstationary nonlinear medium by using a wave equation of the form

$$\frac{\partial^2 A}{\partial t^2} - c^2 \frac{\partial^2 A}{\partial x^2} + \hat{V}(x,t,|A|^2)A = 0, \qquad (1)$$

with A a complex-valued wave field, and $\hat{V}(x,t,|A|^2)$ an integrodifferential operator whose parameters depend on position x, time t, and intensity $|E|^2$ (the relationship between these parameters and $|A|^2$ is not necessarily local).

We assume that the medium is weakly absorbing and that its properties vary smoothly in space (x) and time (t) over the pulse length Λ^* and the period of oscillations τ^*. We also assume that nonlocal nonlinear effects are small compared to local effects. All these restrictions can formally be allowed for by including a small parameter μ in the expression for $\hat{V}$:

$$\hat{V} = V(\mu x, \mu t, |A|^2) + \mu \hat{\delta}(\mu x, \mu t, |A|^2). \qquad (2)$$

Here $V(\mu x, \mu t, |A|^2)$ is a purely real function of the three variables μx, μt, and $|A|^2$, and $\mu \hat{\delta}(\mu x, \mu t, |A|^2)$ is the operator responsible for small losses and small nonlocal nonlinear effects. We are interested in the solutions of Eq. (1) that are asymptotic in the small parameter μ.

Equation (2) shows that in the limit $\mu\equiv 0$, which corresponds to a homogeneous stationary medium with purely local nonlinearity, Eq. (1) becomes the nonlocal Klein–Gordon equation of the type

$$\frac{\partial^2\Psi}{\partial t^2}-c^2\frac{\partial^2\Psi}{\partial x^2}+V(|\Psi|^2)\Psi=0. \tag{3}$$

Suppose that Eq. (3) has solutions that are localized and at rest in space:

$$\Psi=\bar{\Psi}_0(x)\exp(i\omega_0 t+i\varphi_0), \tag{4}$$

where $\bar{\Psi}_0$ is a real function that satisfies the ordinary differential equation

$$\frac{d^2\bar{\Psi}_0}{dx^2}+\frac{\omega_0^2-V(\bar{\Psi}_0^2)}{c^2}\bar{\Psi}_0=0,\quad \int_{-\infty}^{+\infty}\bar{\Psi}_0^2\,dx\neq\infty. \tag{5}$$

Then, in view of its Lorentz invariance, Eq. (3) (along with Eq. (4)) also defines a complete set of traveling solitons,

$$\Psi=\bar{\Psi}_0\left(\frac{x-vt}{\sqrt{1-v^2/c^2}}\right)\exp\left(i\omega_0\frac{t-vx/c^2}{\sqrt{1-v^2/c^2}}+i\varphi_0\right), \tag{6}$$

whose velocity v is lower than c $(v<c)$. In the solution (6), ω_0 is the frequency of field oscillations in the soliton's "proper" reference frame ($t'=(t-vx/c^2)/\sqrt{1-v^2/c^2}$ and $x'=(x-vt)/\sqrt{1-v^2/c^2}$ are the soliton's "proper" time and position); $\omega=\omega_0/\sqrt{1-v^2/c^2}$, $k=\omega v/c^2$, and $v_f=c^2/v$ are the frequency, wave number, and phase velocity of the soliton in the laboratory reference frame; and φ_0 is an arbitrary constant phase shift.

By way of an example we take a medium with cubic nonlinearity

$$V(\bar{\Psi}_0^2)=\omega_p^2(1-\alpha\bar{\Psi}_0^2) \tag{7}$$

characterized by two parameters, ω_p^2 and α (for a plasma ω_p^2 is the square of the plasma frequency, and α is determined by the type of nonlinearity). Here $\bar{\Psi}_0(x')$ has the simple analytic form

$$\bar{\Psi}_0(x')=\sqrt{\frac{2}{\alpha}\left(1-\frac{\omega_0^2}{\omega_p^2}\right)}\Bigg/\cosh\frac{x'\sqrt{\omega_p^2-\omega_0^2}}{c}, \tag{8}$$

where the frequency of oscillations in the "proper" reference frame, ω_0, can vary between 0 and ω_p $(0<\omega_0^2<\omega_p^2)$.

We now return to the original equation (1) with the operator $\hat{V}$ defined by (2).

