Casimir effect in the presence of an elastic boundary

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A boundary condition with dimensional parameter δ simulating the penetration depth is studied in a model of a scalar field ($D=1+1$ and $3+1$) defined in a half-space. The boundary condition gives a canonical energy–momentum tensor which differs by a divergence from the standard tensor. The divergence determines the surface energy and makes it possible to obtain a conserved total energy. Quantization of the model leads to a corpuscular picture, but the scalar "photon" now describes an excitation of both the field and the boundary. The expressions found for the vacuum energy–momentum tensor reveal a new structure of boundary divergences and in special cases lead to known results. The transition to the Dirichlet boundary condition (in the sense $\delta \to 0$) is nontrivial, a result of the fact that the model does not exist in the limit $\delta \to -0$. Specifically, it is found that not only does the divergence contribution to the vacuum energy–momentum tensor not vanish as $\delta \to +0$, but it introduces additional boundary singularities. A procedure is proposed for interpreting the boundary divergences under ideal boundary conditions. This procedure, on the one hand, solves the problem of the divergence contribution and the finiteness of the vacuum energy and, on the other, admits extension to the case $\delta > 0$, if a cutoff parameter is introduced into the theory. The magnitude and meaning of the cutoff parameter should be determined by the physical nature of the boundary. © *1996 American Institute of Physics.* [S1063-7761(96)00109-6]

1. INTRODUCTION

The local vacuum characteristics of a quantum field depend on the geometry and topology of the region where the field is defined. This dependence has been studied in different field models.[1–5] These studies revealed a number of difficulties of the theory which still have not been completely resolved. First, there is the problem of determining the vacuum energy: The vacuum energy density exhibits a nonintegrable singularity on a (flat) boundary.[1–5] This problem arises in conformally noninvariant models (in a space of dimension $D=3+1$). If the boundary is curved, the corresponding divergence also exists in conformally invariant models.[6,7] This is applies to theories with the very simple Dirichlet or Neumann boundary conditions (in the latter case, generally speaking, there is no conformal invariance[7]). In the case of two dielectrics, the energy density of the electromagnetic field also acquires a nonintegrable singularity at the interface, even if the interface is flat. Despite the fact that the field in such a system is not a vacuum field, the procedure for obtaining the energy–momentum tensor from the photon Green's function is still the same.[8,9] The photon Green's function satisfies more general boundary conditions than Dirichlet or Neumann conditions. This is important for understanding the origin of surface divergences.

There remains the problem of interpreting the sign of the energy density of the vacuum fluctuations of the electromagnetic field, for example, in the space between parallel plates.[1)] The negativeness of the energy density was particularly emphasized by DeWitt[1,5] as an example of the breakdown, in the quantum case, of the conditions of the Hawking–Penrose theorem "on the unavoidability of singularities" in the general theory of relativity. However, it has not been ruled out that the energy corresponding to this density is the part of the total energy that depends on the distance l between the conductors and determines the experimentally observed force. Deutsch and Candelas[6] have indicated the existence of corrections to this energy which do not depend on l but do depend on the details of the molecular structure of the boundary.

The solution of the problem of surface divergences in an electromagnetic field in the presence of a conducting, infinitely thin sphere is also, in my opinion, not entirely satisfactory. Although a careful investigation[10–12] has shown that the sum of the vacuum energies inside and outside the sphere is finite, each energy is itself infinite. This infinity will show up if the thickness of the sphere is finite.

It has been stated time and again[2,3,6,7] in connection with the problems listed above that the boundary divergences of the energy–momentum tensor can be eliminated by weakening the boundary conditions, taking account of the permeability of the boundaries at high frequencies. This makes it necessary to construct models of boundaries and to impose appropriate (more general) boundary conditions on the quantum field.

In the present paper, the simplest scalar model (in dimensions 1 + 1 and 3 + 1) is studied for a field defined in the half-space $x \geqslant 0$. The mixed boundary condition (Robin)[2)]

$$\partial_1 \phi(0) = \frac{1}{\delta} \phi(0), \tag{1}$$

corresponding to elastic clamping of the end of a "string" at the origin (the dependence on the time and the other coordinates, if they exist, is implied), is taken as the boundary condition. The parameter δ models the penetration depth and is assumed to be constant. The scalar model on the segment

TABLE I.

	$\langle T_{00}\rangle_{\delta\to+0}$	$\langle \text{div}\rangle_{\delta\to+0}$	$\langle \tilde{T}_{00}\rangle_{\delta\to+0}$
$D=2,\ m\neq0$	$-m^2(2\pi)^{-1}K_0(\rho)$	$-m^2(2\pi)^{-1}K_1'(\rho)$	$m^2(2\pi\rho)^{-1}K_1(\rho)$
$D=4,\ m\neq0$	$m^4(2\pi)^{-2}\rho^{-1}K_2'(\rho)$	$-m^4(2\pi)^{-2}(K_2(\rho)/\rho\prime)$	$m^4(2\pi\rho)^{-2}K_2(\rho)$
$D=2,\ m=0$	0	$1/8\pi x^2$	$1/8\pi x^2$
$D=4,\ m=0$	$-(4\pi x^2)^{-2}$	$\frac{3}{2}(4\pi x^2)^{-2}$	$\frac{1}{2}(4\pi x^2)^{-2}$

$0\leqslant x\leqslant l$ with the boundary condition (1) at one end was studied in Ref. 13 in connection with the Casimir effect. As will be shown below, however, the authors of this work neglected the existence of a surface energy. The surface energy, arising as a divergence correction to the energy density $T_{00}(x)$, has no effect on the conservation law written in differential form,

$$\partial_\mu T^{\mu\nu}=0,\quad T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}[(\partial\phi)^2-m^2\phi^2] \quad (2)$$

($T_{\mu\nu}$ is the "standard" energy–momentum tensor of a scalar field[2,3]). At the same time, in integral form we have the inequality

$$\partial_0 H\equiv\partial_0\int d\mathbf{x}T_{00}(x)\neq 0, \quad (3)$$

which results from the fact that the surface integrals cannot be neglected (the field exchanges energy with the boundary; for example, a packet incident on the boundary is elastically reflected with delay time $\tau\simeq 2\delta/v$, where $\delta\ll\lambda$, v is the velocity of the packet, and λ is the wavelength).

In Sec. 2 below, the variational problem for a system satisfying a boundary condition of the type (1) is discussed and the Lagrangian $\tilde{\mathscr{L}}$ and energy–momentum tensor $\tilde{T}_{\mu\nu}$, which are modified by the boundary condition, are found. The tensor $\tilde{T}_{\mu\nu}$ is identical to the metric tensor $T^{(\xi)}_{\mu\nu}$ (ξ is the conformal coupling parameter; see Refs. 2 and 3) for $\xi=1/4$, and it is therefore different from $T^{(\xi_c)}_{\mu\nu}$ in the conformal limit $m=0$, $\delta=0$ ($\xi_c=0$ for $D=2$ and $\xi_c=1/6$ for $D=4$), in which $\langle T^{(\xi_c)}_{\mu\nu}\rangle_{m=0,\delta=0}=0$ (see Refs. 1–3). Since for $\delta>0$ there is no divergence arbitrariness, the passage to arbitrarily small $\delta\to+0$ leads (in both the quantum and classical theories) to a breaking, which is not proportional to δ, of the conformal symmetry, which can be restored by the divergence transformation, but only for $\delta=0$.

Section 3 is devoted to the canonical quantization of the model. Despite the absence of a field interaction, the canonical quantization is not entirely trivial both because a second derivative in time appears in the Lagrangian and because the Lagrangian is degenerate. The results obtained in Secs. 2 and 3 are general; they are not related to the specific form of the boundary and are valid when δ depends on the (spatial) coordinates.

Section 4 is devoted to calculations of the vacuum energy–momentum tensor for a flat boundary (dimension $D=2$ and 4). Here, specifically, the values of the energy–momentum tensor in the limits $\delta\to0$ and $\delta\to\infty$ are calculated and compared with known results. It is also found that in the limit $\delta\to+0$ the difference between the "renormalized" vacuum averages of $\tilde{T}_{00}$ and T_{00} satisfies

$$\lim_{\delta\to+0}[\langle\tilde{T}_{00}(x)\rangle_\delta-\langle T_{00}(x)\rangle_\delta]>0 \quad (4)$$

(the index δ indicates that the average is taken over the vacuum of the field satisfying the boundary condition (1)), and the contribution, taken literally, of the divergence term on the left-hand side of the inequality (4) to the total energy is not only different from zero, but infinite. Therefore, deviations of the divergence term are added to the surface divergences of the density $\langle T_{00}\rangle_\delta$ (for $\delta\geqslant0$), and on account of (4) there arises the new problem of the uniqueness of the theory for $\delta=0$.

One way to overcome the difficulties is based on the introduction of a cutoff in the integral over the modes, and is studied in detail in the concluding section, where other published methods[7–9,13–16] for solving the problem of boundary divergences are also briefly discussed.

I have found a different, more formal, method for solving these two problems. The method is based on a special interpretation of the singular contributions to the total energy, and does not employ any assumptions about the structure of the boundary. Although on account of the existence of additional singularities for $\delta>0$ the method is only applicable to "ideal" boundary conditions ($\delta=0$ or $\delta=\infty$), it turns out to be consistent with the cutoff procedure. I find for the vacuum energy of the half-space (for $D=4$ the energy is taken per unit area of the boundary):

$$\langle\tilde{H}\rangle_{\delta=0}=\langle H\rangle_{\delta=0}=-\frac{m}{8}\quad(D=2), \quad (5a)$$

$$\langle\tilde{H}\rangle_{\delta=0}=\langle H\rangle_{\delta=0}=\frac{m^3}{48\pi}\quad(D=4). \quad (5b)$$

The corresponding expressions for the Neumann boundary condition ($\delta=\infty$) differ from these expressions by a sign. Note that (5a) and (5b) are respectively positive and negative, despite the opposite being the case for the energy densities $\langle\tilde{T}_{00}(x)\rangle_{\delta=0}$ and $\langle T_{00}(x)\rangle_{\delta=0}$; see Table I.

