Electron energy spectrum and the persistent current in an elliptical quantum ring

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The behavior of electrons in quasi-one-dimensional curvilinear microstructures in a magnetic field is investigated. It is shown in the case of an elliptical ring that a variable curvature makes the energy spectrum fundamentally different from the cases of a linear quantum wire and a circular ring. Gaps appear in the magnetic-field dependence of the energy terms. When the Fermi level passes through the gap regions, the dependence of the magnetization of the system on the magnetic flux changes significantly.
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1. INTRODUCTION

The persistent current in quantum rings has attracted the attention of investigators for more than 10 years (see Refs. 1 and 2). The phenomenon has several general features. For example, under certain conditions the expression for the persistent current does not depend on the electron-electron interaction.[3,4] Furthermore, if the ring is assumed to be an ideal circle (the one-dimensional limit), the results of the following two versions of the problem coincide: 1) a point solenoid that carries a magnetic flux Φ and passes within a ring perpendicularly to its plane (the Aharonov-Bohm geometry) and 2) a uniform magnetic field that provides the same flux through a ring. In both cases the persistent current is described by the same function of the flux Φ (the spin effects are not taken into account), which oscillates with a period $\Phi_0=2\pi\hbar c/|e|$.

The extent of this universality is an interesting question. In the present work we investigate the influence of the geometric shape, specifically, the ellipticity of the ring on the persistent current. It is shown that the Φ_0 periodicity of the current J as a function of the magnetic flux is maintained, but its magnitude and the form of the function $J(\Phi)$ differ from those in the case of a circular ring. We note in this connection that the model often used in the literature of a discrete series of sites with periodic boundary conditions clearly ignores the geometric shape of the ring, which, as we shall show, is significant.

2. EFFECTIVE HAMILTONIAN OF A ONE-DIMENSIONAL RING

We note first of all that if the shape of the ring differs from a circle, the transition to an effective one-dimensional Hamiltonian becomes nontrivial.

Let us consider an electron moving in a narrow channel of elliptical shape. The uniform magnetic field is perpendicular to the plane of the ring and is assigned by the vector potential

$$\mathbf{A}(\mathbf{r})=\tfrac{1}{2}[\mathbf{H}\times\boldsymbol{\rho}]. \tag{1}$$

The Schrödinger equation has the standard form

$$\frac{1}{2m}\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)^2\psi+V(\boldsymbol{\rho})\psi=E\psi, \tag{2}$$

where $V(\boldsymbol{\rho})$ is the potential restricting electron motion and $\boldsymbol{\rho}$ is the two-dimensional radius vector of a point. In accord with our goal of going from (2) to a quasi-one-dimensional equation, we assume that $V(\boldsymbol{\rho})$ ensures a significantly greater frequency for motion across the ellipse than along it.

In the elliptical coordinates u and v (Ref. 5)

$$x=h\cosh u\cos v,\quad y=h\sinh u\sin v$$

(h is half of the distance between the foci of the ellipse) the vector potential (1) has the form

$$A_u=\frac{Hh\sin v\cos v}{2\sqrt{g(u,v)}},\qquad A_v=\frac{Hh\sinh u\cosh u}{2\sqrt{g(u,v)}},$$

where

$$g(u,v)=\sinh^2u+\sin^2v. \tag{3}$$

Now Eq. (2) becomes

$$-\frac{1}{g(u,v)}\left[\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+i\alpha\left(\sin(2v)\frac{\partial\psi}{\partial u}+\sinh(2u)\frac{\partial\psi}{\partial v}\right)-\frac{\alpha^2}{4}(\sin^2(2v)+\sinh^2(2u))\psi\right]+\frac{2mh^2}{\hbar^2}V(u,v)\psi=\frac{2mh^2E}{\hbar^2}\psi. \tag{4}$$

Here

$$\alpha=\frac{|e|Hh^2}{\hbar c}=\frac{1}{2}\left(\frac{h}{l_H}\right)^2,$$

where l_H is the magnetic length.

