Pinning of Abrikosov–Josephson vortices

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A nonlinear perturbation scheme, which describes the displacement of Abrikosov–Josephson vortices under the influence of a local perturbation, is developed for Josephson junctions with a large critical density, in which the Josephson length is shorter than the London length. Nonlinear oscillations of the vortex position and nonlinear relaxation during the evolution of pinning are considered both in the case of a tunnel junction between bulk superconductors and in the case of a tunnel contact between superconducting films. The pinning force for Abrikosov–Josephson vortices, which is determined by the perturbation of the Josephson critical current density, is obtained. [S1063-7761(96)02808-9]

1. INTRODUCTION

1. Investigations of the properties of vortex structures in Josephson junctions with a large critical current density are based on nonlocal electrodynamics,[1] which becomes necessary under conditions such that the Josephson length λ_j is less than the London length λ. A detailed discussion of the experimental conditions under which nonlocal Josephson electrodynamics applies was given in Refs. 2 and 3. These conditions correspond to values of the Josephson critical current density exceeding 10^6 A/cm^2, as well as large values of the Ginzburg–Landau parameter κ. The latter are common in type-II superconductors. A set of exact solutions of this electrodynamics describing both vortex structures at rest[2,3] and vortex structures moving along a tunnel junction[3–6] has been obtained. The energy of such vortices greatly exceeds the energy of ordinary Josephson vortices and approaches the energy of Abrikosov vortices. In the structure of their magnetic field, the new vortices are similar to Abrikosov vortices, although, unlike the latter, they do not have a singular core. The uniqueness of the new vortices and their similarity to Abrikosov vortices together with their Josephson character gave rise to the term Abrikosov–Josephson vortices.[2,7]

Gurevich and Cooley[8] considered the interaction of an Abrikosov vortex with an Abrikosov–Josephson vortex and showed that the interaction allows for strong pinning of the Abrikosov vortex. The study in Ref. 8 established a new possible physical cause for the strong pinning of Abrikosov vortices, which can be of great practical importance. It specifies, in particular, one of the important motivations for an experimental search for manifestations of anomalous Josephson junctions with a large critical current density. There is already an opinion[2,8] in the literature that such anomalous Josephson junctions can form, for example, on twin planes in YBaCuO (Refs. 9–11) or on thin ribbon inclusions of α-Ti in Nb–Ti superconductors,[12–14] which have become widely used in technology. There are clearly numerous similar natural and artificial plane defects (see, for example, Ref. 8).

Since, generally speaking, interacting Abrikosov and Abrikosov–Josephson vortices can move together in some directions in the plane of a junction, it would be interesting to understand the possibility of the pinning of an Abrikosov–Josephson vortex at some site in this plane. However, the pinning of Abrikosov–Josephson vortices themselves has not hitherto been examined, although this question was considered long ago for ordinary Josephson vortices.[15] Below we fill in this gap to a certain extent.

Section 2 discusses a linear perturbation scheme, which makes it possible to elucidate the perturbation determining vortex pinning, for which the linear perturbation scheme is unsuitable. The general form of this perturbation is established in Sec. 3. Section 4 is devoted to a time-independent nonlinear perturbation scheme that makes it possible to determine the displacement of a vortex under the action of a local perturbation of the critical current in a tunnel junction. Section 5 gives a general description of the nonlinear evolution of the position of a vortex when it is pinned. In addition, in the last two sections the general assumptions of the scheme are applied to the case of a solitary vortex (a 2π phase kink) in a tunnel junction between two bulk superconductors. The necessary treatment of a solitary vortex in a tunnel junction between two superconducting films is performed in Sec. 6. Finally, Sec. 7 summarizes the results. The pinning force is considered in the Appendix.

2. In this section we present some assumptions underlying linear perturbation theory, which point out the route to be followed below in considering the pinning of Abrikosov–Josephson vortices.

We write the equation for the phase difference $\varphi(z,t)$ between the Cooper pairs on the two sides of a tunnel junction in the form (compare Ref. 15)

$$\sin\varphi+\frac{1}{\omega_j^2}\frac{\partial^2\varphi}{\partial t^2}-\mathscr{L}[\varphi]=\mu f. \tag{2.1}$$

Here ω_j is the Josephson frequency, and $\mathscr{L}[\varphi]$ is a linear operator, which reduces in local electrodynamics to the differential operator $\lambda_j^2(\partial^2\varphi/\partial z^2)$, where λ_j is the Josephson length, and is an integral operator in the nonlocal electrodynamics of Josephson junctions. Below we shall discuss, in particular, the consequences pertaining to the clearly expressed nonlocal limit, in which the operator $\mathscr{L}[\varphi]$ reduces to $lH[\varphi_z]$, where

$$H[\varphi_z]=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dz'}{z'-z}\frac{\partial\varphi(z',t)}{\partial z'} \tag{2.2}$$

is a Hilbert transformation,[16] and $l=\lambda_j^2/\lambda$ is the characteristic length defining the region where the energy of the Abrikosov–Josephson vortex is localized.