If the soliton solutions (6) exist with the potential $V=V(\mu x_0=\text{const},\mu t_0=\text{const},|\Psi|^2)$, it is natural to assume that in a smoothly inhomogeneous medium with slowly varying parameters $(\mu\to 0)$, certain pulsed signals that are close to (6) in structure can propagate, at least along finite propagation paths. Reasoning from this assumption, we solve Eqs. (1) and (2) in the form of a series that is asymptotic in μ:

$$A=(A_0(\eta,\mu t)+\mu A_1(\eta,\mu t)+\cdots)\exp(i\varphi). \tag{9}$$

Here the field's phase φ, which allows for linear corrections of the phase wavefront of the signal, is described by the following expression:

$$\varphi=\varphi_0+\int_0^t(\omega(\mu t')-k(\mu t')v(\mu t'))dt'-k(\mu t)\eta$$
$$+\mu\int_0^t(\omega_1(\mu t')-k_1(\mu t')v(\mu t'))dt'-\mu k_1(\mu t)\eta,$$

where $\varphi_0=\text{const}$; $\eta=x-x_0(t)$ is measured from the signal's center $x_0(t)$, whose propagation velocity is $v=dx_0/dt$; $\omega(\mu t)$ and $k(\mu t)$ are the frequency and wave number $(\partial\varphi/\partial x|_{\mu=0}=k)$; and $\mu\omega_1$ and μk_1 are small corrections to the frequency and wave number.

We plug (9) into (1) and (2). Then in zeroth-order perturbation theory in the small parameter μ, we have

$$\begin{cases}(v^2(\mu t)-c^2)\dfrac{\partial^2A_0}{\partial\eta^2}-(\omega^2(\mu t)-k^2c^2-V(\mu x_0(t),\\[2mm] \mu t,|A_0|^2)A_0=0,\quad 2i(\omega v-kc^2)\dfrac{\partial A_0}{\partial\eta}=0,\end{cases}$$

which implies

$$A_0=\Psi_0\left(\frac{\eta}{\sqrt{1-v^2(\mu t)/c^2}},\mu t\right),\quad k=\frac{\omega(\mu t)}{c^2}v(\mu t), \tag{10}$$

$$\omega=\frac{\omega_0(\mu t)}{\sqrt{1-v^2(\mu t)/c^2}}, \tag{11}$$

where Ψ_0 is the soliton solution of Eq. (5) with $V=V(\mu x_0(t),\mu t,\Psi_0^2)$.

Actually, the expressions (11) reflect the expected quasisoliton structure of the pulsed signals that at time t are at the point with coordinate $x_0(t)$. Zeroth-order perturbation theory provides no insight into propagation of the signal (11) and the variation of the "proper" frequency $\omega_0(\mu t)$. Hence we write the inhomogeneous equation for $A_1(\eta,\mu t)$ obtained via first-order perturbation theory in μ after plugging (9) into (1) and (2):

$$(v^2-c^2)\frac{\partial^2A_1}{\partial\eta^2}-(\omega^2-k^2c^2-V(\mu x_0(t),\mu t,|A_0|^2))A_1$$
$$+\left(\frac{\partial V(\mu x_0(t),\mu t,A_0^2)}{\partial A_0^2}\right)A_0^2(A_1+A_1^*)=F_2-iF_1, \tag{12}$$

$$F_1=\left(\dot{\omega}+\frac{c^2}{v}\dot{k}\right)A_0+2\omega\frac{\partial A_0}{\partial t}+2\frac{c^2}{v}\dot{k}\eta\frac{\partial A_0}{\partial\eta}$$
$$-\text{Im}(e^{-i\varphi}\hat{\delta}(A_0e^{i\varphi}))_{x=x_0(t)}+2(\omega_1v-k_1c^2)\frac{\partial A_0}{\partial\eta},$$

$$F_2=\dot{v}\frac{\partial A_0}{\partial\eta}+2v\frac{\partial^2A_0}{\partial\eta\partial t}-\frac{2c^2}{v}k\dot{k}\eta A_0-\left(\frac{\partial V}{\partial\bar{x}}\right)_{x=x_0(t)}\eta A_0$$
$$-\text{Re}(e^{-i\varphi}\hat{\delta}(A_0e^{i\varphi}))_{x=x_0(t)}+2\omega(\omega_1-k_1v)A_0.$$

The right-hand side of Eq. (12) is determined by the derivatives of the soliton parameters (11) with respect to the "slow" time $\bar{t}=\mu t$ and the "slow" coordinate $\bar{x}=\mu x$, and the left-hand side is the result of linearizing (10). The homogeneous equation that follows from (12) if we set F_1 and F_2 to zero has two solutions localized in η:

$$A_1^{(1)}=iA_0(\eta,\mu t), \quad A_1^{(2)}=\frac{\partial A_0(\eta,\mu t)}{\partial \eta}. \tag{13}$$

For the inhomogeneous equation (12) to have a solution localized in η, the functions $F_1(\eta,\mu t)$ and $F_2(\eta,\mu t)$, according to the Fredholm alternative theorem,[15] must be "orthogonal" to $A_1^{(1)}(\eta,\mu t)$ and $A_1^{(2)}(\eta,\mu t)$, respectively:

$$\int_{-\infty}^{+\infty} F_1(\eta,\mu t)A_0(\eta,\mu t)d\eta=0,$$
$$\int_{-\infty}^{+\infty} F_2(\eta,\mu t)\frac{\partial A_0(\eta,\mu t)}{\partial \eta}d\eta=0. \tag{14}$$