Since the frequencies of the motion along the coordinates u and v differ markedly, we utilize the adiabatic approximation. Now we must average Eq. (4) with respect to the ground-state wave function for transverse motion. We first consider the case of a rectangular well with infinitely high walls, i.e., we assume that $V(u,v)$ goes to infinity on two contours, viz., $u_1(v)$ and $u_2(v)$, which bound the region of electron motion. Although in the limit $u_2-u_1\equiv\delta\rightarrow0$ this

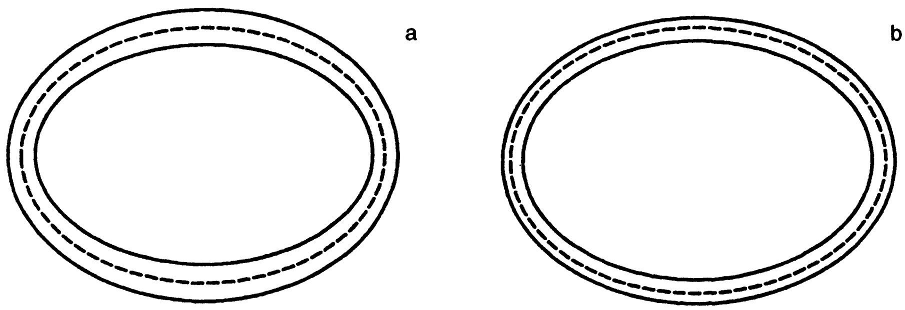

FIG. 1. Two types of elliptical rings: a—boundaries of the annular region in the form of confocal ellipses; b—a ring of constant width.

region shrinks to the ellipse $u=u_0$ $(u_1<u_0<u_2)$, the form of the resultant one-dimensional Hamiltonian is highly dependent on the way in which the limit is reached. For example, the results for the cases in which these contours are confocal ellipses and in which the region between them has the form of an elliptical ring of constant width (Figs. 1a and b) are different.

We represent the wave function in Eq. (4) in the form

$$\psi(u,v)=\eta(u,v)\chi(v), \tag{5}$$

where $\eta(u,v)$ corresponds to the "rapid" transverse motion and satisfies the equation

$$-\frac{1}{g(u,v)}\left[\frac{\partial^2\eta}{\partial u^2}+i\alpha\sin(2v)\frac{\partial\eta}{\partial u}-\frac{\alpha^2}{4}[\sin^2(2v)+\sinh^2(2u)]\eta\right]+\frac{2mh^2}{\hbar^2}V(u,v)\eta=\lambda(v)\eta, \tag{6}$$

in which v is regarded as a parameter and $\lambda(v)$ is the adiabatic value of the transverse energy at a given value of v.

In the case of confocal ellipses (Fig. 1a) the potential in Eq. (6) has the form

$$V(u,v)=\begin{cases}0, & u_1<u<u_2,\\ \infty, & u<u_1, \quad u>u_2,\end{cases} \tag{7}$$

so that for the ground state of the transverse motion we obtain

$$\eta(u,v)=\exp[-i\alpha(u-u_0)\sin v\cos v]\sqrt{\frac{2}{\delta}}\sin\left[\frac{\pi}{\delta}(u_2-u)\right],$$
$$\lambda(v)=\frac{1}{g_0(v)}\left[\frac{\pi^2}{\delta^2}+\frac{\alpha^2}{4}\sinh^2(2u_0)\right]; \tag{8}$$

here and in the following we use the abbreviated notation

$$g_0(v)\equiv g(u_0,v). \tag{9}$$

In the case of a ring of constant thickness (Fig. 1b) the potential in Eq. (6) is explicitly dependent on the slow coordinate v, since the edges of the ring in (7) are no longer specified by a constant value of u, but are given by functions of v, whose form is found in the following manner. Let γ_1 and γ_2 be the distances from points of the ellipse $u=u_0$ measured along normals to it from outside and from inside. Assuming that γ_1 and γ_2 are small, we can identify these distances with the elements ds_1 and ds_2 of the length of the arc of the hyperbola $v=\text{const}$ normal to the ellipse at the same point. Since the quantity $g(u,v)$ introduced by Eq. (3) defines the diagonal components of a metric tensor, we easily obtain $\gamma_{1,2}=h\sqrt{g(u_0,v)}du_{1,2}$. Hence we at once have

$$u_1=u_0-\frac{\gamma_1}{h\sqrt{g_0(v)}}, \quad u_2=u_0+\frac{\gamma_2}{h\sqrt{g_0(v)}}. \tag{10}$$

