Dynamics of charged particles in the field of an intense transverse electromagnetic wave

V. A. Buts and A. V. Buts

Khar'kov Physicotechnical Institute, National Scientific Center, 310108 Khar'kov, Ukraine
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We give a rigorous solution that describes, in the laboratory reference frame, the dynamics of charged particles moving in the field of a fast or slow transverse electromagnetic wave with an arbitrary field strength. We determine the size of the interaction region where the particles effectively exchange energy with the field, and determine the dependence of the frequency of oscillations of passing and captured particles on the wave's field strength. We also formulate the conditions in which the particles are accelerated by the wave and the conditions in which the particles give up some of their energy to the wave. Finally, we show that even in a single interaction with the field, the transfer of energy from a flux of particles with a large energy spread to short-wave radiation can be highly efficient. © *1996 American Institute of Physics.* [S1063-7761(96)00309-5]

1. INTRODUCTION

The interaction of transverse electromagnetic waves and charged particles is a fundamental process lying at the base of many phenomena that emerge in the interaction of radiation and matter. The literature devoted to studies of this interaction is vast. The results of such studies can be found in books, reviews, and monographs (see, e.g., Refs. 1–4 and the literature cited therein). A subject thoroughly studied in this respect is the dynamics of particles interacting with the field of a wave whose amplitude is small. By a small-amplitude wave we mean a wave whose nonlinearity parameter $\mathscr{E}\equiv eE/mc\omega$ is small. Here E and ω are the amplitude and frequency of the wave, e and m are the particle's charge and mass, and c is the speed of light. Advances in laser physics allow for pump waves in which $\mathscr{E}\gg 1$. Such fields make it possible, among other things, to construct new compact accelerators with exceptionally high acceleration rates. Basically, it was the possibility of employing intense laser fields to accelerate charged particles that stimulated research in the dynamics of particles in laser fields. Such research started immediately after the advent of lasers. Apparently the first work in this field was Ref. 5, while the latest are Refs. 6 and 7. Most researches have analyzed acceleration patterns in electromagnetic fields of a fairly complicated structure. For this reason, e.g., in Ref. 8, the analysis of the dynamics of particles in high-intensity fields was done numerically, while analytical studies required using various small parameters. The most natural and commonly used smallness parameter is the nonlinearity parameter $\mathscr{E}$. Rigorous solutions are analyzed infrequently. The reason is that only for the simplest field configurations can such solutions be found. But even here the formulas provide a solution only in implicit form, which complicates analysis. For instance, in Ref. 1 a rigorous solution was found (in the reference frame in which the particle on the average is at rest) for the problem of a charged particle moving in the field of a plane electromagnetic field in a vacuum. The form of these solutions is not always convenient, especially for analyzing the interaction with the field not of a single particle but of a flux of particles, since different reference frames must be assigned to different particles, depending on the phase with which a particle enters the interaction region. Using the Hamilton–Jacobi equation in the same way as it was done in Ref. 1, we arrive at rigorous solutions in any reference frame, including the laboratory one. Much effort is required to obtain similar solutions for the problem in which the particles move in the field of a slow wave. However, as will be shown shortly, these solutions can be obtained in a simpler manner directly from the equations of motion, without resorting to the Hamilton–Jacobi equation.

In this paper we obtain the main formulas describing, in the laboratory reference frame, the dynamics of particles moving in the field of a fast or slow transverse electromagnetic waves of arbitrary strength and polarization. The formulas are used to determine the conditions for effective energy exchange between wave and particles. We will show that, notwithstanding the implicit nature of the exact solutions, their analysis makes possible a better understanding of the various aspects of particle dynamics, especially for particle motion in high-intensity fields, where the nonlinearity parameter $\mathscr{E}$ is not small.

The basic integrals of the motion are obtained in Sec. 2. In Sec 3 we analyze the dynamics of particle motion in a vacuum, find the dimensions of the interaction region needed for efficient energy exchange between wave and particles, and determine the dependence of the oscillation period on the field strength. There we also formulate the conditions in which the particles are accelerated or give up some of their energy to the wave. In Sec. 4 we derive the main expressions describing the dynamics of particle motion in the field of a slow wave. The conditions needed for the particles to be captured are also determined, and so are the frequencies of the capture oscillations. Phase stability of particles in a wave with a variable phase velocity is described in Sec. 5. Finally, the most important results are summarized and discussed in the Conclusion.