In the last section, the reasons why two different densities differing by the singular term (4) appear for $\delta=0$ are investigated. The theory with $\delta=0$ is ill-defined because the model does not exist in the limit $\delta\to-0$. Formally, this lack of uniqueness stems from the fact that the passage to the limit $\delta\to+0$ and the (improper) integral over modes cannot be interchanged when calculating the divergence contribution to the total vacuum energy. One consequence of the proposed interpretation (5) is that this "nonuniqueness" is eliminated. The technical details of the calculations presented in Sec. 4 are assembled in the Appendix.

2. VARIATIONAL PRINCIPLE AND THE ENERGY–MOMENTUM TENSOR

Let $\phi(x)$ be a scalar field over the domain $\mathscr{D}=\mathbb{R}_t\times\mathscr{D}'$ with an arbitrary, sufficiently smooth, boundary $\partial\mathscr{D}$ in Minkowski spacetime. We assume that the boundary is stationary (i.e., $\partial\mathscr{D}=\mathbb{R}_t\times\partial\mathscr{D}'$, where $\partial\mathscr{D}'$ is the spatial section of $\partial\mathscr{D}$) and that the normal derivative

$$\phi_{,n}(x)=F(x)\phi(x), \qquad \partial_0 F(x)=0, \quad x\in\partial\mathscr{D}, \tag{6}$$

where $F(x)$ is a function prescribed on the boundary that does not depend on ϕ. At infinity the field ϕ is assumed to satisfy reasonable boundary conditions, for example $\phi=0$.

The Lagrangian density is usually chosen in the form[2,3]

$$\mathscr{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2 \tag{7}$$

and is determined to within a total divergence. In our case, generally speaking, there is no such arbitrariness, since the field does not vanish on the boundary. Therefore, the action

$$\widetilde{W}=\int_{\mathscr{D}}\mathscr{L}\,dx+\Theta(\phi), \tag{8}$$

where $\Theta(\phi)$ is a model-dependent functional specified on the boundary ∂D.

Systems of the type (6)–(8) are systems with a "loaded" boundary.[17] If the functional $\widetilde{W}$ is known, then its variation with respect to ϕ under the condition $\delta\widetilde{W}=0$ with $\delta\phi|_{\partial\mathscr{D}}=0$ yields the equations of motion (the Klein–Gordon–Fock equation in the present case). Assuming, next, that $\delta\phi|_{\partial\mathscr{D}}$ is arbitrary and ϕ satisfies the equation of motion, the field ϕ is found to satisfy a boundary condition which for the "correct" choice of $\Theta(\phi)$ is identical to the condition (6).[17]

Since the form of the functional $\Theta(\phi)$ is not known in advance in our case, we seek a functional $\widetilde{W}$ that is and would be extremal on a class of functions that satisfy the known boundary condition (6). The only possible modification of $\mathscr{L}$ that leads to a linear boundary condition (i.e., a quadratic functional $\Theta(\phi)$) and does not change the equations of motion has the form (c is a constant)

$$\widetilde{\mathscr{L}}=\mathscr{L}-c\,\partial_\mu(\phi\partial^\mu\phi), \qquad \widetilde{W}=\int_{\mathscr{D}}\widetilde{\mathscr{L}}\,dx. \tag{9}$$

We require that

$$\delta\widetilde{W}=0, \tag{10}$$

whereupon, together with Eq. (6),

$$\delta\phi_{,n}(x)=F(x)\delta\phi(x), \qquad x\in\partial\mathscr{D}. \tag{11}$$

The conditions (6), (10), and (11) make it possible to determine the constant

$$c=\frac{1}{2} \tag{12}$$

(in this case the surface contribution to $\delta\widetilde{W}$ can be expressed in terms of the Wronskian $\varphi\delta\varphi_{,n}-\varphi_{,n}\delta\varphi$, which vanishes on the boundary), and to rewrite $\widetilde{\mathscr{L}}$ in the form

$$\widetilde{\mathscr{L}}=-\frac{1}{2}\phi(x)(m^2+\partial^2)\phi(x). \tag{13}$$

The functional $\Theta(\phi)$ in Eq. (8) can now be written as a surface integral:

$$\Theta(\phi)=-c\int_{\partial\mathscr{D}}d\sigma_\mu\phi(x)\partial^\mu\phi(x)$$

$$=-\frac{1}{2}\int_{\partial\mathscr{D}}d\sigma\phi(x)\phi_{,n}(x), \tag{14}$$

where we have employed the constant (12). The Lagrangian $\widetilde{\mathscr{L}}$ contains second derivatives in time, and vanishes for the solutions of the equation of motion

$$(m^2+\partial^2)\phi(x)=0. \tag{15}$$

The energy–momentum tensor $\widetilde{T}_{\alpha\beta}$ for a system with the higher-order derivatives can be determined as follows (see Ref. 18; $\phi_{,\alpha}\equiv\partial_\alpha\phi, \overleftrightarrow{\partial}_\alpha\equiv\overrightarrow{\partial}_\alpha-\overleftarrow{\partial}_\alpha$):

$$\widetilde{T}^\beta_\alpha=\frac{\partial\widetilde{\mathscr{L}}}{\partial\phi_{,\beta}}\phi_{,\alpha}+\frac{\partial\widetilde{\mathscr{L}}}{\partial\phi_{,\beta\nu}}\phi_{,\alpha\nu}-\partial_\nu\left(\frac{\partial\widetilde{\mathscr{L}}}{\partial\phi_{,\beta\nu}}\right)\phi_{,\alpha}-\delta^\beta_\alpha\widetilde{\mathscr{L}}$$

$$=-\frac{1}{2}\phi\overleftrightarrow{\partial}_\alpha\partial^\beta\phi \tag{16}$$

(ϕ satisfies Eq. (15), $c=1/2$). If we had assumed the constant c to be arbitrary, then instead of (16) we would have obtained (see Eq. (2)) for $\widetilde{T}_{\alpha\beta}$

$$\widetilde{T}_{\alpha\beta}=T_{\alpha\beta}+\partial^\lambda f_{\alpha\beta\lambda}, \tag{17}$$

$$f_{\alpha\beta\lambda}=\frac{c}{2}(2g_{\alpha\beta}\phi\partial_\lambda\phi-g_{\alpha\lambda}\phi\partial_\beta\phi-g_{\beta\lambda}\phi\partial_\alpha\phi),$$

$$f_{\alpha\beta\lambda}=f_{\beta\alpha\lambda} \tag{18}$$

or, using the relation

$$\partial_\alpha(\phi\partial_\beta\phi)=\partial_\beta(\phi\partial_\alpha\phi), \tag{19}$$

the simpler expression

$$\widetilde{T}_{\alpha\beta}=T_{\alpha\beta}+c\partial^\lambda(g_{\alpha\beta}\phi\partial_\lambda\phi-g_{\alpha\lambda}\phi\partial_\beta\phi). \tag{20}$$

It is easy to show that by virtue of Eq. (15) and in accordance with Eq. (2),

$$\partial^\alpha\widetilde{T}_{\alpha\beta}=\partial^\alpha T_{\alpha\beta}=0. \tag{21}$$

However, instead of Eq. (3) we now have

$$\partial_0\widetilde{H}\equiv\partial_0\int_{\mathscr{D}'}d\mathbf{x}\widetilde{T}_{00}=-c\int_{\partial\mathscr{D}'}d\sigma_k\overleftrightarrow{\partial}_k\phi_{,0}$$

$$+(1-2c)\int_{\partial\mathscr{D}'}d\sigma_k T_{k0}, \tag{22}$$

where the first term vanishes identically by virtue of the boundary condition (6), and the second term equals zero only if Eq. (12) holds. Therefore for $c=1/2$

$$\partial_0 \tilde{H}=0, \quad \tilde{H}=-\frac{1}{2}\int_{\mathscr{D}'} d\mathbf{x}\phi \overleftrightarrow{\partial}_0 \phi_{,0}. \tag{23}$$

Note that the interpretation of the two terms in $\tilde{\mathscr{L}}$ (9) as "standard" and "surface" terms is based on the correspondence principle. However, the divergence correction, generally speaking, need not be Lorentz-invariant: to cancel the contribution of the surface integrals in Eq. (3), it is sufficient to add the term $(1/2)\partial_k(\phi\partial_k\phi)$ to $\mathscr{L}$, i.e., only the spatial part of the divergence correction in Eq. (9). The Lorentz symmetry is broken at the outset due to the presence of the boundary. Nonetheless, the form (9) is necessary when asymptotic regions exist sufficiently far from the boundary, since the system must possess Lorentz symmetry in these regions.[3)]

An important qualitative result of this section is the clarification of the fundamental role of self-consistency, associated with the boundary condition (6), of the variational problem: the choice $c=1/2$ not only modifies the initial Lagrangian (7), but is also the only choice that conserves total energy.