Thus, we should substitute $u_2=u_2(v)$ from (10) into the expression for the wave function (8) and replace the constant δ in (8) by the function

$$\tilde{\delta}(v)=\frac{\gamma}{h\sqrt{g_0(v)}}, \tag{11}$$

where $\gamma\equiv\gamma_1+\gamma_2$ is the total width of the ring.

Now, to obtain a one-dimensional equation for $\chi(v)$, we substitute the wave function $\psi(u,v)$ in the form (5) into (4), multiply both sides of the equation by $\eta^*(u,v)$, and integrate over u. After some relatively simple transformations, in the case of a ring formed from confocal ellipses we obtain

$$-\frac{1}{g_0(v)}\frac{\partial^2\chi}{\partial v^2}-\frac{i\alpha}{g_0(v)}\sinh(2u_0)\frac{\partial\chi}{\partial v}+\lambda(v)\chi=\frac{2mh^2}{\hbar^2}E\chi, \tag{12}$$

and in the case of a ring of constant width we have

$$-\frac{1}{g_0(v)}\frac{\partial^2\chi}{\partial v^2}-\frac{i\alpha}{g_0(v)}\sinh(2u_0)\frac{\partial\chi}{\partial v}+\tilde{\lambda}(v)\chi+\frac{1}{16g_0^3}\left(\frac{\partial g_0}{\partial v}\right)^2\left[1+\frac{4\pi^2}{3}-4\pi^2\frac{\gamma_1/\gamma_2}{(1+\gamma_1/\gamma_2)^2}\right]\chi=\frac{2mh^2}{\hbar^2}E\chi, \tag{13}$$

where $\tilde{\lambda}$ is defined by (8) with $\tilde{\delta}$ from (11).

The magnetic field is eliminated from Eqs. (12) and (13) by the gauge transformation

$$\chi(v)=\tilde{\chi}(v)\exp(-i\alpha v\sinh u_0\cosh u_0).$$

In addition, after the limiting transition to a one-dimensional elliptical ring, we can employ the standard properties of an ellipse, viz., the eccentricity ε and the length of the major semiaxis a:

$$\varepsilon=\frac{1}{\cosh u_0},\quad a=\frac{h}{\varepsilon}.$$

Now, according to (9), for g_0 we obtain

$$g_0=\frac{1-\varepsilon^2\cos^2 v}{\varepsilon^2}.$$

Thus, the one-dimensional equation for $\tilde{\chi}(v)$ takes on a compact form:

a) for the case of confocal ellipses

$$\frac{\partial^2\tilde{\chi}}{\partial v^2}+\left[\Delta(1-\varepsilon^2\cos^2 v)-\frac{\pi^2}{\delta^2}\right]\tilde{\chi}=0, \tag{14}$$

b) for a ring of constant width

$$\frac{\partial^2\tilde{\chi}}{\partial v^2}+\left[\left(\Delta-\frac{\pi^2 a^2}{\gamma^2}\right)(1-\varepsilon^2\cos^2 v)-\frac{1}{16}\right.$$
$$\left.\times\left(1+\frac{4\pi^2}{3}-4\pi^2\frac{\gamma_1/\gamma_2}{(1+\gamma_1/\gamma_2)^2}\right)\frac{\varepsilon^4\sin^2(2v)}{(1-\varepsilon^2\cos^2 v)^2}\right]\tilde{\chi}=0, \tag{15}$$