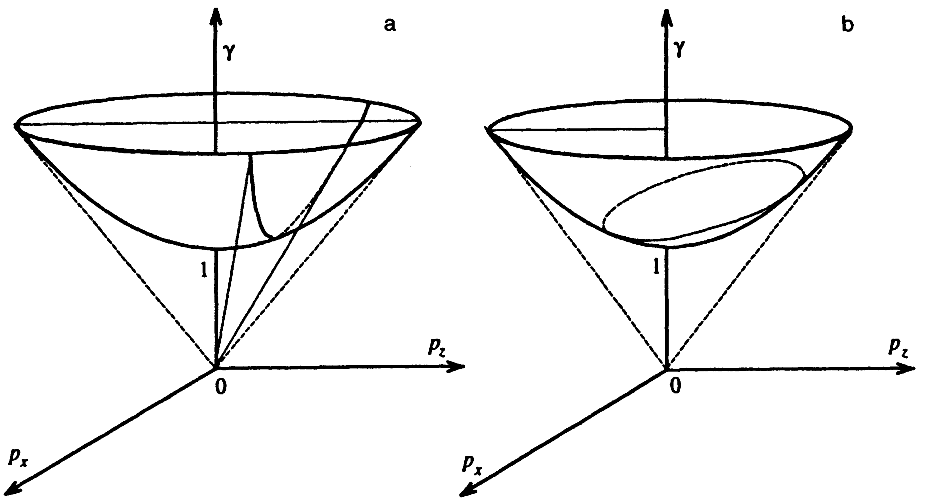

FIG. 1. Intersection of the hyperboloid $\gamma^2-p_x^2-p_z^2=1$ with the integral of longitudinal motion $p_z-k\gamma=p_{z0}-k\gamma_0$ for two cases of particle–wave interaction: (a) the wave is fast ($k\leqslant 1$), and (b) the wave is slow ($k>1$).

2. BASIC EQUATIONS AND INTEGRALS OF THE MOTION

We start with a charged particle moving in the field of a plane electromagnetic wave of arbitrary polarization. The components of the electric and magnetic fields in such a wave can be written as

$$\mathbf{E}=\mathrm{Re}(\mathbf{E}_0 e^{i\psi}), \qquad \mathbf{H}=\mathrm{Re}\frac{[\mathbf{kE}]}{k_0}, \tag{1}$$

where $\psi\equiv\omega t-\mathbf{kr}$, $\mathbf{E}_0=\boldsymbol{\alpha}E_0$, with $\boldsymbol{\alpha}=\{\alpha_x,i\alpha_y,\alpha_z\}$ the wave's polarization vector, $k_0=\omega/c$, and ω and $\mathbf{k}$ are the wave's frequency and wave vector. We introduce the following dimensionless variables:

$$\mathbf{p}_1=\frac{\mathbf{p}}{mc}, \quad \mathbf{k}_1=\frac{\mathbf{k}}{k_0}, \quad \tau=\omega t, \quad \mathbf{r}_1=k_0\mathbf{r},$$

$$\mathscr{E}=\frac{e\mathbf{E}_0}{mc\omega}, \quad \mathbf{v}_1=\frac{\mathbf{v}}{c}, \quad v_{ph1}=\frac{v_{ph}}{c}=\frac{\omega}{kc}.$$

In these variables the equation of motion assumes the form (we drop the subscript "1")

$$\dot{\mathbf{p}}\equiv\frac{d\mathbf{p}}{d\tau}=\mathrm{Re}\{[(1-\mathbf{kv})\mathscr{E}+\mathbf{k}(\mathbf{v}\mathscr{E})]e^{i\psi}\}. \tag{2}$$

It is convenient to augment Eq. (2) by an equation that determines the particle energy and can be found from system (2):

$$\dot{\gamma}=\mathrm{Re}(\mathbf{v}\mathscr{E}e^{i\psi}), \tag{3}$$

where $\gamma=\sqrt{1+p^2}$ is the dimensionless particle energy (measured in units of mc^2).

Equations (2) and (3) have the following well-known integrals of the motion:

$$\mathbf{p}-\mathbf{k}\gamma+\mathrm{Re}(i\mathscr{E}e^{i\psi})=\mathbf{p}_0-\mathbf{k}\gamma_0+\mathrm{Re}(i\mathscr{E}e^{i\psi_0})$$
$$=\mathrm{const}=\mathbf{C}. \tag{4}$$

The subscript "0" labels the initial variables.

Without loss of generality, below we assume that the wave propagates along the z axis, i.e., $\mathbf{k}=\{0,0,k\}$.

For the case of a linearly polarized wave ($\alpha_y=0$) the equations of motion (2) and the integrals of the motion (4) yield another integral of the motion, important for further analysis:

$$(\gamma-\gamma_*)^2-\frac{p_x^2}{1-k^2}=(\gamma_0-\gamma_*)^2-\frac{p_{x0}^2}{1-k^2}, \tag{5}$$

where

$$k\neq 1, \qquad \gamma_*=\gamma_0\gamma_{ph}^2(1-v_{z0}v_{ph}), \qquad \gamma_{ph}^2=\frac{1}{1-v_{ph}^2}.$$

For the case of the interaction in a vacuum ($k=1$) we can write the integral of the motion (5) as

$$\gamma=\gamma_0+\frac{p_x^2-p_{x0}^2}{2(\gamma_0-p_{z0})}. \tag{6}$$

Let us find the region in the energy–momentum space (γ,p) within which the particles can move. We do this for a particle interacting with a linearly polarized wave ($\alpha_y=0$). Then $p_y=p_{y0}=\mathrm{const}$. Particle motion in this case is restricted by the condition

$$\gamma^2-p_x^2-p_z^2=1+p_{y0}^2.$$

The condition specifies a hyperboloid of two sheets. Only its upper sheet ($g\geqslant 1$), which is depicted in Fig. 1, carries physical meaning. The motion is restricted not only by the hyperboloid surface but also by the integral of the motion $p_z-k\gamma=p_{z0}-k\gamma_0$, i.e., real motion takes place along the section of the hyperboloid by this integral of the motion. Figure 1a depicts this section for the case where the particles interact with the field of a fast wave ($k\leqslant 1$), and Fig. 1b depicts the section of the same hyperboloid by the same integral of the motion for $k>1$, i.e., when the wave interacting with the particles is slow. Particle motion takes place along the line of intersection of the integral of the motion and the hyperboloid. While in the first case (Fig. 1a) the section is a hyperbola ($k<1$), in the second (Fig. 1b; $k>1$) it is an ellipse. At $k=1$ the intersection line is a parabola. Qualitatively the difference between the possible trajectories (closed or open) rests within a relatively simple fact: in a vacuum ($k=1$) and in the field of a fast wave ($k<1$) the particles can travel with respect to the wave, which can carry them along (see below), while a slow wave ($k>1$) can capture them. Equations (5) and (6) are the analytical expressions for the projections on the (p_x, γ) plane of the lines of intersection of the integrals of the motion and the hyperboloid.