If these conditions are not met, the additional term $A_1(\eta,\mu t)$ builds up in a resonant manner in the process of the quasisoliton's motion, which rather rapidly leads to a breakdown of the representation (9) with $A_0(\eta,\mu t)$ in the form (11). We therefore require that the conditions (14) be met and substitute the specific expressions for F_1 and F_2. After integrating by parts several times with allowance for (11) and substituting $\eta=\xi\sqrt{1-v^2(\mu t)/c^2}$, $\bar{t}=\mu t$, and $\bar{x}=\mu x$ we arrive at the following expressions for the orthogonality conditions (14):

$$\frac{dN}{d\bar{t}}=-\chi, \tag{15}$$

$$\frac{d}{d\bar{t}}\frac{m_0 v}{\sqrt{1-v^2/c^2}}=\frac{1}{2}\left(\frac{\partial}{\partial \bar{x}}W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})\right)_{\bar{x}=\bar{x}_0,\bar{t}_1=\bar{t}}\sqrt{1-v^2/c^2}$$
$$+\frac{\omega_0 v}{c^2\sqrt{1-v^2/c^2}}\frac{dN}{d\bar{t}}-\Sigma, \tag{16}$$

where we have introduced new notation for the integral characteristics of the field and medium:

$$N=\omega\int_{-\infty}^{+\infty}A_0^2\,d\eta=\omega_0\int_{-\infty}^{+\infty}\Psi_0^2(\xi,\bar{t})d\xi, \tag{17}$$

$$m_0=\int_{-\infty}^{+\infty}\left(\left(\frac{d\Psi_0}{d\xi}\right)^2+\frac{\omega_0^2}{c^2}\Psi_0^2\right)d\xi, \tag{18}$$

$$W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})=\int_{-\infty}^{+\infty}\left(\int_0^{\Psi_0^2(\xi,\bar{t})}V(\bar{x},\bar{t}_1,\Psi_0^2)\right)d\xi, \tag{19}$$

$$\chi=\text{Im}\sqrt{1-v^2/c^2}\int_{-\infty}^{+\infty}(\Psi_0 e^{-i\varphi}\hat{\delta}\Psi_0 e^{i\varphi})\,d\xi, \tag{20}$$

$$\Sigma=\text{Re}\int_{-\infty}^{+\infty}\Psi_0\frac{\partial}{\partial\xi}(e^{-i\varphi}\hat{\delta}\Psi_0 e^{i\varphi})\,d\xi. \tag{21}$$

In what follows we call N the "number of quanta," m_0 the "rest mass," and $W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})$ the "effective potential energy" of the soliton-like signal (9). The quasisoliton's integral parameters N and m_0 and the "proper" frequency are expressed in terms of one another at any fixed values of the arguments $\bar{x}=\bar{x}_0(\bar{t})$ and $\bar{t}$ of the potential $V(\bar{x},\bar{t},\Psi_0^2)$. Both the number of quanta N and the rest mass m_0 are independent of the quasisoliton velocity, and they are completely determined by the proper frequency ω_0 and the properties of the medium at the field location.[2)]

The variation of N is related to the operator $\hat{\delta}$, which takes weak dissipative and nonlocal nonlinear processes into account. However, if this operator acts on the soliton $\Psi_0 e^{i\varphi}$ and does not change the phase φ of the field, χ in Eq. (15) vanishes ($\chi\equiv 0$) and N is conserved in the course of the quasisoliton's motion ($N=\text{const}$).

The left-hand side of Eq. (16) is written in a form common in relativistic dynamics (m_0 is the rest mass, $m=m_0/\sqrt{1-v^2/c^2}$ is the mass in the laboratory reference frame, and $p=mv$ is the momentum). Extending the analogy between the quasisoliton (9) and a particle, we can call the right-hand side of Eq. (16) a force. The first term in the force is expressed in terms of the $\bar{x}$-derivative of $W_{\text{eff}}(\bar{x},\bar{t},\bar{t})$ at point $\bar{x}=\mu x_0(\bar{t})$, which suggests interpreting W_{eff} as the effective potential energy of the signal. Equation (19), which at $\bar{t}_1=\bar{t}$ defines $W_{\text{eff}}(\bar{x},\bar{t},\bar{t})$, is obtained as a result of averaging the soliton's "potential energy density"

$$W=\int_0^{\Psi_0^2(\xi,\bar{t})}V(\bar{x},\bar{t},\Psi_0^2)d\Psi_0^2$$

over the coordinate ξ at time $\bar{t}$ and at point $\bar{x}=\mu x_0(\bar{t})$. The fact that, according to (19), $W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})$ formally depends on two independent times $\bar{t}_1$ and $\bar{t}$ will be needed in deriving the equation for the relativistic mass $m=m_0/\sqrt{1-v^2/c^2}$ that follows from Eqs. (15) and (16) (see below).