The right-hand side of Eq. (2.1), as usual,[15] is the perturbation:

$$\mu f=-\frac{\beta}{\omega_j^2}\frac{\partial\varphi}{\partial t}+\alpha\frac{\partial^3\varphi}{\partial z^2\partial t}-i-\sum_k r_k\delta(z-a_k)\sin\varphi, \tag{2.3}$$

where β is determined by the dissipation caused by the tunneling of normal electrons through the barrier, φ_{zzt} is associated with the dissipation of normal electrons parallel to the barrier, i is the normalized distributed current density making a contribution to the energy, and, finally, the last sum describes the contributions of local regions with an increased Josephson current (microshorts, thin spots, etc.).[15]

We are interested in how the perturbations affect the stationary state $\varphi_0(z)$ specified by the equation

$$\sin\varphi_0(z)-\mathscr{L}[\varphi_0(z)]=0. \tag{2.4}$$

In the case of (2.2) several solutions of such an equation were obtained in Refs. 2 and 3 for Abrikosov–Josephson vortices.

We, first of all, consider the consequences of the linear perturbation scheme, in which

$$\varphi(z,t)=\varphi_0(z)+\delta\varphi(z,t), \quad \delta\varphi\ll\varphi_0. \tag{2.5}$$

Here we define the perturbation $\delta\varphi$ using the linearized equation

$$\delta\varphi\cos\varphi_0(z)+\frac{1}{\omega_j^2}\frac{\partial^2\delta\varphi}{\partial t^2}-\mathscr{L}[\delta\varphi]=\mu f. \tag{2.6}$$

The consequences of this equation are analyzed using the eigenfunctions ψ_n and the corresponding eigenvalues of the equation

$$\psi_n(z)\cos\varphi_0(z)-\mathscr{L}[\psi_n]=\varepsilon_n\psi_n(z) \tag{2.7}$$

that were obtained for several stationary states $\varphi_0(z)$ in Ref. 7. In particular, for the sine-Hilbert equation, in which we have $\mathscr{L}[\varphi]=lH[\varphi_z]$, for the state

$$\varphi_0(z)=\pi+2\arctan(z/l), \tag{2.8}$$

which corresponds to the solitary Abrikosov–Josephson vortex obtained by Gurevich,[2] according to Ref. 7, we have the following system of orthonormalized eigenfunctions:

$$\psi_s(z)=\sqrt{\frac{2}{\pi}}\frac{l^{3/2}}{z^2+l^2}=\sqrt{\frac{l}{2\pi}}\frac{d\varphi_0(z)}{dz}, \quad \varepsilon_s=0, \tag{2.9}$$

$$\left.\begin{aligned}\psi_+(z,q)&=\frac{(z^2-l^2)\cos qz+2lz\sin qz}{\sqrt{\pi}(z^2+l^2)}\\ \psi_-(z,q)&=\frac{(z^2-l^2)\sin qz-2lz\cos qz}{\sqrt{\pi}(z^2+l^2)}\end{aligned}\right\}\quad \begin{aligned}&\varepsilon(q)=1+ql,\\ &q\geq 0.\end{aligned} \tag{2.10}$$

Here the continuum states are doubly degenerate and are represented by even and odd eigenfunctions, respectively.

The solution of Eq. (2.6) can be written in the form

$$\delta\varphi(z,t)=\sum_n C_n(t)\psi_n(z), \tag{2.11}$$

where, according to (2.6) and (2.7), for the expansion amplitudes we have the equation

$$\frac{1}{\omega_j^2}\frac{d^2C_n(t)}{dt^2}+\varepsilon_nC_n(t)=\mu\int_{-\infty}^{+\infty}dz\ \psi_n(z)f, \tag{2.12}$$

which specifies the amplitudes C_n in terms of the assigned right-hand side.

According to Refs. 3 and 7, ε_n is nonnegative for the stable states $\varphi_0(z)$. If ε_n is positive, the solution of Eq. (2.12) for time-independent perturbations has the form

$$C_n=\frac{1}{\varepsilon_n}\mu\int_{-\infty}^{+\infty}dz\ \psi_n(z)f. \tag{2.13}$$

If the spectrum of eigenvalues contains a zero eigenvalue ($\varepsilon_n=\varepsilon_s=0$), as occurs in the case of (2.9), it can be stated that the linear perturbation scheme has a stationary solution only when the following orthogonality condition holds:

$$\mu\int_{-\infty}^{+\infty}dz\ \psi_sf=0. \tag{2.14}$$

A secular time dependence, which violates the condition that $\delta\varphi$ be small as the time increases, appears for the nonstationary perturbation scheme, according to (2.12), when there is a zero eigenvalue ($\varepsilon_n=\varepsilon_s=0$) and when the condition (2.14) is violated. The simple and, to some extent, trivial conclusions of this section will make it possible below to formulate a simple intuitive nonlinear perturbation scheme that describes the pinning of Abrikosov–Josephson vortices.