3. CANONICAL QUANTIZATION

The canonical momentum defined by Ostrogradskiĭ[18,19] is

$$\pi_{1x}=\frac{\partial\tilde{\mathscr{L}}}{\partial\phi_{,0}}-\partial_0\frac{\partial\tilde{\mathscr{L}}}{\partial\phi_{,00}}=\frac{1}{2}\phi_{,0}, \tag{24}$$

while the canonical momentum π_{2x} is

$$\pi_{2x}=\frac{\partial\tilde{\mathscr{L}}}{\partial\phi_{,00}}=-\frac{1}{2}\phi, \tag{25}$$

indicating the special nature of the theory with the Lagrangian (13) as a theory with higher-order derivatives. Switching to an expanded Hamiltonian formalism, we obtain in the notation of Ref. 18

$$\phi_{1x}\equiv\phi(\mathbf{x},t), \qquad \phi_{2x}\equiv\phi_{,0}(\mathbf{x},t), \qquad v_x\equiv\phi_{,00}(\mathbf{x},t), \tag{26}$$

$$\tilde{\mathscr{L}}\equiv\mathscr{L}^v=-\frac{1}{2}\phi_{1x}v_x-V(\phi_{1x}), \tag{27}$$

where the potential energy density

$$V(\phi_{1x})=\frac{1}{2}(m^2\phi_{1x}^2-\phi_{1x}\nabla^2\phi_{1x}). \tag{28}$$

The auxiliary function

$$H^*=\int d\mathbf{x}[\pi_{1x}\phi_{2x}+V(\phi_{1x})]+\int d\mathbf{x}\, v_x\Phi_x^{(1)}, \tag{29}$$

and $\Phi_x^{(1)}$ is a first-class constraint (25):

$$\Phi_x^{(1)}=\pi_{2x}+\frac{1}{2}\phi_{1x}. \tag{30}$$

The only second-class constraint is ($\{\cdot,\cdot\}$ are Poisson brackets)

$$\Phi_x^{(2)}=\{\Phi_x^{(1)}, H^*\}=\frac{1}{2}\phi_{2x}-\pi_{1x}, \tag{31}$$

so that the matrix of constraints

$$\|\{\Phi_x^{(\alpha)},\Phi_{x'}^{(\beta)}\}\|=1_{xx'}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{32}$$

is nondegenerate. Therefore, we have a system with second-class constraints, and the Dirac brackets determine the standard commutation relations for ϕ_{1x} and ϕ_{2x} (see Eq. (26)):

$$\{\phi_{1x},\phi_{2x}\}_D=\delta(\mathbf{x}-\mathbf{x}'), \tag{33}$$

which makes it possible to reproduce, after quantization, the standard algebra of the creation and annihilation operators. The Hamiltonian H^* on the surface of constraints $\Phi=0$ equals

$$\tilde{H}^*=\int d\mathbf{x}\,[\pi_{1x}\phi_{2x}+V(\phi_{1x})]=\int d\mathbf{x}\left[\frac{1}{2}\phi_{2x}^2+V(\phi_{1x})\right]. \tag{34}$$

If ϕ satisfies Eq. (15), it is identical to the Lagrangian energy $\tilde{H}$ introduced in Eq. (3).

Using the complete system of solutions of Eq. (15)[4)] and the standard definition of the creation and annihilation operators[2,3] (a_k^+ and a_k, respectively; k is a collective index representing the quantum numbers and labels the solutions of the problem (16) and (15)), we find for the energy $\tilde{H}$ in Eq. (23)

$$\tilde{H}=\sum_k \frac{1}{2}\omega_k(a_k a_k^+ + a_k^+ a_k), \tag{35}$$

and, with the aid of the quantum analog of Eq. (33),

$$[\phi(\mathbf{x},t),\phi_{,0}(\mathbf{x},t)]=i\delta(\mathbf{x}-\mathbf{x}'), \tag{36}$$

we obtain for a_k and a_k^+ in the corresponding normalization

$$[a_k,a_{k'}^+]=\delta_{kk'}, \qquad [a_k,a_{k'}]=[a_k^+,a_{k'}^+]=0, \tag{37a}$$

and the vacuum state is determined by the condition

$$a_k|0\rangle=0 \text{ for all } k. \tag{37b}$$

If we had left $c\neq 1/2$ (as in Eq. (20)), then the time-dependent correction

$$\left(c-\frac{1}{2}\right)\int_{\partial\mathscr{D}'} d\sigma\phi\phi_{,n}=\left(c-\frac{1}{2}\right)\int_{\partial\mathscr{D}'} d\sigma F(x)\phi^2(x), \tag{38}$$

would have appeared on the right-hand side of Eq. (35). Such a correction would make it impossible to interpret $\tilde{H}$ in terms of the occupation numbers of the states, and the "vacuum" vector (37b) would no longer be an eigenvector of $\tilde{H}$. The Hamiltonian H in Eq. (3) corresponds to the sum of the right-hand sides of (35) and (38) with $c=0$. Hence, it follows that H cannot be used to determine the (stationary) vacuum state. The correction (38) (with $c=0$) prevented the authors of Ref. 13 from obtaining the usual sum of half-frequencies for the vacuum energy. The reason is not the approximate character of the boundary condition (1) (see Ref. 13), but the fact that the energy of the system has been improperly defined.[5)]

In summary, the foregoing construction leads to a rather curious qualitative result: a scalar "photon" describes an excitation of both the field and the boundary, making the separation of the total energy into "energy of the field" and "energy of the boundary" to some degree arbitrary.

As a technical aside, in contrast to T_{00}, asymmetric bilinear combinations of the field ϕ and its derivatives appear in $\tilde{T}_{00}$. After quantization, the property (19), for example, no longer holds. However

$$\partial_\alpha[\phi,\partial_\beta\phi]_+=\partial_\beta[\phi,\partial_\alpha\phi]_+ \tag{39}$$

($[\cdot,\cdot]_+$ is an anticommutator), and analysis shows that the appropriate symmetrization of the operators eliminates the difficulties associated with noncommutativity. For example, defining $\tilde{T}_{\mu\nu}$ in the form

$$\tilde{T}_{\mu\nu}=\frac{1}{2}[\partial_\mu\phi,\partial_\nu\phi]_+-\frac{1}{2}g_{\mu\nu}((\partial\phi)^2-m^2\phi^2)+\partial^\lambda f_{\mu\nu\lambda}, \tag{40}$$

$$f_{\alpha\beta\lambda}=\frac{1}{8}(2g_{\alpha\beta}[\phi,\partial_\lambda\phi]_+-g_{\alpha\lambda}[\phi,\partial_\beta\phi]_+-g_{\beta\lambda}[\phi,\partial_\alpha\phi]_+), \tag{41}$$

yields a symmetric energy–momentum tensor, energy conservation, $\tilde{T}_{0n}=0$ on the boundary, and so on. Equation (39) makes it possible to simplify, similarly to Eq. (20), the last term in Eq. (40):

$$\partial^\lambda f_{\alpha\beta\lambda}=\frac{1}{4}\partial^\lambda(g_{\alpha\beta}[\phi,\partial_\lambda\phi]_+-g_{\alpha\lambda}[\phi,\partial_\beta\phi]_+).$$

4. FIELD IN A HALF-SPACE WITH AN ELASTIC WALL

The complete orthonormal set of functions satisfying Eq. (15) and the boundary condition (1) has the form

$$\tilde{\phi}_{k\pm}(x)=(2\pi\sqrt{2\omega_k})^{-1}\exp(\mp i\omega_k t+i\mathbf{q}\mathbf{x}_\perp)\psi_k(x), \tag{42}$$

where $\omega_k=\sqrt{m^2+\mathbf{q}^2+k^2}$. To simplify the formulas, the index k replaces the complete set (q_1,q_2,k), and the eigenfunctions of the boundary-value problem (1) on the half-axis (see Ref. 20) are

$$\psi_k(x)=\sqrt{\frac{2}{\pi}}\frac{k\delta}{\sqrt{1+k^2\delta^2}}\left(\cos kx+\frac{\sin kx}{k\delta}\right), \tag{43}$$

$x\equiv x_1$, $\mathbf{x}_\perp=(x_2,\ x_3)$, $-\infty<q_{2,3}<\infty$, and $k>0$. The completeness and orthogonality relations for the functions $\psi_k(x)$ have the standard form

$$\int_0^\infty \psi_k(x)\psi_k(x')dk=\delta(x-x'), \tag{44}$$

$$\int_0^\infty \psi_k(x)\psi_{k'}(x)dx=\delta(k-k'). \tag{45}$$

In Eqs. (44) and (45) k, $x>0$, and the normalization of the functions $\tilde{\phi}_{k\varepsilon}$ from Eq. (42) ($\varepsilon=\pm$) is

$$i\int d\mathbf{x}\,\tilde{\phi}^*_{k\varepsilon}\overleftrightarrow{\partial_0}\,\tilde{\phi}_{k'\varepsilon'}=\varepsilon\delta^{(2)}(\mathbf{q}-\mathbf{q}')\delta(k-k')\delta_{\varepsilon\varepsilon'}. \tag{46}$$

We assume below that $\tilde{\phi}_{k+}\equiv\tilde{\phi}_k$, and

$$\phi(x)=\int d\mathbf{q}\int_0^\infty dk[a_k\tilde{\phi}_k(x)+a_k^+\tilde{\phi}_k^*(x)], \tag{47}$$

where the operators a_k and a_k^+ satisfy (37).

In spacetime dimensions $1+1$, there is no factor of $(2\pi)^{-1}$ in the functions $\tilde{\phi}_{k\pm}(x)$ in Eq. (42) and no term $i\mathbf{q}\mathbf{x}_\perp$ in the argument of the exponential, and the frequency $\omega_k=\sqrt{m^2+k^2}$.