We now return to our discussion of the right-hand side of Eq. (16). The second term is related to the variation in the number of quanta (dN/dt). If we put the quantum mass equal to ω/c^2 and the rate at which the quanta disappear equal to dN/dt, then $(\omega c^2 dN/dt)v$ is the variation in the signal's momentum due to this disappearance. Finally, the last term in the relativistic force, Σ, emerges only because the response of the medium to the soliton field, $\hat{\delta}(\Psi_0 e^{i\varphi})$, has an additional term that is asymmetric with respect to this term. In particular, this term is finite if we allow for nonlocal nonlinear effects leading to "self-acceleration" or "self-retardation" of quasisolitons.

Note that the soliton mass $m=m_0/\sqrt{1-v^2/c^2}$ is not simply the product of the number of quanta N and their energy ω/c^2, which corresponds only to the last (second) term in the integral representation (18).

Equations (15) and (16) are sufficient, in principle, to describe the dynamics of the quasisoliton signal (9). By solving them we can find the trajectory $x_0(t)$ and the proper frequency $\omega_0(\mu t)$. However, for a more complete analogy with relativistic mechanics, in addition to Eqs. (15) and (16) we also write an equation for the quasisoliton mass m. Such an equation can be obtained by direct differentiation of $m(\bar{t})=m_0/\sqrt{1-v^2/c^2}$, using Eqs. (15), (16), (10), and (11). We will not perform this straightforward but cumbersome calculation here. Instead we give the final result:

$$\frac{d}{d\bar{t}}\frac{m_0}{\sqrt{1-v^2/c^2}}=-\frac{1}{2c^2}\left(\frac{\partial}{\partial t_1}W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})\right)_{\bar{x}=\bar{x}_0(t),\bar{t}_1=\bar{t}}$$

$$\times\sqrt{1-v^2/c^2}+\frac{\omega_0}{c^2\sqrt{1-v^2/c^2}}\frac{dN}{d\bar{t}}-\frac{v}{c^2}\Sigma. \tag{22}$$

The right-hand side of Eq. (22) contains the partial derivative of $W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})$ with respect to $\bar{t}_1$, which vanishes in stationary media $(V(\mu x,\mu t,|E|^2)=V(\mu x,|E|^2))$. The second term gives the variation of the quasisoliton mass due to creation or annihilation of quanta with energy ω/c^2. The last term, which affects the mass of the signal, is proportional to the work done by the force Σ per unit time, or $v\Sigma$.

For a medium with the cubic nonlinearity (7) and $\alpha=\text{const}$, $\omega_p^2=\omega_p^2(\bar{x},\bar{t})$, the number of quanta N, the rest mass m_0, and the effective potential energy of a quasisoliton are

$$N=4\frac{c}{\alpha}\frac{\sqrt{\bar{\omega}_p^2-\omega_0^2}}{\bar{\omega}_p^2}\omega_0, \tag{23}$$

$$m_0=\frac{N}{c^2}\frac{2\omega_0^2+\bar{\omega}_p^2}{3\omega_0}, \tag{24}$$

$$W_{\text{eff}}=\frac{N}{\omega_0}\omega_p^2(\bar{x},\bar{t}_1)\left(1-\frac{2}{3}\frac{\bar{\omega}_p^2-\omega_0^2}{\bar{\omega}_p^2}\right), \tag{25}$$

where $\bar{\omega}_p^2=\omega_p^2(\bar{x}=\mu x_0(\bar{t}),\bar{t})$ is the square of plasma frequency at the location of the quasisoliton. If the duration of such a signal,

$$\tau=\frac{c}{v}\sqrt{\frac{1-v^2/c^2}{\bar{\omega}_p^2-\omega_0^2}},$$

is much larger than the period of oscillations $(2\pi/\omega)$, i.e., the quasimonochromaticity condition $\tau\omega\gg2\pi$ is met, the expressions (23), (24), and (25) for N, m_0, and W_{eff} simplify, and we can write, for $v\sim c$ and $\omega_p/\omega_0\to1$, the following approximate expressions:

$$N\simeq4\sqrt{2}\frac{c}{\alpha}\sqrt{\frac{\bar{\omega}_p-\omega_0}{\bar{\omega}_p}},\quad m_0\simeq\frac{N}{c^2}\bar{\omega}_p,$$
$$W_{\text{eff}}\simeq\frac{N}{\bar{\omega}_p}\omega_p^2(\bar{x},\bar{t}). \tag{26}$$

In a medium with cubic nonlinearity, a single value of N corresponds to two different values of the proper frequency $\omega_0^{(\pm)}$ and the rest mass $m_0^{(\pm)}$. From (23) and (24) we can easily obtain

$$\omega_0^{(\pm)}=A_{(\pm)}\bar{\omega}_p,\quad m_0^{(\pm)}=\frac{N\bar{\omega}_p}{3c^2}\frac{2A_{(\pm)}^2+1}{A_{(\pm)}}, \tag{27}$$

where

$$A_{(\pm)}=\frac{1}{2}\pm\frac{1}{2}\sqrt{1-\frac{\alpha^2N^2}{4c^2}}.$$

Here the number of quanta N cannot exceed the maximum permissible value $N_{\max}=2c/\alpha$ $(0<N\leqslant N_{\max})$, which implies equality of the proper frequencies $\omega_0^{(\pm)}$ and of the rest masses $m_0^{(\pm)}$:

$$\omega_0^{(\pm)}(N=N_{\max})=\frac{\bar{\omega}_p}{\sqrt{2}},$$
$$m_0^{(\pm)}(N=N_{\max})=\frac{2\sqrt{2}N\bar{\omega}_p}{3c^2}. \tag{28}$$