where the dimensionless energy Δ is defined as

$$\Delta=\frac{2ma^2}{\hbar^2}E.$$

Now the significant difference between the two ways of going over to a one-dimensional elliptical ring is clearly visible. Both equations [(14) and (15)] contain the infinite (in the limit) constants π^2/δ^2 and $\pi^2 a^2/\gamma^2$ corresponding to the energies of the zero-point transverse-motion oscillations. However, in the case of a ring of constant width [Eq. (15)] this constant is calculated from the total energy. Consequently, the frequencies of electron motion along the ellipse are finite. In the case of confocal ellipses [Eq. (14)] these frequencies tend to infinity as $\delta\to 0$, and a reasonable limiting transition to a one-dimensional ring is impossible. This is attributed to purely geometric factors: in the limit $\delta\to 0$, the electron is confined to the "pockets" near the regions of smallest curvature (see Fig. 1a), i.e., the motion along the coordinate v is also localized, the dimensions of this region of localization tending to zero together with δ.

We also note the nontrivial character of the limiting transition in the case of a ring of constant width. The resultant effective potential in (15) depends on the ratio γ_1/γ_2, i.e., on the way in which the limit is reached (to be specific, we shall henceforth assume that $\gamma_1=\gamma_2=\gamma/2$).

The results obtained above are not specified by the details of the rectangular-well potential. Bounding the one-dimensional motion along the ellipse using the parabolic-trough potential

$$V_p(u,v)=\frac{m\Omega^2(v)}{2}(u-u_0)^2,$$

when $\Omega(v)=\Omega_0=\text{const}$, we obtain the following equation in the adiabatic limit for the gauge-transformed wave function of longitudinal motion:

$$\frac{\partial^2\tilde{\chi}}{\partial v^2}+\left[\Delta(1-\varepsilon^2\cos^2 v)-\frac{ma^2\Omega_0}{\hbar}\sqrt{1-\varepsilon^2\cos^2 v}\right.$$
$$\left.-\frac{1}{32}\frac{\varepsilon^4\sin^2(2v)}{(1-\varepsilon^2\cos^2 v)^2}\right]\tilde{\chi}=0.$$

This equation is similar in its physical meaning to (14), i.e., it also corresponds to confinement of the longitudinal electron motion to the points of smallest curvature of the ellipse following the limiting transition $\Omega_0\to\infty$. Similarity to the case of a rectangular trough of constant width will clearly be achieved when we choose a function $\Omega(v)$ such that the energy is independent of the transverse-motion levels on v i.e., when

$$\Omega(v)=\Omega_0\sqrt{1-\varepsilon^2\cos^2 v}.$$

This choice leads to an equation similar to (14):

$$\frac{\partial^2\tilde{\chi}}{\partial v^2}+\left[\left(\Delta-\frac{ma^2\Omega_0}{\hbar}\right)(1-\varepsilon^2\cos^2 v)\right.$$
$$\left.-\frac{1}{8}\frac{\varepsilon^4\sin^2(2v)}{(1-\varepsilon^2\cos^2 v)^2}\right]\tilde{\chi}=0. \tag{16}$$

To conclude this section, we mention more ways to obtain an elliptical ring, which is apparently highly conducive to experimental implementation. Let there initially be a circular ring on the xy plane in the form of a rectangular trough between concentric circles of radii R_1 and R_2. We now consider a uniaxial deformation of the plane ($x'=x$, $y'=\alpha y$, $\alpha\leqslant 1$. As a result of this transformation, the circles R_1 and R_2 become ellipses that have the same eccentricity $\varepsilon=\sqrt{1-\alpha^2}$, but are not confocal. Performing the procedure of the adiabatic approximation in this case ($R_2-R_1\ll R_1$), we obtain the following equation for the longitudinal wave function in the elliptical coordinate system associated with the inner contour:

$$\frac{\partial^2\tilde{\chi}}{\partial v^2}+\left[\Delta(1-\varepsilon^2\cos^2 v)-\frac{4\pi^2 R_1^4}{(R_2^2-R_1^2)^2(1-\varepsilon^2)}(1-\varepsilon^2\cos^2 v)^2\right.$$
$$\left.-\frac{1}{4}\left(1+\frac{4\pi^2}{3}\right)\frac{\varepsilon^4\sin^2(2v)}{(1-\varepsilon^2\cos^2 v)^2}\right]\tilde{\chi}=0. \tag{17}$$

Equation (17) resembles Eq. (14) for the case of confocal ellipses in the sense that the frequencies of longitudinal motion tend to infinity as the width of the ring decreases. One significant difference from (14), however, is that now the electron is trapped in the region of greatest curvature of the ellipse.