3. Thus, it has been shown that the linear perturbation scheme does not permit a consistent description of the perturbation of stationary Abrikosov–Josephson vortices that is associated with the influence on the state with the zero eigenvalue $\varepsilon_n=0$ and is described, in particular, by Eq. (2.9). We now show, first, that the presence of such an eigenstate of Eq. (2.7) is a general assumption and, second, that just this state is of special significance for the question of interest to us, the pinning of Abrikosov–Josephson vortices.

In fact, for the model of an infinitely long junction, in which, for example, (2.2) holds, a stationary vortex state can be described both by the function $\varphi_0(z)$ and by the function $\varphi_0(z+\Delta)$, where Δ is a constant shift. Therefore, along with Eq. (2.4) we can write the equation

$$\sin\varphi_0(z+\Delta)-\mathscr{L}[\varphi_0(z+\Delta)]=0. \tag{3.1}$$

Expansion in powers of Δ in this equation and consideration of Eq. (2.4) make it possible, in particular, to obtain

$$\frac{d\varphi_0(z)}{dz}\cos\varphi_0(z)-\mathscr{L}\left[\frac{d\varphi_0}{dz}\right]=0. \tag{3.2}$$

A comparison of this equation with (2.7) reveals that in the case of an arbitrary stationary vortex state, (3.2) describes the eigenstate of (2.7) with the zero eigenvalue $\varepsilon=0$. Here the general assumption is that the eigenfunction of such a state has the form

$$\psi_s(z)=C\frac{d\varphi_0(z)}{dz}, \tag{3.3}$$

where C is a constant, as occurred in the special case of Eq. (2.9).

The eigenfunctions $\psi_s(z)$ of (2.7) clearly describe the displacement in space (shift) of Abrikosov–Josephson vortices within the perturbation scheme. The corresponding amplitude C_s in the expansion (2.8) determines the magnitude Δ of this spatial displacement:

$$C_s\psi_s(z)=C_sC\frac{d\varphi_0(z)}{dz}=\Delta\frac{d\varphi_0(z)}{dz}. \tag{3.4}$$

The linear perturbation scheme does not allow one to find the amplitudes that determine the displacement of the vortices. Therefore, a nonlinear perturbation-theory approach, in which the expression (3.4) is small compared with unity, will be considered below. For example, in the case of (2.9) this corresponds to the condition $\Delta\ll l$.

4. Unlike (2.6), the nonlinear perturbation scheme requires retention of the next powers of $\delta\varphi$. More specifically, we now use the following equation:

$$\delta\varphi\cos\varphi_0(z)-\frac{1}{2}(\delta\varphi)^2\sin\varphi_0(z)-\frac{1}{6}(\delta\varphi)^3\cos\varphi_0(z)+\frac{1}{\omega_j^2}\frac{\partial^2\delta\varphi}{\partial t^2}-\mathscr{L}[\delta\varphi]=\mu f. \tag{4.1}$$

The next assumption, which lies at the basis of the approximation used, is that the amplitude C_s in the expansion (2.11) is significantly greater than the amplitudes of the remaining eigenstates. This is naturally surmized on the basis of the secular increase in this amplitude when the condition (2.14) is violated or on the basis of the undefined (infinite) nature of the response of the stationary linear perturbation scheme. According to this assumption (which, of course, will be justified below), as an approximation we set

$$\delta\varphi(z,t)=C_s\psi_s(z). \tag{4.2}$$

We first consider the situation of a stationary perturbed state. Then, multiplying Eq. (4.1) by $\psi_z(z)$ and integrating over the coordinates, we obtain

$$AC_s^2+BC_s^3=\mu f_s, \tag{4.3}$$

where

$$\mu f_s=\int_{-\infty}^{+\infty}dz\ \mu f\psi_s(z), \tag{4.4}$$

$$A=-\frac{1}{2}\int_{-\infty}^{+\infty}dz\ \sin\varphi_0(z)\psi_s^3(z), \tag{4.5}$$

$$B=-\frac{1}{6}\int_{-\infty}^{+\infty}dz\ \cos\varphi_0(z)\psi_s^4(z). \tag{4.6}$$

The cubic term is retained in Eq. (4.3) because of the occurrence (see below) of a symmetry in which the coefficient A vanishes. However, even if we have $A\neq 0$, the amplitude of the shift state is of order $\mu^{1/2}$, and if we have $A=0$, it is of order $\mu^{1/3}$ and is thus certainly greater than the amplitudes of order μ described by Eq. (2.13) for the remaining terms in the expansion (2.11). Therefore, the assumption that C_s is much greater than the other expansion amplitudes is justified.