When $N\ll N_{\max}$, the two proper frequencies $\omega_0^{(\pm)}$ and the two rest masses $m_0^{(\pm)}$ differ considerably:

$$\omega_0^{(+)}\simeq\bar{\omega}_p,\quad \omega_0^{(-)}\simeq\frac{N}{2N_{\max}}\bar{\omega}_p\ll\omega_0^{(+)},$$
$$m_0^{(+)}\simeq\frac{N\bar{\omega}_p}{c^2},\quad m_0^{(-)}\simeq2\frac{N_{\max}\bar{\omega}_p}{c^2}\gg m_0^{(+)}. \tag{29}$$

The "light" quasisoliton with rest mass m_0^+ satisfies the quasimonochromaticity condition $(\tau\omega\gg2\pi)$, while the "heavy" quasisoliton with rest mass m_0^- does not.

3. INHOMOGENEOUS NONSTATIONARY MEDIUM WITH LOCAL NONLINEARITY

Let the operator $\hat{\delta}$ in (2) be identically zero ($\hat{\delta}\equiv0$) or, in other words, let the medium be locally nonlinear and nonabsorbing. This means that the number of quanta N is conserved ($N=\text{const}$) and that Eqs. (16) and (22) describing the dynamics of a quasisoliton signal assume the following form:

$$\frac{d}{d\bar{t}}\frac{m_0v}{\sqrt{1-v^2/c^2}}=-\frac{\sqrt{1-v^2/c^2}}{2}\frac{\partial}{\partial\bar{x}}W_{\text{eff}}(\bar{x},\bar{t},\bar{t})\bigg|_{\bar{x}=\bar{x}_0(\bar{t})},$$
$$\frac{d}{d\bar{t}}\frac{m_0}{\sqrt{1-v^2/c^2}}=\frac{\sqrt{1-v^2/c^2}}{2c^2}\frac{\partial}{\partial\bar{t}_1}W_{\text{eff}}(\bar{x}_0(\bar{t}),\bar{t}_1,\bar{t})\bigg|_{\bar{t}_1=\bar{t}}. \tag{30}$$

We see that here the two simplest limiting cases are an inhomogeneous stationary medium (in which case $\partial W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})/\partial\bar{t}_1=0$) and a homogeneous nonstationary medium (in which case $\partial W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})/\partial\bar{x}=0$), whereupon the quasisoliton's mass $m=m_0/\sqrt{1-v^2/c^2}$ or momentum $p=vm=vm_0/\sqrt{1-v^2/c^2}$ is conserved, respectively.

In a stationary situation ($m=\text{const}$) the medium "does not perform work on the quasisoliton," and the medium's inhomogeneity only changes the velocity v, which for media with cubic nonlinearity is equivalent to preservation of the frequency ω ($\omega=\text{const}$) and alteration of the wave number

$$k=\frac{\omega v}{c^2}=\frac{\omega}{c}\sqrt{1-\frac{\omega_0^2}{\omega^2}}=\frac{\omega}{c}\sqrt{1-A_{(\pm)}^2\frac{\bar{\omega}_p^2}{\omega^2}}.$$

Here the turning point (point of reflection) of such quasisolitons can be found from the condition $k=0$ (or $v=0$). Its location depends not on the number of quanta N but on the rest mass m_0. If $N\ll N_{\max}$, then as (29) clearly shows, the "light" quasimonochromatic signals are reflected from the point x^* where $\omega_p(x^*)=\omega$, while the "heavy" ones are

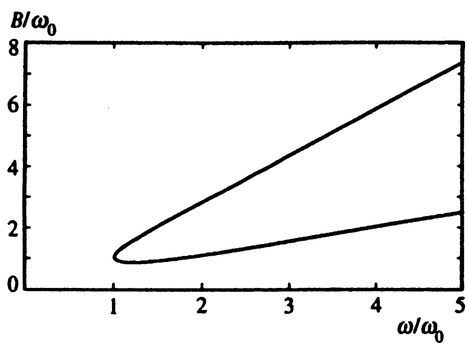

FIG. 1. The dependence of B/ω_0 on ω/ω_0 at $u^2/c^2=0.5$.

reflected from the layers where $\omega_p=2N_{\max}\omega/N\gg\omega$. When $N=N_{\max}$, the condition for reflection has the form $\omega_p=\omega/\sqrt{2}$, which corresponds to a supercritical plasma with concentration twice the critical value.