Before proceeding to specific calculations, consider the general properties of the vacuum energy–momentum tensor. The boundary condition (1) does not destroy the Poincaré symmetry in the variables x_0, x_2, and x_3, and therefore, just as in the case of ideal boundary conditions,[1,5] the corresponding components of the ("renormalized") vacuum energy–momentum tensor must be proportional to the metric tensor $g_{\mu\nu}$, and $\langle\tilde{T}_{\mu\nu}\rangle_\delta$ as a whole can depend only on $x_1\equiv x$. The boundary condition (1), though different from the Dirichlet condition, does not introduce anything new. In complete analogy with Refs. 3 and 5, we therefore have

$$\langle\tilde{T}_{\mu\nu}\rangle_\delta=f(x)g_{\mu\nu}+g(x)n_\mu n_\nu=f(x)\mathrm{diag}(1,\bar{g},-1,-1), \tag{48}$$

where n_μ is a vector normal to the plane of the boundary, and the functions $f(x)$ and $\bar{g}(x)\equiv(g-f)/f$ satisfy

$$\partial_1(f\bar{g})=0, \tag{49}$$

which follows from Eq. (21). Since the tensor $\langle\tilde{T}_{11}(x=\infty)\rangle$ must equal zero, we have $f\bar{g}=0$, whence $\bar{g}=0$, if $f\neq 0$. The fact that f is different from zero is due to the lack of conformal symmetry, which is broken on account of the presence of the two dimensional parameters ($m\neq 0$, $\delta\neq 0$). The parameter δ is not invariant under conformal transformations.[6)]

The structure of the vacuum averages of the two terms on the right-hand side of (40), taken separately, is also similar to Eq. (48) (with a different function f and 0 instead of $\bar{g}$). No subtractions need be performed in the divergence term on account of the presence of the total derivative (the local divergence in Minkowski space appears upon averaging only in $T_{\mu\nu}$, and therefore in $\tilde{T}_{\mu\nu}$).

For what follows, it is convenient to introduce the functions

$$A_2(\mu,\rho)=\mu^2\int_0^\infty\frac{dk}{2\omega_k}\left(\psi_k^2-\frac{1}{\pi}\right)=\frac{\mu^2}{4\pi}\int_{-\infty}^\infty\frac{dk}{\omega_k}\exp[i\Phi(k\delta)+2ikx], \tag{50}$$

$$B_2(\mu,\rho)=\frac{\delta^2}{4\pi}\int_{-\infty}^\infty dk\ \omega_k\exp[i\Phi(k\delta)+2ikx], \tag{51}$$

which appear in two-dimensional calculations, and their analogs for the four-dimensional case

$$A_4(\mu,\rho)=\delta^4\int\frac{d^2q}{4\pi^2}m_\perp^2\int_0^\infty\frac{dk}{2\omega_k}\left(\psi_k^2-\frac{1}{\pi}\right)=\frac{\delta^4}{16\pi^3}\int d^2q m_\perp^2\int_{-\infty}^\infty\frac{dk}{\omega_k}\exp[i\Phi(k\delta)+2ikx], \tag{52}$$

$$B_4(\mu,\rho)=\frac{\delta^4}{16\pi^3}\int d^2q\int_{-\infty}^{\infty}dk\omega_k\exp[i\Phi(k\delta)+2ikx]. \quad (53)$$

In (50)–(53), the function $\psi_k(x)$ defined in Eq. (43) is employed, and the following notation is introduced:

$$\mu\equiv m\delta,\qquad \rho\equiv 2mx,\qquad m_\perp^2=m^2+\mathbf{q}^2,$$

$$\exp[i\Phi(k\delta)]=\frac{k\delta-i}{k\delta+i}, \quad (54)$$

where $\Phi(k\delta)+2kx$ is the phase difference between the reflected and incident monochromatic waves at the point x.

Using Eqs. (37), (40), and (41) we obtain for the renormalized vacuum averages

$$\|\langle\tilde{T}_{\mu\nu}\rangle_\delta\|_{D=2}=\delta^{-2}B_2(\mu,\ \rho)\mathrm{diag}(1,0), \quad (55)$$

$$\|\langle\tilde{T}_{\mu\nu}\rangle_\delta\|_{D=4}=\delta^{-4}B_4(\mu,\ \rho)\mathrm{diag}(1,0,-1,-1) \quad (56)$$

(the renormalized average $\langle\dots\rangle_\delta\equiv\langle\dots\rangle_\delta^{ur}-\langle\dots\rangle_M$; the subscript M signifies that the corresponding quantity is taken in Minkowski space, and the index ur signifies that the quantum average over the vacuum (37b) is not renormalizable; regularization is performed by a modewise subtraction procedure[2–4] (and does not present any difficulties). At the same time

$$\|\langle T_{\mu\nu}\rangle_\delta\|_{D=2}=\delta^{-2}A_2(\mu,\rho)\mathrm{diag}(1,0), \quad (57)$$

and $\|\langle T_{\mu\nu}\rangle_\delta\|_{D=4}$ has the more complicated form

$$\|\langle T_{\mu\nu}\rangle_\delta\|_{D=4}=\delta^{-4}\mathrm{diag}\Big(A_4(\mu,\ \rho),0,\frac{\mu^2}{6\pi}(1+2\partial_\rho^2)B_2(\mu,\rho),\ \times\frac{\mu^2}{6\pi}(1+2\partial_\rho^2)B_2(\mu,\rho)\Big). \quad (58)$$

Using the relations (A11) and (A12) from the Appendix, the last expression can be rewritten in a form similar to Eq. (57):

$$\|\langle T_{\mu\nu}\rangle_\delta\|_{D=4}=\delta^{-4}A_2(\mu,\rho)\mathrm{diag}(1,0,-1,-1). \quad (59)$$

The properties of the functions A_i and B_i and their relationship to the important function $I_{MT}(\mu,\ \rho)$ introduced in Ref. 14 are described in the Appendix. We analyze the limiting properties of the matrices (55)–(58) with respect to their arguments μ and ρ.

The functions $\delta^{-i}A_i(\mu,\rho)$ and $\delta^{-i}B_i(\mu,\ \rho)$ change sign on changing from $\delta=0$ to $\delta=\infty$:

$$\langle\tilde{T}_{\mu\nu}(x)\rangle_{\delta=\infty}=-\langle\tilde{T}_{\mu\nu}(x)\rangle_{\delta=0},$$

$$\langle T_{\mu\nu}(x)\rangle_{\delta=\infty}=-\langle T_{\mu\nu}(x)\rangle_{\delta=0}, \quad (60)$$

the indicated limits being taken only at finite x, since all functions $A_2,\ \dots,\ B_4$ are singular on the boundary. The equality (60) means that in the first quadrant of the (x,δ) plane there exist "lines of zero vacuum polarization" $\tilde{x}=\tilde{x}(\delta)$ and $x=x(\delta)$ where $\langle\tilde{T}_{\mu\nu}(x)\rangle_\delta$ and $\langle T_{\mu\nu}(x)\rangle_\delta$ vanish, respectively.

Far from the boundary, all four functions are exponentially small ($x\gg m^{-1}$; see Eqs. (A1)–(A4) and (A19)).

The asymptotic expressions of the functions $\delta^{-i}A_i(\mu,\rho)$ and $\delta^{-i}B_i(\mu,\ \rho)$ on the boundary for ($\delta>0$)

$$x\ll\delta,\qquad x\ll 1/m \quad (61)$$

can be obtained on the basis of (A1), (A2) and (A27), (A28) for $D=2$:

$$\langle\tilde{T}_{00}\rangle_\delta=\delta^{-2}B_2=-\frac{m^2}{2\pi}\Big[\rho^{-2}-2(\mu\rho)^{-1}+\Big(\frac{1}{2}-2\mu^{-2}\Big)\times\ln\Big(\frac{\gamma_E\rho}{2}\Big)-\frac{1}{4}+2(1-\mu^{-2})I_{MT}(\mu^{-1},0)\Big], \quad (62)$$

$$\langle T_{00}\rangle_\delta=\delta^{-2}A_2=-\frac{m^2}{2\pi}\Big[\ln\Big(\frac{\gamma_E\rho}{2}\Big)+2I_{MT}(\mu^{-1},0)\Big], \quad (63)$$

and (A3), (A4), (A27), and (A28) for $D=4$:

$$\langle\tilde{T}_{00}\rangle_\delta=\frac{m^4}{6\pi^2}\Big[-3\rho^{-4}-(\mu^{-2}+\mu^{-1}\partial_\rho)\Big[\rho^{-2}+\Big(\frac{3}{2}-\mu^{-2}\Big)\times\ln\Big(\frac{\gamma_E\rho}{2}\Big)\Big]+\frac{3}{4}\rho^{-2}+\frac{3}{16}\ln\Big(\frac{\gamma_E\rho}{2}\Big)-\frac{9}{64}+\frac{1}{4\mu^2}+(1-\mu^{-2})^2I_{MT}(\mu^{-1},0)\Big], \quad (64)$$

$$\langle T_{00}\rangle_\delta=\frac{m^4}{6\pi^2}\Big[6\rho^{-4}-\frac{4}{\mu}\rho^{-3}+2(\mu\rho)^{-2}-\frac{2}{\mu^3\rho}+\Big(\frac{3}{8}-2\mu^{-4}\Big)\ln\Big(\frac{\gamma_E\rho}{2}\Big)-\frac{3}{32}-\frac{1}{2\mu^2}+(1+\mu^{-2}-2\mu^{-4})I_{MT}(\mu^{-1},0)\Big], \quad (65)$$

ln $\gamma_E=0.577\dots$ is Euler's constant, and $I_{MT}(\mu^{-1},\ 0)$, defined in Eq. (A6), is finite. Comparing the leading terms of the asymptotic expansions (64) and (65), we can see that as $x\to 0$ the lines of zero polarization $\tilde{x}(\delta)$ and $x(\delta)$ coincide: $x=3\delta/4$. The lines are also different in this range of x when $D=2$.