To make the equations that follow specific, we assume the following stationary perturbation of the vortex structure

$$\mu f=-\sum_k r_k\delta(z-a_k)\sin\varphi_0(z). \tag{4.7}$$

Then

$$\mu f_s=-\sum_k r_k\sin\varphi_0(a_k)\psi_s(a_k). \tag{4.8}$$

We now consider the role of the perturbation (4.7) in determining the position of the solitary stationary Gurevich vortex (2.8), whose position is given according to (2.9) and (3.4) by the formula

$$\Delta=\sqrt{\frac{l}{2\pi}}C_s. \tag{4.9}$$

In this case we have

$$B=\frac{5}{32\pi l}, \tag{4.10}$$

$$\mu f_s=\sqrt{\frac{8}{\pi l}}\rho_0=\sqrt{\frac{8}{\pi l}}\sum_k\frac{r_ka_kl^3}{(l^2+a_k^2)^2}, \tag{4.11}$$

and $A=0$. The latter leads to the relation

$$C_s=(\mu f_s/B)^{1/3}\equiv C_s(\infty), \tag{4.12}$$

$$C_s(\infty)=2^{1/2}\pi^{1/6}\left(\frac{32}{\pi}\right)^{1/3}l^{1/2}\left(\frac{\rho_0}{l}\right)^{1/3}. \tag{4.13}$$

As a result of the pinning action of the perturbation (4.7), the Gurevich vortex (2.8) is displaced by the distance

$$\Delta=\left(\frac{32}{5\pi}\right)^{1/3}l\left(\frac{\rho_0}{l}\right)^{1/3}. \tag{4.14}$$

If the perturbation takes place at only one site and we have $a_k\gg l$, then $\Delta=(32r_kl^2/5\pi)^{1/3}(l/a_k)$.

5. We now turn to the temporal evolution of a vortex under the assumption that a perturbation having the form (4.7) acts at $t>0$. To complete the description of this evolution, we separate the term with a first derivative with respect to time from (2.3) as the main part of the equation describing the evolution of the phase difference φ. Within the assumptions previously advanced, this allows us to write the following equation instead of (4.1):

$$\delta\varphi\cos\varphi_0(z)-\frac{1}{2}(\delta\varphi)^2\sin\varphi_0(z)-\frac{1}{6}(\delta\varphi)^3\cos\varphi_0(z)+\frac{\beta}{\omega_j^2}\frac{\partial\delta\varphi}{\partial t}+\frac{1}{\omega_j^2}\frac{\partial^2\delta\varphi}{\partial t^2}-\mathscr{L}[\delta\varphi]=\mu f, \tag{5.1}$$

where the right-hand side of (5.1) is given by Eq. (4.7). According to the approximation (4.2), for the case of $A=0$ we obtain the following equation for the evolution of the amplitude of the shifted state $C_s(t)$:

$$\frac{1}{\omega_j^2}\frac{d^2C_s}{dt^2}+\frac{\beta}{\omega_j^2}\frac{dC_s}{dt}+BC_s^3=\mu f_s, \tag{5.2}$$

where μf_s is defined by Eq. (4.4).

When β is not very large, the dissipative term in (5.2) can be neglected at moderate evolution times. Then the evolution of C_s is described by the equation

$$\frac{1}{\omega_j^2}\frac{d^2C_s}{dt^2}+BC_s^3=\mu f_s, \tag{5.3}$$

which is similar to the equation describing the nonlinear oscillations of a particle with mass ω_j^{-2} in a field with the potential energy

$$U(C_s)=\frac{1}{4}BC_s^4-\mu f_s C_s. \tag{5.4}$$

We note that the bottom of the potential well (5.4) has the value $C_s=C_s(\infty)$, which is described by Eq. (4.12).

The nonlinear vortex displacement oscillations are characterized by the following solution of Eq. (5.3):

$$\sqrt{2}\omega_j t=\int_{C_s(0)}^{C_s}\frac{dy}{[E-U(y)]^{1/2}}, \tag{5.5}$$

where E is an integration constant and $C_s(0)$ is the initial value of the amplitude of the shift state.