In a homogeneous nonstationary medium the quasisoliton's "momentum" $p=mv$ is conserved ($p=$const) but the mass m changes. For media with cubic nonlinearity this means that the wave vector $k=\omega v/c^2$ remains constant ($k=$const); however, there is a change in frequency ω described by $\omega^2=\omega_0^2+c^2k^2$, where $\omega_0(\bar{t})$ is the proper quasisoliton frequency proportional to $\bar{\omega}_p$ [the proportionality factor $A_{(\pm)}$ depends on the number of quanta N and the "heaviness" of the quasisoliton, which is determined by the choice of sign, plus or minus, in Eqs. (27)].

One more example considered here is the interaction of a quasisoliton signal with a wave in the parameters of the medium, traveling with constant velocity.[3] For the sake of definiteness we restrict our study to media with cubic nonlinearity, for which

$$\omega_p=\omega_p(\bar{x}+u\bar{t}),\quad u=\text{const},\quad \alpha=\text{const}. \tag{31}$$

Allowing for (31) and (23)–(25), we can easily integrate Eqs. (30) and obtain the following relationship linking the quasisoliton velocity v and the quasisoliton coordinate $\bar{x}_0$:

$$\omega_p(\bar{x}+u\bar{t})\frac{1+vu/c^2}{(1-v^2/c^2)^{1/2}}=\text{const}. \tag{32}$$

Bearing in mind that $v^2/c^2=1-\omega_0^2/\omega^2$ and $\omega_0\sim\omega_p(\bar{x}_0+u\bar{t})$, we are able to transform (32) into

$$\omega\pm\frac{u}{c}(\omega^2-\omega_0^2)^{1/2}=B=\text{const}, \tag{33}$$

where B is a constant, and the $\pm$ signs indicate the direction of motion of the soliton and the wave in the parameters of the medium (the + sign corresponds to motion in opposite directions, with $uv>0$, and motion in the same direction to the − sign, when $uv<0$).

Figure 1 depicts the dependence of B/ω_0 on $\omega/\omega_0=(1-v^2/c^2)^{-1/2}$. Note that ω is the quasisoliton's frequency in the laboratory reference frame, and ω_0 is the quasisoliton's proper frequency, proportional to ω_p. The upper part of the diagram describes the quasisoliton's motion opposite the motion of the parameter wave ($uv>0$); the lower part applies when the two are moving in the same direction ($uv<0$). Asymptotically both curves become straight lines with slopes $1+u/c$ and $1-u/c$, respectively. If initially the quasisoliton was moving toward the wave of medium parameters, $\omega_p(\bar{x}+u\bar{t})$, away from an extremely transparent plasma, where both ω_p and ω_0 are much lower than $\omega=\omega_1$, it will slow down as ω_p increases and its frequency ω will increase monotonically (see Fig. 1). At $\omega=B=\omega_1(1+u/c)$ the quasisoliton velocity v vanishes, with $\omega=\omega_0$. After passing the turning point the quasisoliton begins to move in the same direction as the wave $\omega_p(\bar{x}+u\bar{t})$, but its velocity is still lower than u. In the process it penetrates layers of the plasma with ever increasing density.

The greatest depth of plasma penetration of the quasisoliton can be found from the condition $\omega_0=B/(1-u^2/c^2)^{1/2}=\omega_1((1+u/c)/(1-u/c))^{1/2}$. Then the quasisoliton overtakes the wave of medium parameters and, "sliding" off it into the region of tenuous plasma, finally acquires the frequency $\omega_2=\omega_1(1+u/c)/(1-u/c)$ corresponding to the ordinary Doppler shift.

The above suggests that the quasisoliton behaves like a linear signal described by spacetime geometrical optics (see Ref. 4), the only difference being that the plasma frequency ω_p must be replaced by the proper frequency ω_0, which is related to ω_p by Eq. (27) and can assume two distinct values (both lower than ω_p) for each value of N. The layers of the plasma that the quasisoliton penetrates are denser than those penetrated by the corresponding ray of linear spacetime geometrical optics.

In principle, the wave $\omega_p(\bar{x}+u\bar{t})$ can move with a velocity $u>c$. The quasisoliton, and for that matter any other signal, is not reflected by such a wave (at all frequencies the quasisoliton velocity is lower than c). If the wave $\omega_p(\bar{x}+u\bar{t})$ is a plasma concentration barrier with a jump in ω_p from $\omega_p^{(1)}$ to $\omega_p^{(2)}$, where for the sake of definiteness we assume that $\omega_p^{(2)}$ is much higher than both $\omega_p^{(1)}$ and ω_1 (here ω_1 is the initial quasisoliton frequency, and $u>c$), then after interacting with such a wave the quasisoliton frequency will be close to $\omega_2=\omega_0^{(2)}/(1-c^2/u^2)^{1/2}$, where $\omega_0^{(2)}=A_{(\pm)}\omega_p^{(2)}$, and will be independent of the initial value ω_1.