3. INTERACTION IN A VACUUM

The most thorough analysis of the dynamics of the particle motion can be performed when the particles interact with the field in a vacuum. Bearing in mind the relation between the phase ψ and the integrals of the motion (4), we find that

$$\gamma\dot{\psi}=\text{const}=-C_z. \tag{7}$$

Combining (7) with the system of equations (2), we easily arrive at the following general expressions for the particle's momentum components and energy:

$$\begin{aligned} p_x&=p_{x0}+\mathscr{E}_x(\sin\psi-\sin\psi_0),\\ p_y&=p_{y0}+\mathscr{E}_y(\cos\psi-\cos\psi_0),\\ p_z&=p_{z0}\pm\frac{(p_x^2+p_y^2)-(p_{x0}^2+p_{y0}^2)}{2\gamma\dot{\psi}},\\ \gamma&=\gamma_0\pm(p_z-p_{z0}), \end{aligned} \tag{8}$$

where $\mathscr{E}_{x,y}\equiv eE_0\alpha_{x,y}/mc\omega$; the upper sign (+) in the expressions for γ and p_z corresponds to the case where the wave and particle propagate in the same direction ($k=1$), and the lower sign ($-$) to the case where they propagate in opposite directions ($k=-1$).

Combining (8) and (7), we can easily find the expressions for the particle coordinates and the period of particle oscillations in the field of the wave. To avoid cumbersome formulas, they are written below only for the particular case of a linearly polarized wave ($\alpha_y=0$):

$$x=x_0+\frac{1}{\gamma\dot{\psi}}[(\psi-\psi_0)(p_{x0}-\mathscr{E}_x\sin\psi_0)-\mathscr{E}_x(\cos\psi-\cos\psi_0)],\qquad y=y_0+\frac{p_{y0}(\psi-\psi_0)}{\gamma\dot{\psi}},$$

$$z=z_0+\frac{p_{z0}(\psi-\psi_0)}{\gamma\dot{\psi}}\pm\frac{1}{2(\gamma\dot{\psi})^2}\left\{(\psi-\psi_0)\left[\mathscr{E}_x^2\left(\frac{1}{2}+\sin^2\psi_0\right)-2\mathscr{E}_xp_{x0}\sin\psi_0\right]-\mathscr{E}_x^2\left[\left(\frac{1}{4}\sin 2\psi-\sin 2\psi_0\right)-2\sin\psi_0(\cos\psi-\cos\psi_0)\right]-2\mathscr{E}_xp_{x0}(\cos\psi-\cos\psi_0)\right\},$$

$$T=\frac{2\pi}{\gamma\dot{\psi}}\left\{\gamma_0+\frac{1}{2\gamma\dot{\psi}}\left[\mathscr{E}_x^2\left(\frac{1}{2}+\sin^2\psi_0\right)-2\mathscr{E}_xp_{x0}\sin\psi_0\right]\right\}. \tag{9}$$

In the reference frame in which the particle is on the average at rest Eqs. (8) and (9) yield

$$p_x=\mathscr{E}_x\sin\psi,\qquad p_y=0,\quad p_z=-\frac{\mathscr{E}_x^2\cos 2\psi}{4\gamma\dot{\psi}},$$

$$x=-\frac{\mathscr{E}_x\cos\psi}{\gamma\dot{\psi}},\qquad y=0,\quad z=-\frac{\mathscr{E}_x^2\sin 2\psi}{8(\gamma\dot{\psi})^2},$$

$$\gamma\dot{\psi}=\sqrt{1+\frac{1}{2}\mathscr{E}_x^2}.$$

The expressions coincide with those obtained in Ref. 1.

It is easily shown that in the same reference frame where the particle is on the average at rest the expressions for the momenta and coordinates of the particle in the case of circular polarization also coincide with those derived in Ref. 1.

Note that the choice of the reference frame where a particle is on the average at rest depends on the phase with which the particle enters the interaction region, i.e., on ψ_0. For this reason using such a reference frame is inconvenient if we are studying the energy exchange between a large number of particles and the field. It is much simpler to study the dynamics of the interaction using Eqs. (8) and (9) and the integral of the motion (7).