In the initial stage, when dissipation has not yet entered, it can be assumed that at $t=0$, C_s and dC_s/dt are equal to zero. This corresponds to $E=0$. Then Eq. (5.5) gives (see Ref. 17, 3.166, 23, p. 277):

$$\frac{t}{t_0}=\int_0^u\frac{dx}{(x-x^4)^{1/2}}=\frac{1}{3^{1/4}}F\left(\arccos\frac{1-(1+\sqrt{3})u}{1+(-1+\sqrt{3})u},k\right), \tag{5.6}$$

where $F(\varphi,k)$ is an elliptic integral of the first kind,

$$t_0=\frac{1}{(2B)^{1/6}(\mu f_s)^{1/3}\omega_j}, \tag{5.7}$$

$$u=\frac{C_s(t)}{2^{2/3}C_s(\infty)},\quad k=\frac{\sqrt{2-\sqrt{3}}}{2}\approx 0.26. \tag{5.8}$$

Equation (5.6) enables us to describe the nonlinear vortex oscillations explicitly in the following manner:

$$C_s(t)=2^{2/3}C_s(\infty)\frac{1-\mathrm{cn}(\tau,k)}{1-\mathrm{cn}(\tau,k)+\sqrt{3}[1+\mathrm{cn}(\tau,k)]}, \tag{5.9}$$

where

$$\tau=3^{1/4}(t/t_0). \tag{5.10}$$

The elliptic cosine $\mathrm{cn}(\tau,k)$ has a period $4\mathbf{K}(k)$, which is determined by the complete integral of the first kind.

Equation (5.10) allows us to write the following condition for neglecting the dissipation in Eq. (5.2):

$$\beta \ll t_0^{-1}. \tag{5.11}$$

The coefficient on the right-hand side of Eq. (5.9) differs from the stationary amplitude of the shift mode by a factor of $2^{2/3}$. This means that the oscillatory deviations of the vortex from its initial state exceed the time-independent deviation, and oscillations occur about this position. The characteristic time of the nonlinear oscillations of the position of the Gurevich vortex then appearing is given by the formula

$$t_0=\frac{1}{\omega_j}\left(\frac{2\pi^2}{5}\right)^{1/6}\left(\frac{l}{\rho_0}\right)^{1/3}. \tag{5.12}$$

According to (4.13), an estimate of this time can also be written in the following simple form:

$$t\sim l/\omega_j\Delta. \tag{5.13}$$

With the passage of time, damping decreases the amplitude of the nonlinear vortex oscillations at times $t\gg t_0$ Accordingly, the amplitude approaches the bottom of the potential well (5.4), which corresponds to the stationary value $C_s(\infty)$. Then, it becomes posible to have small weakly damped oscillations for

$$\delta C_s(t)=C_s(t)-C_s(\infty), \tag{5.14}$$

which can be described by the equation

$$\frac{d^2\delta C_s}{dt^2}+\beta\frac{d\delta C_s}{dt}+\omega^2\delta C_s=0 \tag{5.15}$$

near the stationary value. The frequency of these linear oscillations equals

$$\omega=\omega_j\sqrt{3}B^{1/6}(\mu f_s)^{1/3}=\frac{1}{t_0}\frac{3^{1/2}}{2^{1/6}}, \tag{5.16}$$

and the damping rate equals $\beta/2$.

On the other hand, a qualitatively different situation, in which the conductivity of the tunnel junction is so great that the inequality opposite to (5.11) holds, is possible. Then the second derivative in Eq. (5.2) can be neglected. Accordingly, the nonlinear relaxation of the vortex displacement is then described by the equation (for $A=0$)

$$\frac{\beta}{\omega_j^2}\frac{dC_s}{dt}+BC_s^3=\mu f_s. \tag{5.17}$$

The solution of this equation corresponding to the initial condition $C_s(t=0)=0$ is described by the relation (see Ref. 17, 2.143, 4, p. 77)

$$\frac{t}{t_r}\equiv\frac{\omega_j^2}{\beta}(\mu f_s)^{2/3}B^{1/3}t=\frac{1}{3}\ln\frac{\sqrt{1+y+y^2}}{1-y}+\frac{1}{\sqrt{3}}\arctan\frac{y\sqrt{3}}{2+y}, \tag{5.18}$$

where

$$C_s(t)=C_s(\infty)y(t). \tag{5.19}$$

For a sufficiently large time,

$$t\gg\frac{\beta}{\omega_j^2}\frac{1}{(\mu f_s)^{2/3}B^{1/3}}\equiv t_r, \tag{5.20}$$

the stationary solution is established according to a linear relaxation law:

$$C_s(t)=\left(\frac{\mu f_s}{B}\right)^{1/3}\left\{1-\sqrt{3}\exp\left[\frac{\pi}{2\sqrt{3}}-\frac{3t}{t_r}\right]\right\}. \qquad (5.21)$$