4. CONVERSION OF QUASISOLITON FREQUENCY INITIATED BY NONLOCAL NONLINEAR PROCESSES

We analyze the self-action of quasisoliton signals in initially stationary homogeneous media,

$$\frac{\partial}{\partial\bar{t}_1}W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})=0,\quad \frac{\partial}{\partial\bar{x}_1}W_{\text{eff}}(\bar{x},\bar{t}_1,\bar{t})=0,$$

which simultaneously exhibit both local and nonlocal nonlinear properties ($\hat{\delta}\neq 0$). By way of example, we examine the propagation of relativistic quasisolitons[16] in a plasma in the presence of weak additional ionization. It is convenient to express the electric field **E**, the magnetic field **H**, and the velocity $\mathbf{v}_e$ of ordered electron motion in this problem in terms of the vector potential $\mathbf{A}(x,t)=A_y(x,t)\mathbf{y}_0+A_z(x,t)\mathbf{z}_0$:

$$\mathbf{H}=\operatorname{curl}\mathbf{A},\quad \mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\quad \mathbf{v}_e=\frac{e}{mc}\mathbf{A}, \tag{34}$$

where m_e and e are the electron mass and charge.

We ignore bulk electron losses during the passage of the pulse and describe the relative perturbation of plasma concentration due to additional ionization, $\tilde{n}=\Delta n/n$, where n is the plasma concentration, with the following model equation:[17]

$$\frac{\partial\tilde{n}}{\partial t}=\nu_0\frac{(\mathbf{E})^2}{E_i^2}, \tag{35}$$

where ν_0 is a coefficient with dimensions of frequency, and E_i is the characteristic field, which governs the intensity of additional ionization. By assuming that the relative perturbation of the electron mass caused by relativistic motion, $\tilde{m}=v_e^2/2c^2$, is much larger than $\tilde{n}$ $(\tilde{m}\gg\tilde{n})$, which means that nonlinear relativistic effects are much stronger than ionization effects, we can easily derive the following equation for $\mathbf{A}(x,t)$:

$$\frac{\partial^2\mathbf{A}}{\partial t^2}-c^2\frac{\partial^2\mathbf{A}}{\partial x^2}+\omega_p^2\left(1-\frac{1}{2}\left(\frac{e\mathbf{A}}{m_ec^2}\right)^2\right)\mathbf{A}+\omega_p^2\times\int_{-\infty}^{t}\tilde{n}\frac{\partial\mathbf{A}}{\partial t}dt=0, \tag{36}$$

where

$$\mathbf{j}_i=\frac{c}{4\pi}\omega_p^2\int_{-\infty}^{t}\tilde{n}\frac{\partial\mathbf{A}}{\partial t}dt$$

is the current of the electrons produced with zero velocity[6,8] and $\omega_p=4\pi e^2N/m_e$ is the plasma frequency.

For a circularly polarized wave, in which $A_y=|A|\cos\varphi$ and $A_z=|A|\sin\varphi$, instead of a vector equation for $\mathbf{A}$ we can write an equivalent equation for the complex-valued amplitude $A=A_y+iA_z=|A|\exp(i\varphi)$:

$$\frac{\partial^2A}{\partial t^2}-c^2\frac{\partial^2A}{\partial x^2}+\omega_p^2\left(1-\frac{1}{2}\left|\frac{eA}{m_ec^2}\right|^2\right)A+\mu\hat{\delta}A=0, \tag{37}$$

where

$$\mu\hat{\delta}A=\omega_p^2\int_{-\infty}^{t}\tilde{n}\frac{\partial A}{\partial t}dt,\quad \frac{\partial\tilde{n}}{\partial t}=\frac{\nu_0}{c^2E_i^2}\left|\frac{\partial A}{\partial t}\right|^2.$$

According to Eqs. (37) and (15)–(21), when relativistic quasisolitons propagate in a plasma, additional weak polarization results in a simultaneous decrease in the number of quanta N and conversion of the central frequency ω of the signal, since $\chi\neq0$ and $\Sigma\neq0$. If Eqs. (15), (16), and (22) are applied in the quasimonochromatic limit ($\omega\tau\gg1$, with τ the duration of the signal),[4)] we easily obtain the following equations for N and ω:

$$\frac{dN}{dt}=-2\gamma N^3,\quad \frac{d\omega}{dt}=\gamma\omega N^2, \tag{38}$$

where

$$\gamma=\frac{1}{24}\frac{\nu_0e^2\omega_p^2}{E_i^2m_e^2c^8}.$$

The solution of the system (38) with initial conditions $N(t=0)=N(0)$ and $\omega(t=0)=\omega(0)$ is

$$N=\frac{N(0)}{(1-4\gamma N^2(0)t)^{1/2}},\quad \omega=\omega^{(0)}(1+4\gamma N^2(0)t)^{1/4}. \tag{39}$$

We see that additional ionization can considerably increase the frequency of a soliton-like signal whose field structure is preserved by relativistic nonlinearity. The signal remains quasimonochromatic, and as the central frequency of the signal increases, the signal's temporal spectrum narrows and its duration increases accordingly. Significant conversion of the signal frequency without broadening the temporal spectrum sets the problem considered here apart from that studied in Ref. 17, where ionization nonlinearity dominates and no quasisolitons can exist.