Let us now analyze the above expressions. First, they clearly suggest that the wave carries the particles along, which becomes especially evident when one examines the case of a wave interacting with particles that were initially at rest. Putting $p_{x0}=p_{z0}=0$ and $x_0=z_0=\phi_0=y_0=0$ in Eqs. (8) and (9), we find that

$$p_x=\mathscr{E}_x\sin\psi,\qquad p_z=\pm\frac{1}{4}\mathscr{E}_x^2(1-\cos 2\psi),$$

$$\gamma=1+\frac{1}{4}\mathscr{E}_x^2(1-\cos 2\psi),\qquad x=-\mathscr{E}_x\cos\psi,$$

$$z=\pm\left[\frac{\mathscr{E}_x\tau}{4+\mathscr{E}_x^2}-\frac{1}{8}\mathscr{E}_x^2\sin 2\psi\right],\qquad T=2\pi\left(1+\frac{1}{4}\mathscr{E}_x^2\right), \tag{10}$$

where T is the oscillation period. We see that, while oscillating, the particle is carried along by the wave. The oscillation period depends on the field strength and for $\mathscr{E}_x\gg 1$ is considerably longer than the wave's period. If the region in which the field interacts with the particles is large, Eqs. (8)–(10) can be averaged over the oscillations. As a result we obtain the average coordinates, energy, and momentum of the particle carried along by the wave. For a particle that was initially at rest ($p_0\ll 1$) these quantities are

$$\langle x\rangle=\langle p_x\rangle=0,\quad \langle p_z\rangle=\pm\frac{\mathscr{E}_x}{4},$$

$$\langle\gamma\rangle=1+\frac{\mathscr{E}_x^2}{4},\quad \langle z\rangle=\pm\frac{\mathscr{E}_x^2\tau}{4+\mathscr{E}_x^2}.$$

Now we analyze the problem in detail. Note that although the general expressions (8) and (9) appear to be simple, they are really complex because the phase ψ is a function of the initial phase ψ_0. Below we examine the most interesting particular cases amendable to simple analysis.

Let $p_{x0}=0$. Then, as Eqs. (8) imply, the energy and longitudinal momentum of a particle vary periodically but always remain larger than, or equal to, the initial values γ_0 and p_{z0}. For a relativistic particle ($\gamma_0\gg1$) the energy can vary within the following limits:

$$\gamma_0\leqslant\gamma\leqslant\gamma_0(1+4\mathscr{E}^2).$$

Employing the integral of motion (7), we can easily find the field–particle interaction time t_a or, which is the same, the length l_a of the field–particle interaction region in which the particle acquires the maximum energy $\gamma=\gamma_0(1+4\mathscr{E}^2)$. For instance, for $\gamma_0\gg1$ these two quantities are

$$t_a=T_0\gamma_0^2\left(1+\frac{3}{2}\mathscr{E}^2\right),\qquad l_a=\lambda\gamma_0^2\left(1+\frac{3}{2}\mathscr{E}^2\right),\qquad(11)$$

where λ and T_0 are the wavelength and period of the wave.

If the particle is nonrelativistic ($\gamma_0=1$), the maximum energy is $\gamma=1+2\mathscr{E}^2$. A particle acquires this energy over a distance of

$$l_a=\frac{1}{2}\lambda v_{z0}+\frac{3}{2}\lambda\mathscr{E}^2.$$

Thus, if the field–particle interaction region is limited (the characteristic size being l_a), then after the particle flux has travelled through this region it will on the average acquire some energy. The main drawback of this acceleration mechanism is that there is a spread in particle energy at the exit of the interaction region and that the particles have transverse velocity. The maximum transverse velocity at the exit is $|p_x|=2\mathscr{E}_x$, and in the course of the field–particle interaction the particle shifts in the transverse direction by $l<\lambda\gamma_0\mathscr{E}_x$.

Note that the optimum transverse size of the interaction region (see (11)) is simply the distance over which the phase of the wave on the electron path changes by π. For $\mathscr{E}^2\ll1$ this distance is $2\gamma^2$ times greater than one-half of the wavelength, $\lambda/2$, and is determined solely by kinematics and the relativistic Doppler effect. When $\mathscr{E}>1$, the effect of a particle being carried along by the wave begins to play an important role, and for $\mathscr{E}^2\gg1$ this can essentially increase the size of the region of effective energy exchange.

Suppose that $p_{x0}\neq0$ and that $2\mathscr{E}_x\ll p_{x0}\ll p_{z0}\sim\gamma_0\gg1$. The maximum variation of the particle energy, $\Delta\gamma=\gamma-\gamma_0$, is then determined by the following expression:

$$\Delta\gamma=\frac{4\gamma_0p_{x0}\mathscr{E}}{1+p_{x0}^2}.\qquad(12)$$

The size of the interaction region along the z axis in which the particle acquires or gives up an amount of energy equal to (12) is

$$l_z=\frac{\lambda\gamma_0^2}{1+p_{x0}^2}\left(1-\frac{2p_{x0}\mathscr{E}}{1+p_{x0}^2}\right).$$

Note that while at $p_{x0}=0$ a particle can only acquire energy after interacting with the field, in the case at hand it the can acquire or give up energy, i.e., the particle can be accelerated or the intensity of the wave can grow due to the energy from the particle beam.