For a Gurevich vortex

$$t_r=\frac{\beta}{\omega_j^2}\left(\frac{4\pi^2}{5}\right)^{1/3}\left(\frac{l}{\rho_0}\right)^{1/3}. \qquad (5.22)$$

Thus, there are two possible regimes for establishing the stationary displacement of a vortex under the perturbation (4.7). The first is the weak-dissipation regime corresponding to the inequality (5.11), under which nonlinear oscillations described by the comparatively simple laws of a nonlinear oscillator appear over the course of a comparatively long time. The amplitude and period of the oscillations vary simultaneously. Then, the dynamics of a linear oscillator with weak damping is observed in the final stage of the relaxation process. The second regime corresponds to strong dissipation and is realized under conditions that are the reverse of inequality (5.11), under which the vortex displacement relaxes according to the nonlinear nonexponential law (5.18), which naturally transforms into a linear exponential law in the final stage of the relaxation process.

6. Let us now consider the pinning of an Abrikosov–Josephson vortex in a Josephson junction between superconducting electrodes of finite thickness. As was shown in Ref. 18, in this case we have

$$\mathscr{L}[\varphi]=\frac{l}{4L_n}\int_{-\infty}^{+\infty}\frac{dz'}{\sinh\left[\frac{\pi}{4L_n}(z'-z)\right]}\frac{d\varphi(z')}{dz'}. \qquad (6.1)$$

Here $2L_n$ is the distance between the superconducting electrodes between which there is a Josephson tunnel junction of negligible thickness.

It was shown in Ref. 18 that the solution of Eq. (6.1), which describes a solitary Abrikosov–Josephson vortex at rest, has the form of the following 2π phase-difference kink of the superconducting pairs:

$$\varphi_0(z)=\pi+2\arctan\left[\frac{\sinh(\pi z/D)}{\cos(2\pi L_n/D)}\right], \qquad (6.2)$$

with $D>4L_n$ and is found from the equation

$$\frac{\pi l}{D}=\cot\left(\frac{2\pi L_n}{D}\right). \qquad (6.3)$$

Since

$$\frac{d\varphi_0(z)}{dz}=\frac{4\pi}{D}\frac{\cos(2\pi L_n/D)\cosh(\pi z/D)}{\cos(4\pi L_n/D)+\cosh(2\pi z/D)}, \qquad (6.4)$$

it is not difficult to prove that

$$\psi_s(z)=\frac{1}{\sqrt{\frac{4\pi}{D}\left(1+\frac{2\pi^2 lL_n}{D^2}+\frac{2L_n}{l}\right)}}\frac{d\varphi_0(z)}{dz}. \qquad (6.5)$$

In the limit $L_n\gg l$, where $D\to 4L_n$, these equations correspond to the theory for a Gurevich vortex. In the opposite limit,

$$L_n\ll l, \qquad (6.6)$$

which corresponds to a tunnel junction between superconducting films, the properties of an Alfimov–Popkov vortex are especially simple, since (2.1) becomes the sine-Gordon equation when $\mathscr{L}[\varphi]=2lL_n\varphi_{zt}$. We now treat specifically this limit, in which

$$D=\pi\sqrt{2L_n l}. \qquad (6.7)$$

In this limit

$$\varphi_0(z)=\pi+2\arctan[\sinh(\pi z/D)], \qquad (6.8)$$

and

$$\sin\varphi_0(z)=-\frac{2\sinh(\pi z/D)}{[\cosh(\pi z/D)]^2}. \qquad (6.9)$$

Accordingly,

$$\frac{d\varphi_0(z)}{dz}=\frac{2\pi}{D}\frac{1}{\cosh(\pi z/D)}. \qquad (6.10)$$

Here the normalized shift eigenfunction has the form

$$\psi_s(z)=\sqrt{\frac{D}{8\pi}}\frac{d\varphi_0}{dz}. \qquad (6.11)$$

According to (3.4), the latter formula enables us to characterize the displacement of an Alfimov–Popkov vortex subject to pinning in the following manner:

$$\Delta=\sqrt{\frac{D}{8\pi}}C_s. \qquad (6.12)$$

Since, as in the case of a Gurevich vortex, we have $A=0$, to apply the theory developed above we must know the quantities (4.4) and (4.6). Equations (6.9)–(6.11) allow us to obtain

$$B=\frac{\pi}{30D}, \qquad (6.13)$$

$$\mu f_s=\sqrt{\frac{8\pi}{D}}\rho_1, \qquad (6.14)$$

where

$$\rho_1=\sum_k\frac{r_k\sinh(2\pi a_k/D)}{[1+\cosh(2\pi a_k/D)]^2}. \qquad (6.15)$$