Summarizing the content of this section, we can make the following statements. When we go from a circular ring to an elliptical ring, there are two factors that modulate electron motion along the ring: the variations of the width and curvature of the latter. When the elliptical ring is assigned in different ways, these factors combine differently and can lead to qualitatively different types of electron motion along the ring.

3. ELECTRON ENERGY SPECTRUM: GENERAL PROPERTIES

As we know, the magnetic-field dependence of the persistent current is determined by the dependence of the energy levels of the orbiting motion of the electrons in the ring on the magnetic field. In this context we shall ascertain which features of the energy spectrum vary significantly when we pass from a circular to an elliptical ring. Our analysis is limited to the situation in which the librational character of the electron motion is maintained in the elliptical ring and there is no longitudinal confinement. Therefore, we must consider the case of a ring of constant width [Eq. (15)]. When the results are extended to the case of confocal ellipses [Eq. (14)], we shall have to require that the ratio ε/δ be finite (and sufficiently small).

Formally, Eq. (15) is the Hill equation,[6] and it can be written in the standard form

$$\frac{\partial^2\tilde{\chi}}{\partial v^2}+\left[\theta_0+2\sum_{n=1}^{\infty}\theta_n\cos(2nv)\right]\tilde{\chi}=0,$$

where in our case

$$\theta_0=\tilde{\Delta}-\beta^2\xi\left[1+\sum_{k=1}^{\infty}\left(\frac{\beta}{2}\right)^{2k}\left(2k+1-\frac{2k-1}{\beta^2}\right)C_{2k}^{k}\right],$$

$$\theta_{2l-1}=-\frac{\beta\tilde{\Delta}}{2}\delta_{l,1}-\beta^2\xi\sum_{k=l}^{\infty}\left(\frac{\beta}{2}\right)^{2k-1}\left(2k-\frac{2k-2}{\beta^2}\right)\times C_{2k-1}^{k-l},$$

$$\theta_{2l}=-\beta^2\xi\sum_{k=l}^{\infty}\left(\frac{\beta}{2}\right)^{2k}\left(2k+1-\frac{2k-1}{\beta^2}\right)C_{2k}^{k-l}.$$

Here the C_n^l are binomial coefficients, and the following abbreviated notation is used:

$$\tilde{\Delta}=\left(\Delta-\frac{\pi^2a^2}{\gamma^2}\right)\left(1-\frac{\varepsilon^2}{2}\right),\quad \beta=\frac{\varepsilon^2}{2-\varepsilon^2},$$

$$\xi=\frac{1}{4}\left(1+\frac{\pi^2}{3}\right). \qquad (18)$$

According to the general theory of the Hill equation,[6] its solution has the form

$$\tilde{\chi}(v)=\tilde{\chi}_\mu(v)=e^{i\mu v}\varphi(v), \qquad (19)$$

where $\varphi(v)$ is a periodic function with period π, and the characteristic exponent μ is specified in terms of the coefficients θ_i by the equation

$$\cos(\pi\mu)=1-2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)\det(\hat{M}), \qquad (20)$$

where $\hat{M}$ is an infinite matrix with the elements

$$M_{mn}=-\frac{\theta_n}{(2m)^2-\theta_0}.$$

In our case the requirement of uniqueness imposed on the elliptical contour of the wave function $\chi(v)$ leads to the following condition for $\tilde{\chi}(v)$:

$$\tilde{\chi}(v+2\pi)=\exp(2\pi i\alpha\sinh u_0\cosh u_0)\tilde{\chi}(v),$$

which, together with (19), specifies μ in the form

$$2\pi\mu=2\pi\alpha\sinh u_0\cosh u_0+2\pi n=\pi\frac{h^2}{l_H^2}\sinh u_0\cosh u_0+2\pi n=\frac{S_e}{l_H^2}+2\pi n=2\pi(\tilde{\Phi}+n),$$