For particles with nonzero transverse velocity at the entrance to the field–particle interaction region ($p_{x0}\neq0$) an important characteristic is the minimum transverse size of the interaction region needed for effective energy exchange. If the above inequalities hold, this size is

$$l_x=\frac{\lambda\gamma_0p_{x0}}{1+p_{x0}^2}.$$

More interesting from the practical viewpoint is the case where the wave interacts with a flux of particles rather than a single particle. If all the particle have the same velocity and are uniformly distributed in space, the efficiency of energy exchange between the particle flux and the field can be found by averaging the expression for the energy in (8) over the phases ψ_{0i} of the particles entering the region of interaction with the field. The difficulty of such averaging is caused by the fact that the phase of the ith particle is related to the initial phase ψ_{0i} implicitly. Generally, the relation between ψ and ψ_0 can easily be obtained from (9) if we specify either the longitudinal or the transverse dimensions of the interaction region. In some cases it is more convenient to relate ψ and ψ_0 through a fixed longitudinal size. For instance, if $p_{x0}=0$ and $\mathscr{E}_x^2\ll1$, from (9) we see that for z this relationship becomes quite simple (linear):

$$\psi=\psi_0+\frac{\gamma\dot{\psi}}{p_{z0}}(z-z_0).\qquad(13)$$

If in (13) we put $z_0=0$ and $\gamma_0\gg1$ and assume that the coordinate z is equal to the optimum size, $z=z_{\text{opt}}=2\pi\gamma_0^2$, the relation between the phase at the exit from the interaction region and the phase at the entrance to that region is especially simple:

$$\psi=\psi_0+\pi.$$

Using this relationship, we can easily find the average energy and the average momenta of the particles in the beam acquired by the particles after interacting with the field:

$$\langle\gamma\rangle=\gamma_0+2\gamma_0\mathscr{E}^2,\qquad\langle p_x\rangle=0.$$

The relationship between ψ and ψ_0 is also simple if $p_{x0}\gg\mathscr{E}_x$ for an arbitrary value of $\mathscr{E}_x$:

$$\psi=\psi_0+\xi+\varepsilon\xi\sin\psi_0,\qquad(14)$$

where

$$\varepsilon=\frac{\mathscr{E}}{p_{x0}},\qquad\xi\equiv\frac{1}{p_{x0}}(x-x_0)(\gamma_0-p_{z0})\geqslant\pi.$$

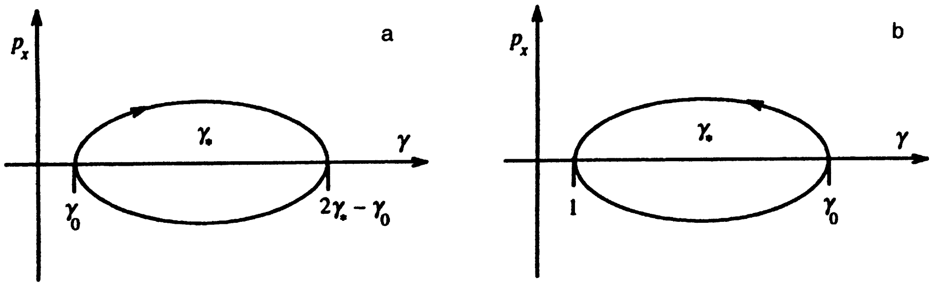

FIG. 2. Projection of the line of intersection of the energy hyperboloid and the integral of motion onto the (γ, p_x) plane when a particle interacts with a slow wave: (a) for $v_{ph} > v_0$, and (b) for $v_{ph} < v_0$. In the latter case the particle may give up all of its energy to the wave.

Plugging (14) into the expression for the energy in (8) and averaging over the initial phase, we arrive at the following expression for the average velocity:

$$\langle\gamma\rangle = \gamma_0 + \frac{\gamma_0(1+v_{z0})}{2(1+p_{x0}^2)}\left\{\mathscr{E}_x^2\left[1 - \cos\xi(J_0(\varepsilon\xi)+J_2(\varepsilon\xi)) - \frac{1}{2}J_2(2\varepsilon\xi)\cos 2\xi\right] - 2p_{x0}\mathscr{E}_x J_1(\varepsilon\xi)\sin\xi\right\}, \qquad (15)$$

where J_n is the nth order Bessel function of the first kind.

Note that in deriving (15) we only used the fact that the parameter ε is small, i.e., both $\varepsilon\xi$ and ε_x can be arbitrary.

Equation (15) shows, in particular, that by appropriate choice of the beam parameters and the size of the interaction region not only can the beam be accelerated but also the opposite process can be initiated, i.e., some of the beam energy can be given up to the wave. For instance, for $\mathscr{E}=0.3$, $\xi=5\pi/2$, and $v_{z0}\simeq 1$ from Eq. (15) it follows that the beam gives up 15% of its energy to the wave in the course of one pass through the interaction region. The minimum dimensions of the interaction region for this case are $l_z = 5\lambda\gamma_0^2/4$ and $l_x = 5\lambda\gamma_0/4$. An important feature of such energy transfer from particle beam to wave is the fact that the efficiency of such transfer is independent of the beam quality. Indeed, the acceptable beam energy spread $\delta\gamma$ can be estimated from the condition that $\delta\xi \leqslant \pi$. Combining this with the assumption that the dimensions of the interaction region are given yields $\delta\gamma \leqslant \gamma$.

4. INTERACTION OF PARTICLES WITH THE FIELD OF A SLOW WAVE

As noted earlier, the dynamics of particles interacting with the field of a slow ($k>1$) transverse electromagnetic wave may drastically differ from that of particles interacting with a fast wave ($k \leqslant 1$). Figures 1a and b clearly illustrate this distinction. The field of a slow wave may capture a particle, which manifests itself in the fact that the integral curves along which the particle moves become closed (see Eqs. (5) and (6)). Figures 2a and b depict the projections onto the (γ, p_x) plane of these curves for the particular cases of $p_{x0}=0$ with $1>v_{ph}>v_0$ and $p_{x0}=0$ with $1>v_0>v_{ph}$, respectively.