In accordance with Eq. (4.12), the stationary vortex shift (6.8) is determined by the stationary amplitude

$$C_s(\infty)=\left(120\sqrt{\frac{D}{2\pi}}\rho_1\right)^{1/3}, \qquad (6.16)$$

and the stationary shift itself has the form

$$\Delta=\frac{15^{1/3}}{(2\pi)^{2/3}}D^{2/3}\rho_1^{1/3}. \qquad (6.17)$$

Finally, we write the relation that specifies the characteristic time t_0, which determines the nonlinear regime of the vortex oscillations (5.9):

$$t_0=\frac{1}{\omega_j}\frac{15^{1/6}}{2^{1/2}\pi^{1/3}}\left(\frac{D}{\rho_1}\right)^{1/3}. \tag{6.18}$$

According to (5.16), this time also determines the frequency of the small linear oscillations (5.15).

Finally, for the strong-dissipation vortex relaxation time (6.8) we have

$$t_r=\frac{\beta}{\omega_j^2}\frac{1}{B^{1/3}(\mu f_s)^{2/3}}=\frac{B}{\omega_j^2}\left(\frac{15}{4\pi^2}\right)^{1/3}\left(\frac{D}{\rho_1}\right)^{2/3}. \tag{6.19}$$

Thus, the vortex pinning in a Josephson junction between superconducting films obeys the general laws of the nonlinear perturbation scheme established and is characterized by its own, qualitatively different vortex size parameter (D) and perturbation parameter (ρ_1).

7. Summarizing the material presented, we can say that we have studied the pinning of Abrikosov–Josephson vortices caused by local regions of a tunnel junction with an enhanced Josephson current (microshorts, thin spots), which are characterized by the perturbation (4.8). Within the perturbation scheme described, the displacements of the Abrikosov–Josephson vortices associated with are small compared with the characteristic scale of the variation of the phase difference between the superconducting pairs. In the case of a Gurevich vortex this scale equals $l=\lambda_j^2/\lambda$, and in the case of the vortex (6.8) it equals $D=\pi\lambda_j\sqrt{2L_n/\lambda}$. In our treatment these scales are always much smaller than the Josephson length λ_j. Despite the comparatively small vortex displacements, the dynamics of these displacements is significantly nonlinear. In the weak-dissipation limit with a small conductivity in the tunnel junction, the nonlinear oscillations are weakly damped, and after a large number of such oscillations they go over to a linear oscillation regime. In the strong-dissipation limit with a large conductivity the relaxation regime is also nonlinear (5.18), and only when stationary vortex displacement is approached does the relaxation law become purely exponential. The qualitative difference between the temporal characteristics of the evolution of a Gurevich vortex and the vortex (6.8) is determined primarily by the difference between their scales (l and D). In addition, the qualitative difference between these vortices is manifested in the dependence of the perturbations ρ_0 and ρ_1 on the distance from the vortex to the site of the perturbation in the Josephson current. For example, while in the case of a Gurevich vortex ρ_0 decreases according to the power law $r_k(l/a_k)^3$, in the case of an Alfimov–Popkov vortex this decrease is characterized by the exponential law $\rho_1\sim r_k\exp(-2\pi a_k/D)$. This difference is due to the different types of localization of the 2π phase-difference kink in the tunnel junction: according to a power law in the case of (2.8) and according to an exponential law in the case of (6.8). A theory describing the pinning force of Abrikosov–Josephson vortices is presented in the Appendix. We stress that the entire uniqueness of the Abrikosov–Josephson vortices is associated with the assumption of a large value for the Josephson critical current that we used, under which the Josephson length is shorter than the London length for the penetration of a magnetic field into a superconductor.

APPENDIX A: PINNING FORCE OF AN ABRIKOSOV–JOSEPHSON VORTEX

To determine the force acting on a vortex due to a perturbation of the form (4.7), we turn to a treatment of the energy of the interaction of a vortex in a tunnel junction with such a perturbation. This energy is specified by the perturbation $\delta_j c$ of the Josephson critical current density according to the equation (see for example, Ref. 15)

$$\delta E=\frac{\hbar}{2|e|}\int_{-\infty}^{+\infty}dz\ \delta j_c(z)[1-\cos\varphi]. \tag{A1}$$

Equation (4.7) corresponds to

$$\delta j_c=j_c\sum_k r_k\delta(z-a_k). \tag{A2}$$

To simplify the presentation, we assume that the spatial dependence for the unperturbed phase difference corresponds to $\varphi(z-z_0)$, where a_0 corresponds to the position of the middle of the vortex on the z axis. Then, when (A2) is taken into account, Eq. (A1) can be written in the form

$$\delta E=\frac{\hbar j_c}{2|e|}\sum_k r_k[1-\cos\varphi_0(a_k-z_0)]. \tag{A3}$$

We note that (A1) and (A3) give the energy per unit of length along the y axis.