The asymptotic formulas (62)–(65) have the following features in common: 1) the dominant singularities are independent of δ; 2) the lowest-order singular contributions are proportional to inverse powers of δ; and, 3) the dominant singularities in the tensors $\tilde{T}_{\mu\nu}$ and $T_{\mu\nu}$ have opposite signs. The lack of dependence on δ, however, is misleading: the conditions (61) make it possible to pass in the formulas (62)–(65) to the limit $\delta\to\infty$ but not the limit $\delta\to 0$, where, according to Eq. (60), the leading boundary divergence changes sign.

The property (2) makes it possible to eliminate the leading divergences in all expressions (62)–(65). For example, for $D=2$ and $\delta>0$, a finite result can be obtained (see (A1) and (A21)) by means of the following subtraction:

$$\langle T_{00}\rangle_s=\langle T_{00}\rangle_\delta-\langle T_{00}\rangle_{\delta=\infty}=\langle T_{00}\rangle_\delta^{ur}-\langle T_{00}\rangle_{\delta=\infty}^{ur}=-\frac{m^2}{\pi}I_{MT}(\mu^{-1},\rho). \quad (66)$$

$\langle T_{00}\rangle_s$ differs by a factor of 2 from the vacuum energy density of the scalar field ($D=1+1$) interacting with a concentrated potential $V(x)=2\lambda\,\delta(x)$, if δ in Eq. (66) is replaced by λ^{-1} (λ is the interaction constant, $|x|<\infty$; see Refs. 4 and 14).[7)] Note that going from $\langle T_{00}\rangle_\delta$ to $\langle T_{00}\rangle_s$ is the same as changing the zero point of total energy: the zero point is not half the "vacuum energy" in Minkowski space, but the vacuum energy of the half-space[8)] with Neumann boundary conditions for the field ϕ. $\langle T_{00}\rangle_\delta$ and $\langle \widetilde{T}_{00}\rangle_s$ vanish for $\delta=\infty$, and for $\delta=0$ they are identical to the surface singularities $\langle T_{00}\rangle_{\delta=0}$ and $\langle \widetilde{T}_{00}\rangle_{\delta=0}$ with doubled coefficients; see Eq. (60).

The subtraction procedure (66) corresponds to discarding the leading terms of the ultraviolet asymptotic expansions in the integrands of (50)–(53), so that the transition to quantities of the type $\langle T_{00}\rangle_s$ is tantamount to taking account of the real dependence of the penetration depth on k: $\delta_{phys}(k)\to+\infty$ as $k\to\infty$. Despite its physical attractiveness, this method does not completely eliminate the divergences (for $\delta>0$): the boundary singularities in $\langle T_{00}\rangle_\delta$ and $\langle \widetilde{T}_{00}\rangle_\delta$, which are proportional to inverse powers of δ, remain in $\langle T_{00}\rangle_s$ and $\langle \widetilde{T}_{00}\rangle_s$.

Using (55)–(59) and (A1)–(A4), (A29) and (A30) from the Appendix, it is easy to calculate the explicit form of the component of the vacuum energy–moment tensor in the limit $\delta\to+0$. The results are assembled in Table I, where for comparison the corresponding expressions in the massless limit are written out separately; $\langle \mathrm{div}\rangle_\delta$ denotes an average of the (0, 0) component of the last term in Eq. (40), and $\rho\equiv 2mx$.

The formulas in the first column of the table have appeared in the literature (see Refs. 2–4) in a similar context, specifically in an analysis of a scalar model on the (half)axis with the conformal coupling parameter $\xi=0$ and zero boundary conditions at the origin. We can see that in accordance with Eq. (4) the transition to Dirichlet boundary conditions leaves nonzero (divergence) corrections to $\langle T_{00}\rangle_\delta$, with a divergent integral over the half-space.[9)]

Despite their singular character, the expressions in the first two rows of Table I make a finite contribution to the total energy if Eq. 2.16.2(2) of Ref. 21 is used to interpret this contribution:

$$\int_0^\infty \rho^{\alpha-1}K_\nu(\rho)d\rho=2^{\alpha-2}\Gamma\left(\frac{\alpha+\nu}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right). \tag{67}$$

The answer for the energies

$$\langle H\rangle=\int_0^\infty dx\langle T_{00}\rangle,\quad \langle \widetilde{H}\rangle=\int_0^\infty dx\langle \widetilde{T}_{00}\rangle$$

with $\delta=0$ is presented in Eq. (5). As a result of the relations (60), the Neumann values of $\langle H\rangle$ and $\langle \widetilde{H}\rangle$ differ from Eq. (5) by a sign.

We now demonstrate that the divergence contribution to the total energy vanishes, for example, for $D=4$. To within a multiplicative factor, this contribution equals (see Table I):

$$\int_0^\infty d\rho(-\rho^{-2}K_2+\rho^{-1}K_2')$$
$$=\int_0^\infty d\rho\left(-\rho^{-2}K_2-\frac{1}{2}\rho^{-1}K_1-\frac{1}{2}\rho^{-1}K_3\right)$$
$$=-\frac{1}{8}\left[\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{3}{2}\right)+\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)\right.$$
$$\left.+\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right)\right]=0. \tag{68}$$

It is noteworthy that the boundary divergences are interpreted in the integral characteristics of the field. This is consistent with the "nonlocalizability" of the scalar photon which described the excitation in the "field + boundary" system. Another important point is that the mass is different from zero ($\rho=2mx$ in Eq. (67)!). Of course, after the integrations are performed according to the scheme (67), the mass can be set equal to zero, so that we obtain an answer which is characteristic for conformally symmetric theories: $\langle \widetilde{H}\rangle=0$. Unfortunately, the method presented here does not yield finite results for $\delta>0$. In the concluding section, we return to a discussion of the problem of obtaining finite expressions for the energy.

5. $\delta\to 0$ LIMIT: DETAILED ANALYSIS

The results presented in Table I demonstrate the singular nature of the limit $\delta\to 0$. This is seen from the fact that the theory with the Lagrangian (7) and Dirichlet boundary conditions "knows nothing" about the divergence transformation (9), since the surface integral (14) vanishes, and therefore after the theory with the Lagrangian (7) is quantized, the result presented in the first column of Table I, rather than the last, should be obtained for the vacuum energy–momentum tensor.

It will be shown below that the local characteristics of the present model exhibit a strong singularity with respect to the parameter δ/x ($x>0$) at $\delta=0$, and a square-root singularity with respect to $\mu\equiv m\delta$ at $\delta=-1/m$, which for $-1/m<\delta<0$ result in a breakdown of unitarity (if an additional interaction, signifying a redefinition of the model, is not introduced into the Lagrangian). Accordingly, the expansion of the vacuum energy–momentum tensor in positive powers of $\delta>0$ at zero can only be asymptotic.

Let us replace δ in the boundary condition (1) by $-\delta_1$ ($\delta_1>0$) and, for simplicity, let us study the two-dimensional case. The set of functions $\psi_k(x)$ (43) assumes the form

$$\psi_k(x)=\sqrt{\frac{2}{\pi}}\,\frac{k\delta_1}{\sqrt{1+k^2\delta_1^2}}\left(\cos kx-\frac{\sin kx}{k\delta_1}\right),\qquad k>0. \tag{69}$$

Now, however, the functions $\psi_k(x)$ do not form a complete set, since together with Eq. (69) there appears a bound (Tamm) mode (see Ref. 20)

$$\widetilde{\phi}_b(x,t)=\frac{1}{\sqrt{2\omega_b}}e^{-i\omega_b t}\psi_b(x),$$

$$\psi_b(x)=\sqrt{\frac{2}{\delta_1}}e^{-x/\delta_1}, \qquad \omega_b=\delta_1^{-1}\sqrt{\mu_1^2-1}, \tag{70}$$

$\mu_1 \equiv m\delta_1$. Instead of Eq. (47) we have ($D=2$)

$$\phi(x)=a_b\tilde{\phi}_b+a_b^+\tilde{\phi}_b^*+\int_0^\infty dk[a_k\tilde{\phi}_k+a_k^+\tilde{\phi}_k^*], \tag{71}$$

where a and a^+ are, once again, annihilation and creation operators, and the function ψ_b is orthogonal to the continuum modes. The contribution of the bound mode to the vacuum characteristics of the field need not be renormalized, and it can easily be calculated using the expressions (40) and (41) ($\mu_1>1$):

$$\langle\tilde{T}_{00}\rangle_\delta^{(b)}=\delta_1^{-2}\sqrt{\mu_1^2-1}e^{-2x/\delta_1}, \tag{72a}$$

$$\langle T_{00}\rangle_\delta^{(b)}=\frac{m^2}{\sqrt{\mu_1^2-1}}e^{-2x/\delta_1}, \tag{72b}$$

$$\langle \mathrm{div}\rangle_\delta^{(b)}=-\frac{\delta_1^{-2}}{\sqrt{\mu_1^2-1}}e^{-2x/\delta_1}. \tag{72c}$$