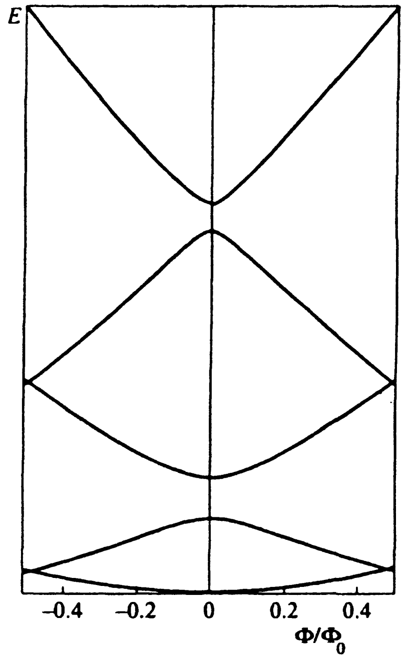

FIG. 2. Qualitative form of the dependence of the librational energy terms on the magnetic flux through the ring in the reduced-zone scheme.

where S_e is the area of the ellipse, $\tilde{\Phi}$ is the magnetic flux through this area measured in units of the magnetic flux quantum $\Phi_0=2\pi\hbar c/|e|$, and n is an integer.

Thus, Eq. (20) defines the dependence of the energy of the quantized levels of longitudinal electron motion in an elliptical ring (this energy $\tilde{\Delta}$ appears in the coefficients θ_0 and θ_1) on the magnetic flux through the ring. Since the flux appears in (20) only in the form

$$\cos(\pi\mu)=\cos[\pi(\tilde{\Phi}+n)], \qquad (21)$$

it can be asserted at once that the desired terms are even periodic functions of $\tilde{\Phi}$ with a period equal to 2. In addition, according to (21), the terms can be classified according to whether n in this equation is even or odd. However, since the shift $\tilde{\Phi}\to\tilde{\Phi}+1$ is equivalent to the replacement $n\to n+1$, the energy of an "even" term for a given $\tilde{\Phi}$ is equal to the energy of the "odd" term for $\tilde{\Phi}\pm 1$, and the even and odd terms cross at the points $\tilde{\Phi}=k+1/2$ (where k is an integer).

As a result, we can expect a qualitative picture of a spectrum that is reduced to the "Brillouin zone" $\tilde{\Phi}\in(-1/2,1/2)$ depicted in Fig. 2 with energy gaps in the center of the zone and "conical" points on its edges.

4. ENERGY GAP

Since the presence of energy gaps is the main special feature distinguishing the spectrum of an elliptical ring from the spectrum of a circular ring, let us consider the nature of these gaps in greater detail. We restrict ourselves here to small values of ε, in which each gap is still fairly narrow and can be investigated analytically in an approximation.

The existing methods for an approximate solution of Eq. (20)[7] do not have any special advantages over ordinary perturbation theory, which can be applied directly to Eq. (15). Therefore, we shall focus on the latter, more graphic method. To construct the correct perturbative scheme based on the smallness of ε^2 in (15), we should transform this equation somewhat. First of all, we rewrite it using the notation (18) in the form

$$-\frac{1}{1-\beta\cos(2v)}\frac{d^2\tilde{\chi}}{dv^2}+\xi\frac{\beta^2\sin(2v)}{[1-\beta\cos(2v)]^3}\tilde{\chi}=\tilde{\Delta}\tilde{\chi}. \tag{22}$$