In the first case (Fig. 2a) the particle velocity is lower than the wave's velocity. The particle can only be accelerated, as in the case of interaction in a vacuum. Here, if $\mathscr{E}_x^2<(\gamma_0-\gamma_*)^2/4\gamma_0^2 v_{ph}^2$ the particles are not carried along by the wave, and the motion occurs only along the left half of the ellipse ($\gamma<\gamma_*$) and qualitatively is no different from the motion of particles that interact with the wave in a vacuum. But if the field strength in the wave is so high that $\mathscr{E}_x^2>(\gamma_0-\gamma_*)^2/4\gamma_{ph}^2 v_{ph}^2$ the particles are captured by the wave and move along the ellipse, i.e., they perform capture oscillations in the field of the transverse wave.

In the second case the particles overtake the wave ($v_0>v_{ph}$) and are slowed down. More than that, it can easily be shown that if the wave's phase velocity and the initial velocity of the particle are related through the formula

$$v_{ph} = \frac{v_{z0}\gamma_0}{\gamma_0+1}$$

and the particle proves to be captured, the particle gives up all of its energy to the wave ($\gamma=1$). Figure 2b corresponds to this case.

The dynamics of particles in the field of a slow wave is much more complicated for analysis than in the case of field-particle interaction in a vacuum. The primary reason is that (7) is no more an integral of motion. Instead we have

$$\gamma\dot{\psi} + (k^2+1)\gamma = -C_z. \qquad (16)$$

Below we give the main formulas describing the dynamics of a particle in the field of a slow wave. As with field-particle interaction in a vacuum, the expressions for the momenta and energy of the particle can easily be found for the general case:

$$p_x = p_{x0} + \mathscr{E}_{x0}(\sin\psi - \sin\psi_0),$$

$$p_z = p_{z0} + k(\gamma_* - \gamma_0)\left\{1 \pm \left[1 - \kappa^2(\sin\psi - \sin\psi_0)^2 - \kappa^2\frac{2p_{x0}}{\mathscr{E}_x}(\sin\psi - \sin\psi_0)\right]^{1/2}\right\}, \qquad (17)$$

$$\gamma = \gamma_0 + \frac{1}{k}(p_z - p_{z0}),$$

where

$$\kappa^2 = \frac{\mathscr{E}_x^2\gamma_{ph}^2 v_{ph}^2}{(\gamma_* - \gamma_0)^2}.$$

The upper sign (+) in front of the square brackets in (17) corresponds (for acceleration; Fig. 2a) to motion along the right half of the ellipse ($\gamma>\gamma_*$), and the lower sign (−) to the motion along the left half ($\gamma<\gamma_*$). The expressions for the particle coordinates can be found by integrating (17). Generally these are cumbersome expressions, but can easily be derived nevertheless. Rather than writing them in full, we give an idea of the characteristic dimensions of the region of effective energy exchange and the characteristic time scales by writing these expressions for the special case of $\psi_0=0$, $p_{x0}=0$, and $\psi\leqslant\frac{1}{2}\pi$:

$$z=z_0-\gamma_{ph}^2 v_{ph}\left[\psi-\left(1+\frac{v_{ph}p_{z0}}{\gamma_*-\gamma_0}\right)\right.$$
$$\left.\times\begin{cases}F(\psi,\kappa), & \kappa\leqslant 1\\ \kappa^{-1}F(\varphi,\kappa^{-1}), & \kappa>1\end{cases}\right], \quad (18)$$

$$x=x_0$$
$$-\frac{\mathscr{E}}{\kappa(k^2-1)(\gamma_*-\gamma_0)}\ln\frac{\{|\kappa\cos\psi+\sqrt{1-\kappa^2\sin^2\psi}|}{1+\kappa}, \quad (19)$$

$$\tau=-\frac{\psi}{k^2-1}+\frac{\gamma_*}{(\gamma_*-\gamma_0)(k^2-1)}\begin{cases}F(\psi,\kappa), & \kappa\leqslant 1,\\ \kappa^{-1}F(\varphi,\kappa^{-1}), & \kappa\geqslant 1,\end{cases} \quad (20)$$

where $F(\psi,\kappa)$ is the elliptic integral of the first kind,

$$\varphi=\arcsin(\kappa\sin\psi),\qquad \kappa^2\sin^2\psi\leqslant 1,$$

with $\kappa<1$ corresponding to the case of passing particles and $\kappa>1$ to the case of captured particles, and τ is the time it takes the particle to reach phase ϕ.

These formulas make it possible, among other things, to find the frequency of small phase oscillations of the captured particles by taking into account the symmetry of the motion. In such oscillations $\psi\ll 1$, $\kappa\gg 1$, $\kappa\sin\psi\simeq 1$, and $\varphi\leqslant\frac{1}{2}\pi$. Plugging these values into (20), we find $\omega_B=\omega\mathscr{E}_x/\gamma_{ph}^2 v_{ph}^2$. As we get closer to the separatrix separating the passing particles from the captured, we see that the frequency of the phase oscillations tends to zero:

$$\omega_B=\omega\mathscr{E}_x\frac{\gamma_{ph}}{\gamma_*}\frac{\pi}{2}\left(\ln\frac{4}{\sqrt{1-\kappa^2}}\right)^{-1}.$$

5. PHASE STABILITY

It would be interesting to establish the possibility of phase stability in a transverse wave and to find the special features of this phenomenon. To this end we assume that the parameters of the medium together with the wave's phase velocity slowly vary along the direction z in which the wave propagates. The phase of the wave can then be written as

$$\psi=\tau-\int_0^z k(z')dz'.$$

Clearly, both the equations of motion (2) and the integrals of the motion (4) retain their form. As in accelerator theory, we introduce the concept of a synchronous particle, whose velocity varies in the same way as does the phase velocity of the wave ($v_s=v_{ph}$).