The formulas (70) and (72) demonstrate the aforementioned strong singularity with respect to $\delta=-\delta_1$ (for $x>0$) and a square-root singularity with respect to μ_1. For $\mu_1=1$ the density $\langle\tilde{T}_{00}\rangle_\delta^{(b)}$ vanishes, while $\langle T_{00}\rangle_\delta^{(b)}$ and $\langle\mathrm{div}\rangle_\delta^{(b)}$ diverge separately. It is also obvious from Eq. (70) that the values $\mu_1<1$ are inadmissible, since the frequency ω_b would be complex and the vacuum energy density would acquire an imaginary part (corresponding to the last term in Eq. (73a))

$$\langle\tilde{T}_{00}\rangle_{-\delta_1}=-\frac{m^2}{2\pi\rho}K_1(\rho)+\frac{1}{\pi\delta_1^2}(\mu_1\partial_\rho-1)K_0(\rho)+\frac{\mu_1^2-1}{\pi\delta_1^2}\times\begin{cases}\displaystyle\oint_1^\infty\frac{dte^{-t\rho}}{(\mu_1t-1)\sqrt{t^2-1}}+\frac{1}{2}\omega_b\psi_b^2(x), & 0<\mu_1<1,\\ \displaystyle\int_1^\infty\frac{dte^{-t\rho}}{(\mu_1t-1)\sqrt{t^2-1}}, & \mu_1>1.\end{cases} \tag{73a,b}$$

In Eq. (73b) the contribution of the surface mode exactly cancels the contribution of the pole appearing in the integration path over k after rotation of the contour $k\to itm$. In (73a) there is no such cancellation, since for $\mu_1<1$ the pole falls on the cut and does not contribute to the integral. We note also that for $\mu_1=1+0$ the pole and the square-root singularity in the integrand coincide, but $\langle\tilde{T}_{00}\rangle_{-\delta_1}$ does not become infinite, due to the factor μ_1^2-1 multiplying the integral.

Here, it is opportune to reconsider the analogy of a scalar field interacting with a singular potential $V(x)=2\lambda\delta(x)$. For $\lambda<0$ the potential admits one bound state, whose contribution to the vacuum energy–momentum tensor was found in Refs. 4 and 14, and which differs from Eq. (72b) by the fact that the δ_1^{-1} is replaced by $|\lambda|$ and that there is a common factor of 1/2. The case $0<\mu_1<1$ (i.e., $-1<\mu<0$ in the previous notations) corresponds to $\lambda<-m$ examined in Refs. 4 and 14. Without repeating all the arguments presented there, I only point out that the appearance of a stationary state localized in a region smaller than the Compton length results in instability of the vacuum state against pair creation. To properly take account of this, terms describing the self-action of the field ϕ must be added to the Lagrangian (13).

The analytic properties of the four-dimensional model with $\delta<0$ are similar to those studied in the two-dimensional case and will not be discussed here. We note that boundary divergences result only from the contribution of the continuum to the vacuum energy–momentum tensor ($D=2,4$), while the surface mode makes a finite contribution.

The foregoing considerations explain the physical reason for the indeterminacy of the limit $\delta=0$ — it is impossible to join the $\delta>0$ and $\delta<0$ theories at $\delta=0$ (in contrast to $\delta=\infty$). Formally, the appearance of two densities $\langle T_{00}(x)\rangle$ and $\langle\tilde{T}_{00}(x)\rangle$ at $\delta=0$ and of the corresponding vacuum energies $\langle H\rangle$ and $\langle\tilde{H}\rangle$ (which are infinite, if Eq. (67) is not used) is a consequence of the fact that the integral over the modes and the passage to the limit $\delta\to+0$ cannot be interchanged when calculating the divergence contribution to the total vacuum energy. This contribution (see Eqs. (1), (14), and (43)) is a surface energy and equals ($D=2$, $\delta>0$)

$$\langle\tilde{H}\rangle_\delta-\langle H\rangle_\delta=\frac{1}{2\delta}\int_0^\infty\frac{dk}{2\omega_k}\psi_k^2(0)=\frac{1}{2\pi}\int_0^\infty\frac{dk}{\omega_k}\frac{k^2\delta}{1+k^2\delta^2}. \tag{74}$$

This integral diverges for $\delta>0$, but if the limit $\delta\to0$ is taken in the integrand, then it vanishes.

6. CONCLUSIONS

The interpretation proposed in Sec. 4 for the boundary divergences does not solve the problem of determining the vacuum energy, if $0<\delta<\infty$. In this regard, I shall briefly discuss a way, proposed in the literature, to solve this problem. In Refs. 14–16, the boundary conditions were replaced by singular potentials simulating infinitely thin, semipermeable walls (see footnote 7). In the case of two walls[2,16] the singular contribution of each wall to the vacuum energy–momentum tensor was dropped completely, being independent of the distance between them. This approach makes it possible to determine the Casimir force between macroscopic bodies, but it leaves open the question of the vacuum energy of the field. Since it depends on the parameter in the boundary condition, this latter energy cannot be regarded as an insignificant constant. The idea of eliminating the surface divergences by including them in the renormalization of the singular potential (Ref. 15, see also Ref. 4) encounters, in our opinion, a fundamental difficulty: the structure of the counterterms is different from that of the bare interaction. A similar idea was invoked somewhat earlier in Ref. 7: the geometric nature of the coefficients in the singular terms of the energy–momentum tensor of a scalar field (interacting with a background gravitational field) makes it possible to include these terms in the renormalization of the corresponding terms of the surface gravity (Gibbons–Hawking). A critical discussion of this approach can be found in Ref. 6; see also Ref. 3.

The idea of eliminating the surface divergences by taking account of boundary permeability at high frequencies is repeated in essentially all work on the problem of determining the vacuum energy. This idea has been confirmed only in part. In Refs. 8 and 9 it was shown for the electromagnetic field that boundary permeability, which is associated with the asymptotic behavior of the dielectric constant ($\varepsilon(\omega\rightarrow i\infty)=1$), does not eliminate the boundary divergence, and reasonable results can be obtained only by introducing a physical cutoff reflecting the discrete nature of matter.[10)] In spite of the fact that the cutoff is introduced manually, for a large number of materials the answers obtained for the vacuum energy are in semiquantitative agreement with the experimental data.[8,9] This agreement can be interpreted as a qualitative solution of the vacuum-energy problem (in QED).

In contrast to electrodynamics, in the model considered here the cutoff parameter (Λ in the momentum, x_0 in the position) is not defined internally. Nonetheless, a formal application of this idea gives reasonable answers — the total energy is finite, the energies $\langle\widetilde{H}\rangle_\delta$ and $\langle H\rangle_\delta$ are equal at $\delta=0$ and ∞, and a new type of relation is obtained between the spatial (x_0) and momentum (Λ) cutoffs.

As an example, let us now examine the two-dimensional case. A finite expression, which vanishes if $\delta=0$ or $\delta=\infty$, is obtained for the surface energy:

$$\frac{1}{2\delta}\langle\phi^2(0,t)\rangle_{\delta,\Lambda}=\frac{1}{2\pi}\int_0^\Lambda\frac{dk}{\omega_k}\frac{k^2\delta}{1+k^2\delta^2},\tag{75}$$

The total energy is

$$\langle\widetilde{H}\rangle_{\delta,\Lambda}=\int_0^\infty dx\langle\widetilde{T}_{00}\rangle_{\delta,\Lambda}=-\frac{m}{8}+\frac{\delta}{2\pi}\int_0^\Lambda dk\frac{\sqrt{m^2+k^2}}{1+k^2\delta^2}.\tag{76}$$

A remarkable property of the representation (76) is that the limits $\delta=0$ ($\Lambda\delta\ll1$, $m\delta\ll1$) and $\delta=\infty$ ($\Lambda\delta\gg1$, $m\delta\gg1$) agree with the result (5), which was obtained with the aid of Eq. (67). Similar statements also hold for the four-dimensional case. There expressions of the type (76) can be viewed as a "continuation" of the interpretation procedure (67) to the case $0<\delta<\infty$.

It is somewhat more difficult to see the same correspondence in the case of the spatial cutoff ($\rho_0\equiv2mx_0$). With the aid of Eqs. (A2), (A25), and (67) we find ($x_0\ll m^{-1}$)

$$\langle\widetilde{H}\rangle_{\delta,x_0}=\int_{x_0}^\infty dx\langle\widetilde{T}_{00}\rangle_\delta=\frac{m}{8}\left(1-\frac{2}{\mu^2}\right)-\frac{m}{2\pi}\left(1-\frac{1}{\mu^2}\right)J(\mu)+\frac{1}{2\pi\delta}K_0(\rho_0),\tag{77}$$

where $J(\mu)\equiv\int_0^\infty d\rho I_{MT}$ and K_0 is a modified Bessel function; see Eq. (A.16). The limit $\delta\rightarrow\infty$ does not present a problem (and yields $+m/8$), but in the limit $\delta\rightarrow+0$ we have

$$\langle\widetilde{H}\rangle_{\delta,x_0}=-\frac{m}{8}-\frac{1}{2\pi\delta}\ln\frac{2\gamma_Ex_0}{\delta}.\tag{78}$$

The last term vanishes only if $2\gamma_Ex_0=\delta$. This nontrivial limitation (which is mass independent!) can be understood as follows. Let the mass m approach zero (in which case Eq. (78) is exact). Then, comparing Eq. (75) or (76)[11)] to Eq. (78), it can be concluded that[12)]

$$-\frac{1}{4\pi\delta}\ln\left[\left(\frac{2\gamma_Ex_0}{\delta}\right)^2\right]=\frac{1}{4\pi\delta}\ln(1+\Lambda^2\delta^2),\tag{79}$$

i.e.,

$$x_0=\frac{\delta}{2\gamma_E}\frac{1}{\sqrt{1+\Lambda^2\delta^2}}.\tag{80}$$

Hence, it is evident that the correspondence $x_0\sim\Lambda^{-1}$, which is standard in a quantum field theory with no boundaries, holds only for $\Lambda\delta\gg1$. In the opposite limit $x_0\sim\delta$.