We note that the one-dimensional Hamiltonian (22) [like the original equation (15)] is formally non-Hermitian and calls for the use of the weight function $1-\beta\cos(2v)$, which itself depends on the perturbation parameter, to calculate the matrix elements. To circumvent this problem, in (22) we seek the wave function $\tilde{\chi}(v)$ in the form

$$\tilde{\chi}(v)=\frac{\tilde{\chi}_c(v)}{\sqrt{1-\beta\cos(2v)}}.$$

Here the function $\tilde{\chi}_c(v)$ is defined by the equation

$$-\frac{d^2\tilde{\chi}_c}{dv^2}-\frac{\beta\cos(2v)}{1-\beta\cos(2v)}\frac{d^2\tilde{\chi}_c}{dv^2}+\frac{2\beta\sin(2v)}{[1-\beta\cos(2v)]^2}\frac{d\tilde{\chi}_c}{dv}$$
$$+\left[\frac{2\beta\cos(2v)}{[1-\beta\cos(2v)]^2}+(\xi-3)\frac{\beta^2\sin^2(2v)}{[1-\beta\cos(2v)]^3}\right]\tilde{\chi}_c=\tilde{\Delta}\tilde{\chi}_c, \tag{23}$$

which already enables us to treat the β-containing terms within ordinary perturbation theory.

In the zeroth approximation

$$(\tilde{\chi}_c)_n^{(0)}=\frac{1}{\sqrt{2\pi}}\exp[i(n+\tilde{\Phi})v],\quad \tilde{\Delta}_n^{(0)}=(n+\tilde{\Phi})^2. \tag{24}$$

The matrix elements $(\delta H)_{n'n}$ of the perturbation operator between the states (24) are calculated in the Appendix. These matrix elements are nonzero only if $n-n'=2m$ is an even number. Their order of magnitude is specified in the principal approximation as β^m. Restricting ourselves to an accuracy $\sim\beta$, we obtain the basic matrix elements in the form

$$(\delta H)_{n'n}=\frac{\beta}{2}(n+\tilde{\Phi})(n'+\tilde{\Phi})(\delta_{n',n+2}+\delta_{n',n-2}). \tag{25}$$

In the reduced-zone scheme used in the preceding section the matrix elements (25) correspond to removal of the degeneracy between the terms $\tilde{\Delta}_1^{(0)}(\tilde{\Phi})$ and $\tilde{\Delta}_{-1}^{(0)}(\tilde{\Phi})$ at their crossing point for $\tilde{\Phi}=0$, i.e., the lowest of the gaps in Fig. 2. The behavior of the terms near this gap is determined in the standard manner[8] and is expressed by the equation

$$\tilde{\Delta}_\pm=\tilde{\Phi}^2+1\pm\sqrt{4\tilde{\Phi}^2+\beta^2/4}. \tag{26}$$

Within the approximation used, precisely the same result is observed for the case of a parabolic trough (16). For the cases of rings of variable width [(14) and (17)] the form of (26) is maintained with the following changes within the proviso regarding the smallness of ε made at the beginning of Sec. 3. In the case of confocal ellipses (14), instead of (18) we should set

$$\tilde{\Delta}=\Delta\left(1-\frac{\varepsilon^2}{2}\right)-\frac{\pi^2}{\delta^2},$$

and multiply the second term under the radical sign in (26) by $1+\pi^2/\delta^2$. In the case of uniaxial compression of the ring

$$\tilde{\Delta}=\Delta-\frac{4\pi^2R_1^4(1+\varepsilon^2/2)}{(R_2^2-R_1^2)^2},$$

and the same term is multiplied by $1-4\pi^2R_1^4(1+\varepsilon^2/2)/(R_2^2-R_1^2)^2$. In both cases the added terms will naturally be dominant. For the higher gaps the second term under the radical sign in the equations like (26) is a significantly different function of the number of the gap in different variants.

We note that at sufficiently large values of the energy ($\tilde{\Delta}\geqslant 1/\beta$) the perturbative scheme in (23) becomes inapplicable even for small β. However, we are then in the region where the quasiclassical approximation is valid, and we can say, in analogy to Ref. 9, that the width of the gaps lying in this distant energy range is exponentially small: in the limit $\beta\ll 1$ it is of order $\exp[-(1/\beta)(\sqrt{\tilde{\Delta}}/2)]$, and for $\beta\sim 1$ it is of order $\exp(-\sqrt{\tilde{\Delta}}/2)$.