Using the integrals of motion (4) to determine the phase ψ, we can easily derive the following equation:

$$\ddot{\psi}-\frac{\dot{k}}{k}\dot{\psi}-\frac{k_z}{\gamma^2}\mathscr{E}_x\left[\frac{\mathscr{E}_x}{2\gamma}(\sin 2\psi-\sin 2\psi_s)+\frac{1}{\gamma}(C_x\cos\psi\right.$$
$$\left.-C_{xs}\cos\psi_s)\right]=0. \quad (21)$$

We analyze only the behavior of small deviations of the phase from the synchronous. This means that we put $\psi=\varphi_s+\varphi$, with $\varphi\ll 1$, and linearize Eq. (21) with respect to φ. As a result we arrive at the equation of damped oscillations of a pendulum for determining φ:

$$\ddot{\varphi}+\frac{\dot{v}_{ph}}{v_{ph}}\dot{\varphi}+\Omega^2\varphi=0,$$

where

$$\Omega^2=\frac{1}{\gamma^2 v_{ph}^2}(1-v_{z0}v_{ph})\mathscr{E}_x(\mathscr{E}_x\cos^2\psi_s-p_{x0}\sin\psi_s).$$

An expression for the frequency of phase oscillations at $p_{x0}=0$ was derived in Ref. 9, while expressions for Ω^2 for $p_{x0}=0$ and $p_{x0}\gg\mathscr{E}_x$ were derived in Ref. 10.

6. CONCLUSION

An important conclusion that can be drawn from the above results is that restricting the region of field–particle interaction leads to efficient exchange of energy between field and particles. The following remark is apparently in order here. In Ref. 6 a categorical statement was made to the effect that it is impossible to accelerate charged particles with an electromagnetic field in a vacuum by a force proportional to the first power of the electric field strength E. This is of course true if the region where the particles and the field interact has no boundaries. But in reality the region is limited, and there can be efficient exchange of energy between wave and particles. The same result follows from the formulas of Ref. 6 if one assumes the field–particle interaction time to be finite ($0\leqslant t\leqslant T$) rather than infinite ($-\infty<t<\infty$), as is done in Ref. 6.

If the particle and wave move in the same direction ($p_{x0}=0$), the particle always acquires energy from the wave, with the energy increment proportional to the particle energy and the square of the electric field strength: $\Delta\gamma\propto\gamma_0\mathscr{E}^2$. Note that the quadratic dependence on the field strength is in no way related to the oscillatory motion, with the result that all restrictions on the particle energy related to bremsstrahlung, which limit the possibility of particles being accelerated by the Miller force,[6] are lifted. Moreover, when a ponderomotive potential accelerates a particle, the energy increment is inversely proportional to the energy ($\Delta\gamma\propto\mathscr{E}^2/\gamma_0$), with the result that at high energies ($\gamma_0\gg 1$) the energy-exchange mechanism considered here is more effective.

If $p_{x0}\neq 0$, a particle may be accelerated but it may also be slowed down. An important feature of the transfer of energy from particle to wave is the weak dependence of the efficiency of energy transfer on the beam energy spread ($\delta\gamma\sim\gamma$). This feature of the scheme of amplification of short-wave radiation has an advantage over other schemes of free electron lasers, in which the spread must be small ($\delta\gamma \leq 10^{-3}\gamma$ for amplifying a wave with $\lambda \simeq 10\ \mu$m; see Ref. 11). Amplifying shorter waves requires beams of even higher quality ($\delta\gamma \sim 10^{-4}\gamma$). This result can easily be explained. In traditional schemes of free electron lasers a self-consistent dynamics of particles and field is considered. Excitation (amplification) of a wave occurs as a result of development of a collective beam instability. The process begins at the linear stage, when the field amplitude is low. Here only high-quality beams can efficiently (with a hydrodynamic increment) excite short-wave radiation. In the scheme of energy transfer from particles to wave considered here the field of the wave does not change, i.e., the fixed-field approximation is valid, and the field strength is high so that the field completely determines the dynamics of the particles. Clearly, such an approximation is valid if in the course of the time it takes a particle to pass through the interaction region all variations in the field strength can be ignored. It is also assumed that the field is in a steady-state regime, i.e., the increment in field energy is balanced by the field leaving the interaction region. Thus, when such an approximation holds, low-quality beams ($\delta\gamma \leq \gamma$) can be used to excite short-wave radiation.