In closing, I list the basic results of this work. First, it was found that the surface energy of the scalar field plays a nontrivial role. Taking account of this energy secures conservation of the total energy, and it is necessary to do so in order to quantize the theory correctly. This example is instructive in that, among other things, it demonstrates that the prescription for obtaining the vacuum energy–momentum tensor from the Green's function depends on the boundary condition satisfied by the Green's function. It is easy to show that together with the well-known expression

$$\langle T_{\mu\nu}(x)\rangle_\delta=-i\mathscr{T}_{\mu\nu}[G_c(x,x')-G_0(x,x')],\tag{81}$$

we also have

$$\langle\widetilde{T}_{\mu\nu}(x)\rangle_\delta=-i\widetilde{\mathscr{T}}_{\mu\nu}[G_c(x,x')-G_0(x,x')],\tag{82}$$

where the operator (see Refs. 3 and 4)

$$\mathscr{T}_{\mu\nu}=\lim_{x\rightarrow x'}\frac{1}{2}(\partial_\mu\partial'_\nu+\partial_\nu\partial'_\mu+m^2g_{\mu\nu}-g_{\mu\nu}\partial'_\sigma\partial^\sigma),\tag{83}$$

and $\widetilde{\mathscr{T}}_{\mu\nu}$ is given by

$$\widetilde{\mathscr{T}}_{\mu\nu}=\lim_{x\rightarrow x'}\frac{1}{4}(\partial_\mu-\partial'_\mu)(\partial'_\nu-\partial_\nu),\tag{84}$$

which follows from Eqs. (40) and (41). The causal Green's function $G_c(x,x')$ can be obtained in the standard manner from Eqs. (37), (42), and (43) and G_0 is the causal Green's function of the scalar field for Minkowski space; (see Refs. 2–4).

A comparison of Eqs. (62) with (63) and (64) with (65) demonstrates the difference in the boundary divergences (for $\delta>0$) for the tensors $\langle\widetilde{T}_{\mu\nu}(x)\rangle_\delta$ and $\langle T_{\mu\nu}(x)\rangle_\delta$. The existence of a point $\bar{x}(\delta)$ where the vacuum polarization vanishes is also of interest (see text after Eq. (60)). The special role of the Dirichlet condition in the model considered is manifested by the model's not existing in the limit $\delta\rightarrow-0$ (Sec. 5), and by an additional constraint $x_0\sim\delta$ on the cutoff parameter, which is necessary for an adequate interpretation of the surface divergences. The interpretation procedure proposed in the case of ideal boundary conditions (see Eqs. (67) and (68)) is apparently new, and is consistent with the previously described regularization procedure. However, the use of an additional dimensional parameter (cutoff parameter) for

$\delta > 0$ leaves the question of its physical meaning open, since (in contrast to QED) the nature of the boundary is unknown here.

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7. APPENDIX

The functions A_i and B_i, $i=2, 4$, introduced in Eqs. (50)–(53) can be represented in the form

$$A_2(\mu,\rho)=\frac{\mu^2}{2\pi}[K_0(\rho)-2I_{MT}(\mu^{-1},\rho)], \tag{A1}$$

$$B_2(\mu,\rho)=-\frac{\mu^2}{2\pi}[\rho^{-1}K_1(\rho)+2I_{MT}-2I''_{MT}], \tag{A2}$$

$$A_4(\mu,\rho)=\frac{\mu^4}{6\pi^2}\left[-\frac{3}{2}\rho^{-1}K'_2(\rho)+I_{MT}+I''_{MT}-2I^{iv}_{MT}\right], \tag{A3}$$

$$B_4(\mu,\rho)=-\frac{\mu^4}{6\pi^2}\left[\frac{3}{2}\rho^{-2}K_2(\rho)-I_{MT}+2I''_{MT}-I^{iv}_{MT}\right]. \tag{A4}$$

Here $K_\nu(\rho)$ are modified Bessel functions, primes denote derivatives of the corresponding order with respect to $\rho\equiv 2mx$, the function I_{MT}, introduced in Ref. 14, is given by

$$I_{MT}(\mu^{-1},\rho)=\frac{i}{2}\int_{-\infty}^{\infty}du\frac{\exp(2iu\tilde{x})}{(u+i)\sqrt{u^2+\mu^2}}$$
$$=\int_1^\infty dt\frac{\exp(-\rho t)}{(\mu t+1)\sqrt{t^2-1}}, \tag{A5}$$

where $\tilde{x}\equiv x/\delta$ and in the last equality the contour of integration has been rotated. In contrast to the functions $A_2, \ldots, B_4$, the function I_{MT} (but not its derivatives with respect to ρ) is finite on the boundary ($\rho=0$):

$$I_{MT}(\mu^{-1},0)=\begin{cases}\dfrac{1}{\sqrt{\mu^2-1}}\tan^{-1}(\sqrt{\mu^2-1}), & \mu\geq 1, \quad \text{(A6)}\\[2ex] \dfrac{1}{\sqrt{1-\mu^2}}\ln\dfrac{1+\sqrt{1-\mu^2}}{\mu}, & 0<\mu\leq 1. \quad \text{(A6a)}\end{cases}$$

Therefore, the boundary singularities of the tensors $\langle\tilde{T}_{\mu\nu}(x)\rangle_\delta$ and $\langle T_{\mu\nu}(x)\rangle_\delta$ are determined by the asymptotic behavior of the modified Bessel functions and the derivatives of $I_{MT}(\mu^{-1}, \rho)$ with respect to ρ in the limit $\rho\rightarrow 0$.

In deriving the formulas (A3) and (A4) from the defining integral representations (52) and (53), the integrals over the transverse momentum $\mathbf{q}$ were calculated first. These formally diverge, but they admit an interpretation in the sense of analytic continuation in the arguments of the B-function

$$\frac{1}{2}\mathrm{B}(p, q)=\int_0^\infty dx\frac{x^{2p-1}}{(1+x^2)^{p+q}}, \tag{A7}$$

(see Eq. 1.5.1(12) in Ref. 23).

The functions A_i and B_i are related to one another. This can be shown by using their integral representations, the properties of the modified Bessel functions,[22] and the equality (A13) given below. For example, $B_4(\mu, \rho)$ and $B_2(\mu, \rho)$ are related by the dispersion relation

$$B_4(\mu, \rho)=\frac{1}{2\pi}\int_\mu^\infty \mu_\perp B_2(\mu_\perp,\rho)d\mu_\perp, \tag{A8}$$

as well as the formula (here and below $\partial_\rho\equiv\partial/\partial\rho$)

$$B_4(\mu,\rho)=\frac{\mu^2}{6\pi}(\partial_\rho^2-1)B_2(\mu,\rho), \tag{A9}$$

which was used to derive Eq. (59) from Eq. (58). A dispersion relation of the form (A8) also relates the function $A_4(\mu, \rho)$ to $A_2(\mu, \rho)$. The cross relations have the form

$$A_4(\mu,\rho)=B_4(\mu,\rho)-\frac{\mu^2}{2\pi}B''_2(\mu,\rho)=-2B_4(\mu,\rho)$$
$$-\frac{\mu^2}{2\pi}B_2(\mu,\rho), \tag{A10}$$

$$B_2(\mu,\rho)=-(\partial_\rho^2-1)A_2(\mu,\rho), \tag{A11}$$

and the relation, similar to Eq. (A9), between A_4 and A_2 is

$$A_4(\mu,\rho)=\frac{\mu^2}{6\pi}(\partial_\rho^2-1)(1+2\partial_\rho^2)A_2(\mu,\rho). \tag{A12}$$

The differential equation

$$\partial_\rho I_{MT}(\mu^{-1},\rho)=\mu^{-1}[I_{MT}(\mu^{-1},\rho)-K_0(\rho)], \tag{A13}$$

plays a fundamental role in determining the asymptotic properties of $I_{MT}(\mu^{-1}, \rho)$. Applying this relation recursively we obtain the asymptotic series

$$I_{MT}(\mu^{-1},\rho)\simeq\sum_{n=0}^{\infty}\mu^n K_0^{(n)}(\rho). \tag{A14}$$

The domain of applicability of this expansion are the values of μ and ρ determined by the pairs of inequalities

$$\mu\ll 1, \qquad 1\ll\rho<\infty, \tag{A15a}$$

$$\mu\ll\rho, \qquad \rho\ll 1. \tag{A15b}$$

Under the conditions given by (A15), the asymptotic properties of the functions $K_0^{(n)}(\rho)$ for $\rho\gg 1$ and $\rho\ll 1$ make it possible to obtain both the well-known[14] asymptotic formulas for I_{MT} for $\mu\ll 1$ and the corrections to these formulas. The corresponding expressions can be easily obtained using the following formulas (see Ref. 22):

$$K_0(\rho)\simeq-\ln(\gamma_E\rho/2), \qquad \rho\ll 1, \tag{A16}$$

$$K_0^{(n)}(\rho)\simeq(-1)^n\begin{cases}(n-1)!\rho^{-n}, & n\geq 1, \quad \rho\ll 1, \quad \text{(A17a)}\\ \sqrt{\pi/2\rho}e^{-\rho}, & n\geq 0, \quad \rho\gg 1, \quad \text{(A17b)}\end{cases}$$

$\ln\gamma_E=0.577\ldots$ is Euler's constant. The asymptotic character of the expansion (A14) (for $\rho\ll 1$) is apparent from the factorial growth of the coefficient of $(\mu/\rho)^n$; see Eq. (A17a), whence it is also seen, in accordance with Eq. (A15b), that the expansion (A14) does not enable one to determine the character of the boundary divergences for finite μ.