5. CONCLUSION

The self-consistent description of one-dimensional soliton-like signals in smoothly homogeneous and nonstationary nonlinear media developed in this work reduces the wave problem to the solution of a closed system of ordinary differential equations for the quasisoliton velocity and the integral characteristics of the wave field. It has proved possible to write this system of equations in a familiar relativistic form, and to interpret the quasisoliton as a set of coupled quanta to which an integral parameter, acting as the effective mass, can be assigned. The number of quanta, the effective mass, and the quasisoliton velocity uniquely determine the field's frequency. The theory makes it possible to examine specific problems, such as the possibility of conversion of the carrier frequency of pulsed wave signals without broadening their temporal spectra in media with combined local and nonlocal nonlinearities, and the interaction of quasisolitons with waves in the parameters of the medium.

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1)Here are we are dealing with the "traditional" quasioptics method, in which the carrier frequency and the field's "central" wave vector are fixed. The ideas of generalizing the quasioptical approximation to arbitrary smoothly inhomogeneous stationary media proposed in Refs. 13 and 14 are close to the situation analyzed in the present work.

2)Multiplying the real and imaginary parts of Eq. (12) by $A_0(\eta,\mu t)$ and $\partial A_0(\eta,\mu t)/\partial\eta$ and integrating the result with respect to η from $-\infty$ to $+\infty$, we arrive at relationships determining the small corrections to the frequency ω_1 and the wave number k_1 in terms of the integral perturbation characteristics $e^{-i\varphi}\hat{\delta}(A_0e^{i\varphi})$ and μA_1. If the conditions (15) and (16) are met, there is no buildup of these corrections in the course of quasisoliton's motion.

3)In the linear limit these aspects are examined in Refs. 1–3.

4)The expression for $\mu\hat{\delta}A$ in (37) simplifies for a quasimonochromatic signal. After integrating by parts twice we find that

$$\mu\hat{\delta}A=\omega_p^2\left(\tilde{n}A-\int_{-\infty}^{t}\frac{\partial\tilde{n}}{\partial t}Adt\right)=\omega_p^2\left(\tilde{n}A+\frac{1}{i\omega}\frac{\partial\tilde{n}}{\partial t}A+O(\omega^2\tau^2)\right),$$

where $O(\omega^2\tau^2)$ stands for terms that are second order in $\omega\tau$.

[1] L. A. Ostrovskiĭ and N. S. Stepanov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **14**, 489 (1971).
[2] N. S. Stepanov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **19**, 960 (1976).
[3] V. I. Semenova, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **10**, 1077 (1967); **15**, 665 (1972); Fiz. Plazmy 3, 824 (1977).
[4] Yu. A. Kravtsov and Yu. I. Orlov, *Geometrical Optics of Inhomogeneous Media* [in Russian], Nauka, Moscow (1980).
[5] A. I. Smirnov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **24**, 1503 (1981).
[6] V. B. Gil'denburg, V. A. Krupnov, and V. E. Semenov, Pis'ma Zh. Tekh. Fiz. **14**, 1695 (1988) [Sov. Phys. Tech. Phys. Lett. **14**, 738 (1988)].
[7] A. V. Kim, S. F. Lirin, A. M. Sergeev, E. V. Vanin, and L. Stenflo, Phys. Rev. A **42**, 2493 (1990).
[8] V. B. Gildenburg, A. V. Kim, V. A. Krupnov *et al.*, IEEE Trans. Plasma Sci. **21**, 34 (1993).
[9] S. C. Wilks, J. M. Dawson, W. B. Mori, T. Katsouleas, and M. E. Jones, Phys. Rev. Lett. **62**, 2600 (1989).
[10] V. A. Mironov, A. M. Sergeev, E. V. Vanin, G. Brodin, and J. Lundberg, Phys. Rev. A **46**, R6178 (1992).
[11] V. V. Kurin and G. M. Fraĭman, Fiz. Plazmy **7**, 716 (1981).
[12] E. M. Gromov, L. V. Piskunova, and V. I. Talanov, Zh. Éksp. Teor. Fiz. **108**, 36 (1995) [JETP **81**, 19 (1995)].
[13] A. I. Smirnov and G. M. Fraĭman, Zh. Éksp. Teor. Fiz. **83**, 1287 (1982) [Sov. Phys. JETP **56**, 737 (1982)].
[14] G. V. Permitin and A. I. Smirnov, Zh. Éksp. Teor. Fiz. **109**, 736 (1996) [JETP **82**, 395 (1996)].
[15] G. Korn and T. Korn, *Mathematical Handbook*, 2nd ed., McGraw-Hill, New York (1968).
[16] V. A. Kozlov, A. G. Litvak, and E. V. Suvorov, Zh. Éksp. Teor. Fiz. **76**, 148 (1979) [Sov. Phys. JETP **49**, 75 (1979)].
[17] Yu. P. Raizer, *Laser-Induced Discharge Phenomena*, Consultants Bureau, New York (1977).

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