The above possibilities of energy exchange between particles and field are rough models of real schemes that can be employed. To bring the models closer to reality we must allow for inhomogeneity in the transverse structure of the field and study the process of energy exchange not in a single isolated interaction region but in a set of such regions organized in some manner. The results of such studies can be found in Refs. 7, 12, and 13. To accelerate charged particles, Sugihara[12,13] suggested using a laser flux consisting of a large number of parallel elementary beams, with the phase of each beam shifted by a certain value in relation to the phase of the previous beam. The effective field acting on particles that cross such a laser flux at a certain angle is equivalent to the field of a slow electromagnetic wave. Allowing for the inhomogeneous structure of the field of the elementary beams does not drastically change the dynamics of the particles. Another interesting way of organizing of the elementary regions was proposed by Apollonov *et al.*[7] In their method a laser beam is successively reflected from many mirrors mounted in a pattern similar to that of electrodes in a photomultiplier, and the path traveled by a laser beam is similar to that traveled by electrons in a photomultiplier. The accelerated particles move along the system's axis between the mirrors, successively intersecting each laser beam in its focal cross section. This method of organizing the elementary regions, with allowance for the above-described processes in each region, can apparently be used as a guide for building real accelerators and free electron lasers.

In addition to allowing for the transverse structure in real systems we must also analyze the dynamics of the particles at the entrance to, and exit from, the interaction region. In these local regions the field structure differs from that of a plane transverse electromagnetic wave. Indeed, even in such a region as that between the mirrors of an open cavity, the plane structure of the field is distorted in the vicinity of the openings in the cavity through which the particles get into the cavity.

As the laser field becomes more intense, the efficiency of energy exchange between particles and field grows, and so does the rate of particle acceleration. High field strengths are produced by focusing the radiation. A focused field is spatially inhomogeneous, which leads to the emergence of a ponderomotive force. This force must be taken into account since it may have a considerable influence on the dynamics of the accelerated particles and the properties of the medium. Generally, high field strengths of even a homogeneous field lead to destruction of the medium. Indeed, in experiments the intensity of focused laser radiation has reached a value of 3×10^{10} W cm^{-2} (see Ref. 14). The strength of the electric field of such radiation exceeds that of interatomic fields. The state of the medium in such fields changes—the medium is transformed into a plasma. Inevitable density fluctuations lead to the appearance of high-frequency pressure forces and to an increase in the degree of the medium's inhomogeneity. As a result, modulation instability develops. Hence an acceleration scheme with the medium (plasma) as an element has potential, apparently, only for pulsed mode operation. The pulse length in such schemes is limited by $T=\min\{d/\mathscr{E}c;\Gamma^{-1}\}$, where d is the diameter of the laser focus, and Γ is the modulation instability increment. These difficulties have led to a situation in which more and more attention is being paid to developing schemes of laser acceleration without any medium. One example is the inverted free electron laser. Apollonov *et al.*[7] developed a new approach to this problem that has many advantages over other approaches. They suggested using a combination of a laser field and an external static magnetic field to accelerate charged particles. The use of an external magnetic field to create conditions for effective energy transfer between particles and field has a long history. Many schemes of accelerating particles and generating electromagnetic waves have been developed over the years. However, Apollonov *et al.*[7] took into account an important physical fact, i.e., that if the magnitude of the external transverse static magnetic field exceeds a certain value, proportional to the particle energy, then the frequency of the laser field oscillations along the electron path varies with time. As a result, in passing through the interaction medium the particles, on the average, acquire energy. Estimates of the maximum increment in the particle energy in a single pass through the interaction region yield (in our variables) $\Delta\gamma \simeq \gamma_0 \mathscr{E}^2 \alpha$ ($\mathscr{E}<1$ and $\alpha>1$), which is a quantity of the same order as the one obtained in our scheme. There are three important features determining the advantages of the approach developed in Ref. 7. First, and the most important, is that the exchange of energy between particle and wave is not strictly resonant. Hence the limitations related to the nonlinear shift in the phase of the accelerating wave with respect to the particle (limitations critical in schemes of the inverted free electron laser type) are unimportant here. Second, acceleration takes place in a vacuum, and all the problems associated with the special features of the interaction of intense laser radiation with matter (with plasma, in particular) have no meaning. Third, the external transverse magnetic field can bring the accelerated particles back to the field–particle interaction region, i.e., multiple cyclic particle acceleration becomes possible. We have listed the advantages of the approach developed in Ref. 7 because they are inherent in the acceleration schemes that can be proposed on the basis of the results obtained in the present paper.

We would also like to emphasize the importance of taking into account the integrals of motion (5) or (6). In various schemes of acceleration of charged particles the complexity of the structure of the accelerating field forces one to analyze the longitudinal (phase) motion of particles and the transverse motion separately. For longitudinal motion it is sufficient in many cases to examine only one integral of the motion in (4). But in some cases the analysis of this integral of the motion without taking into account (5) or (6) may lead to an error. For example, let us consider the motion of a particle along the z axis, with the angle at which the slow electromagnetic wave ($k>1$) moves with respect to this axis being such that $k_z=1$. We analyze only one integral of the motion in (4), the one characterizing the longitudinal motion of a particle:

$$p_z-\gamma-\mathscr{E}_z \sin \psi=C_z.$$

This integral of the motion implies, in particular, that the particle can be accelerated without limit ($\gamma\rightarrow\infty$) by a field with a finite amplitude $\mathscr{E}_z$. The same integral of the motion can be used to find the conditions needed for securing the particles in the limitless acceleration regime and other characteristics of particle motion. But, as the integral of the motion (5) shows, limitless acceleration in this case is impossible: the paths in the (γ,p_x) plane are always closed and the quantities γ and p_x are always finite.

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