The regular asymptotic expansion of I_{MT} for $\mu \geqslant 0$ can be obtained by writing the solution of Eq. (A13) in the form

$$I_{MT}(\mu^{-1},\rho)=\mu^{-1}e^{\rho/\mu}\int_{\rho}^{\infty}e^{-x/\mu}K_0(x)dx, \qquad \text{(A18)}$$

so that the asymptotic expansion of $K_0(\rho)$ (Eq. 7.13.1(7) in Ref. 22) for $\rho \gg 1$ gives ($\Psi(a,b,c)$ is the confluent hypergeometric function)[23]:

$$I_{MT}\simeq\sqrt{\frac{\pi}{2\mu(\mu+1)}}e^{-\rho}\left[\Psi\left(\frac{1}{2},\frac{1}{2},\rho(1+\mu^{-1})\right)-\frac{\mu+1}{8\mu}\Psi\left(\frac{3}{2},\frac{3}{2},\rho(1+\mu^{-1})\right)+\ldots\right]$$
$$=\sqrt{\frac{\pi}{2\rho}}\frac{e^{-\rho}}{\mu+1}\left[1-\frac{5\mu+1}{8(\mu+1)\rho}+\ldots\right]. \qquad \text{(A19)}$$

The first term of the expansion in square brackets in Eq. (A19) was obtained in Ref. 14.

The structure of the boundary divergences of the tensors $\langle\tilde{T}_{\mu\nu}(x)\rangle_\delta$ and $\langle T_{\mu\nu}(x)\rangle_\delta$ is determined by the asymptotic properties of the functions A_i and B_i under the conditions

$$\rho \ll 1, \qquad 1 \ll \mu < \infty, \qquad \text{(A20a)}$$

$$\rho \lesssim \mu, \qquad \mu \ll 1. \qquad \text{(A20b)}$$

An expansion which is an obvious analog of Eq. (A14) is obtained for I_{MT} with the aid of Eq. (A5):

$$I_{MT}(\mu^{-1},\rho)=\sum_{n=0}^{\infty}\mu^{-n-1}K_0^{(-n-1)}(\rho), \qquad \text{(A21)}$$

where

$$D_\rho^{-n}K_0(\rho)\equiv K_0^{(-n)}(\rho), \qquad \text{(A22)}$$

and the integration operator D_ρ^{-1} has the form

$$D_\rho^{-1}=-\int_\rho^\infty d\rho\ldots,$$

$$D_\rho^{-2}=\int_\rho^\infty d\rho'\int_{\rho'}^\infty d\rho''\ldots, \quad \text{and so on.} \qquad \text{(A23)}$$

In contrast to Eq. (A14), the series Eq. (A21) is also meaningful on the boundary. This is attributable to the finiteness of $K_0^{(-n)}$ for $\rho=0$ and $n\geqslant 1$. Note that the second and fourth ρ derivatives of I_{MT}, which appear in Eqs. (A2)–(A4), give rise to boundary divergences in the first few terms of the series (A21) as soon as $2-n-1\geqslant 0$ or $4-n-1\geqslant 0$, respectively. Since

$$K_0^{(-n)}(\rho)\simeq(-1)^{-n}\sqrt{\frac{\pi}{2\rho}}e^{-\rho} \qquad \text{(A24)}$$

for $\rho \gg 1$ (by induction), the result obtained in Eq. (A19) can be checked. The latter agreement is not surprising, despite the fact that for $\rho \gg 1$ the condition (A28) breaks down, merely by virtue of the existence of Eq. (A19) for arbitrary μ and the convergence of the series (A21).

Differentiating Eq. (A21) term by term with respect to ρ, we obtain

$$I''_{MT}=\mu^{-2}I_{MT}-\mu^{-2}K_0(\rho)-\mu^{-1}K'_0(\rho), \qquad \text{(A25)}$$

$$I^{iv}_{MT}=\mu^{-4}I_{MT}-\mu^{-4}K_0(\rho)-\mu^{-3}K'_0(\rho)-\mu^{-2}K''_0(\rho)-\mu^{-1}K'''_0(\rho), \qquad \text{(A26)}$$

so that using the well-known expansion of $K_0(\rho)$ at zero (Eqs. 7.2.5(38) in Ref. 22), we finally find the nonvanishing terms I''_{MT} and I^{iv}_{MT} in the expansions under the conditions (A20):

$$I''_{MT}(\mu^{-1},\rho)=\mu^{-2}I_{MT}(\mu^{-1},0)+(\mu\rho)^{-1}+\mu^{-2}\ln\frac{\gamma_E\rho}{2}, \qquad \text{(A27)}$$

$$I^{iv}_{MT}(\mu^{-1},\rho)=\mu^{-4}I_{MT}(\mu^{-1},0)+(2\mu)^{-2}+\frac{2}{\mu\rho^3}-(\mu\rho)^{-2}+\mu^{-4}\left(1+\frac{1}{2}\mu^2\right)\left(\frac{\mu}{\rho}+\ln\frac{\gamma_E\rho}{2}\right). \qquad \text{(A28)}$$

The asymptotic series (A14) leads to the limiting ($\delta\to+0$) expressions for the functions $\delta^{-i}A_i(\mu,\rho)$ and $\delta^{-i}B_i(\mu,\rho)$, $i=2$, 4. For example,

$$\delta^{-4}A_4(\mu,\rho)=\frac{m^4}{6\pi^2}\left[-\frac{3}{2}\rho^{-1}K'_2(\rho)+K_0(\rho)+K''_0(\rho)-2K^{iv}_0(\rho)\right], \qquad \text{(A29)}$$

since

$$I^{(n)}_{MT}(\infty,\rho)=K^{(n)}_0(\rho). \qquad \text{(A30)}$$

After some straightforward calculations based on the relations between the modified Bessel functions,[22] the expressions of the type (A29) acquire the compact form presented in Table I. Equations (A27) and (A28), as well as the asymptotic properties of the functions K_ν in Eqs. (A1)–(A4), lead to the main result of Sec. 4 — Eqs. (62)–(65).

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1)A. I. Nikishov drew my attention to this problem.

2)The system of units in which $c=1$, $\hbar=1$, $\alpha=e^2/4\pi\hbar c$ is used; the metric possesses the signature $(+,-,-,-)=\mathrm{diag}\|g_{\mu\nu}\|$; and, $\partial_\mu\equiv\partial/\partial x^\mu$.

3)There is also a purely technical drawback to the proposed unsymmetric modification of $\mathscr{L}$: The tensor $\tilde{T}_{\alpha\beta}$ will no longer be symmetric ($\tilde{T}_{0i}\neq\tilde{T}_{i0}$, $\partial^\alpha\tilde{T}_{\alpha\gamma}\neq\partial^\beta\tilde{T}_{\gamma\beta}$, and so on).

4)The existence of such a system is guaranteed by the self-adjointness of the boundary-value problem associated with Eq. (15) and the boundary condition (6) if the function $F(x)$ in the boundary condition is real.

5)However, this did not affect the answer, since the authors discarded at the appropriate point the contribution of the correction (38).

6)Moreover, as shown above (see Eqs. (20)–(23)), the requirement that energy be conserved fixes the value of the constant $c=1/2$ in Eq. (20), while a conformal energy–momentum tensor $T^{(\xi_c)}_{\mu\nu}$ discussed in the Introduction, corresponds to, for example, $c=1/3$ in case $D=3+1$,[2,3] since even for $m=0$, $\delta=0$ we have $\langle\tilde{T}_{\mu\nu}\rangle\neq 0$.

7)This is no mere coincidence. The presence of a $\delta(x)$ interaction leads to a boundary condition[14] for the field $\phi(x)$ at the origin. This boundary condition actually has the same form as the boundary condition (1) with $\lambda=1/\delta$.

8) It will be shown below that the densities $\langle T_{00}(x)\rangle_{\delta=\infty}$ and $\langle \tilde{T}_{00}(x)\rangle_{\delta=\infty}$ lead to an identical (and finite) shift of the zero point for the energy.

9) Note that the second and fourth formulas in the last column of the table are present implicitly in Eq. (3.81) of Ref. 2 and they can be obtained from it with $\xi=1/4$ (see Introduction) after simple transformations. If course, the value $\xi=1/4$ in Ref. 2, just as in other works on this subject, was not distinguished in any way.

10) The momentum cutoff for the metal–vacuum interface is $\Lambda_{\text{phys}}=\omega_p/v_F\sqrt{2}$, where ω_p is the plasma frequency and v_F is the Fermi velocity of the valence electrons.[9] The frequency corresponding to Λ_{phys} is two orders of magnitude higher than ω_p.

11) For $m=0$ we have $\langle H\rangle_\delta=0$ $(D=2)$.

12) The right-hand side of Eq. (79) does not agree with the expression $(2\pi\delta)^{-1}[(1+\Lambda\delta)\ln(1+\Lambda\delta)-\Lambda\delta]$, presented in Eq. (21) of Ref. 13. It can be shown that the discrepancy comes from the improper application of certain operations in Ref. 13 to the divergent integral that respresents the vacuum energy.

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