Equation (A3) enables us to write an expression for the force F exerted by the perturbation (A2) on a vortex per unit of length along the y axis:

$$F=-\frac{\partial\delta E}{\partial z_0}=\frac{\hbar j_c}{2|e|}\sum_k r_k\sin\varphi_0(a_k-z_0)\frac{d\varphi_0(a_k-z_0)}{da_k}. \tag{A4}$$

To make this expression consistent with equations in the main body of the text, we set $z=0$. Then

$$F=\frac{\hbar j_c}{2|e|}\sum_k r_k\sin\varphi_0(a_k)\frac{d\varphi_0(a_k)}{da_k}. \tag{A5}$$

If instead of the usual perturbation (A2) we do not make an assumption regarding the form of $\delta j_c(z)$, in the general case instead of (A5) we obtain

$$F=\frac{\hbar}{2|e|}\int_{-\infty}^{+\infty}dz\ \delta j_c(z)\sin\varphi_0(z)\frac{d\varphi_0}{dz}. \tag{A6}$$

Recalling Eqs. (2.9) and (4.11), we can write the following expressions for the force exerted by the perturbation (A2) on a Gurevich vortex

$$F_g=-\frac{\hbar j_c}{|e|}\frac{\rho_0}{l}=-\frac{\hbar j_c}{|e|}\sum_k\frac{r_k a_k l^2}{(l^2+a_k^2)^2}. \tag{A7}$$

According to (A6), in the general case we have

$$F_g=-\frac{\hbar}{|e|}\int_{-\infty}^{+\infty}dz\ \delta j_c(z)\frac{zl^2}{(l^2+z^2)^2}. \tag{A8}$$

The equations (6.11) and (6.14) allow us to write the expression

$$F_{\mathrm{AP}}=-\frac{\hbar j_c}{|e|}\frac{4\pi\rho_1}{D}=-\frac{\hbar j_c}{|e|}\sum_k\frac{4\pi r_k\sinh(2\pi a_k/D)}{[1+\cosh(2\pi a_k/D)]^2 D} \qquad (A9)$$

for the force acting on an Alfimov–Popkov vortex in a tunnel junction between thin superconducting films. For a general the perturbation of the Josephson critical current density we have

$$F_{\mathrm{AP}}=-\frac{\hbar}{|e|}\int_{-\infty}^{+\infty}dz\ \delta j_c(z)\frac{2\pi\sinh(\pi z/D)}{D\cosh^3(\pi z/D)}. \qquad (A10)$$

In writing both Eq. (A8) and Eq. (A10), we note here that in the general case the expression for μf_s used above can be written in the form

$$\mu f_s=-\frac{c}{j_c}\int_{-\infty}^{+\infty}dz\ \delta j_c(z)\sin\varphi_0(z)\frac{d\varphi_0}{dz}, \qquad (A11)$$

where j_c is the unperturbed position independent Josephson critical current density and C is given by the ratio

$$C=\psi_s(z)/(d\varphi_0/dz) \qquad (A12)$$

according to Eq. (3.3).

We note that for estimates it is convenient to use the quantity

$$\langle\delta j_c\rangle_{\mathrm{eff}}=\int_{-\infty}^{+\infty}dz\ \delta j_c(z)\sin\varphi_0(z)\frac{d\varphi_0}{dz}. \qquad (A13)$$

As was shown above, this quantity is determined not only by the coordinate dependence $\delta j_c(z)$, which determines, in particular, the distance separating the perturbation from the vortex, but also by the coordinate dependence $\varphi_0(z)$, i.e., the structure of the vortex itself. The estimate of the pinning force is written using (A13) in the form

$$F=\frac{\hbar}{2|e|}\langle\delta j_c\rangle_{\mathrm{eff}}.$$

It can be assumed that the present expressions for the pinning force of Abrikosov–Josephson vortices are also suitable when the perturbation (A2) is not small. These expressions are valid at least as long as the vortices are not deformed significantly during their evolution.

Comparing the present treatment of the pinning of Abrikosov–Josephson vortices with the classical treatment of the problem of the pinning of ordinary Josephson vortices,[15] we stress, first, its qualitative uniqueness, which is due to the totally new spatial scale of the new vortices and the short London length, and, second, the qualitative difference between the spatial laws for vortex pinning in a tunnel junction joining bulk superconductors and for vortex pinning in a junction joining superconducting films. Finally, the simplicity of the description of the unique properties obtained in the analytical laws of our treatment should be stressed.

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