5. MAGNETIC-FIELD DEPENDENCE OF THE PERSISTENT CURRENT

In this section we study how the persistent current dependence on the magnetic flux Φ for an elliptical ring. The persistent current $J(\Phi)$ is a thermodynamic characteristic, and knowledge of only the energy spectrum of the system is needed to calculate it. In the case of noninteracting electrons, $J(\Phi)$ is given by the following expression:

$$J(\Phi)=-2c\sum_n\frac{\partial E_n}{\partial\Phi}f(E_n)=\frac{e\hbar}{2\pi ma^2}\sum_n\frac{\partial\tilde{\Delta}_n}{\partial\tilde{\Phi}}f(E_n),$$

where $f(E)$ is a Fermi distribution function.

In Ref. 10 the persistent current in a circular ring with an impurity was expressed in terms of the transmission coefficient $T(E)$ of the corresponding scattering problem at an energy equal to the Fermi energy. Our situation differs from Ref. 10 in that the coefficient in front of the term with the kinetic energy in the wave equation depends on the position, or, stated differently, the mass of the particle depends on the position. This raises the question of correctly determining the transmission (and reflection) coefficient for this case. This can be accomplished fairly simply in the quasiclassical approximation or by perturbation theory, within which Eq. (26) is applicable. However, the calculation of $T(E)$ itself is not a bit easier than finding the corrections $\Delta_\pm$ to the spectrum, which determine the persistent current. In addition, the derivation of the equations in Ref. 10 made extensive use of an expansion in $1/L$ (L is the perimeter of the ring), i.e., one more small parameter is required in perturbation theory. The calculation method used in the present work is based only on the smallness of the eccentricity of the ellipse. Of course, for $\varepsilon \sim 1$ both approaches require the use of numerical methods.

We restrict ourselves to consideration of the situation in which the chemical potential of the electronic system is determined by external conditions (extrapolation of the results to the case of a fixed number of electrons in the ring is fairly obvious). It was shown in Sec. 3 that in the expressions for the energy spectrum the shift $\Phi \rightarrow \Phi + \Phi_0$ is equivalent to the replacement $n \rightarrow n+1$. Since the current is determined by the sum over all the quantum states n, it can be concluded at once that $J(\Phi)$ is a periodic function of Φ with period Φ_0. The oddness of the current follows immediately from the evenness of the spectrum with respect to Φ. Thus, ellipticity does not alter the period of the function $J(\Phi)$: just as in a circular ring, it is equal to the flux quantum. At the same time, the electron energy spectrum in an elliptical ring near the gaps differs significantly from the spectrum in a circular ring. Therefore, considerable changes can be expected in the dependence of the persistent current on Φ when the chemical potential is in a gap region.

We performed some numerical calculations of plots of $J(\Phi)$ for the case of an elliptical ring of constant width with neglect of the scattering and at a temperature equal to zero. The energy spectrum given by Eq. (26) was used in the calculations. Figure 3 shows the variation of the plots of $J(\theta)$ when the Fermi level E_F passes through the lowest gap (the middle of this gap corresponds to a value of the dimensionless energy $\tilde{\Delta}$ equal to unity). The curves were obtained for an eccentricity $\varepsilon = 1/2$, the gap width being equal to 1/8.

When the Fermi level μ $(\mu \equiv E_F 2ma^2/\hbar^2)$ is sufficiently far from the lower and upper edges of the gap, the plots of $J(\Phi)$ have a saw-tooth shape and do not differ from those for the case of a circular ring. However, when μ is in a gap region, the situation changes radically (Figs. 3a, b, and c): smoothing of the "teeth" occurs at the sites of the discontinuities, and when μ is within a gap, the current passes through zero at $\Phi = 0$, and the slope changes sign, i.e., the sign of the magnetic susceptibility changes, near $\Phi = 0$. In other words, all other conditions being equal (i.e., at the same values of μ and Φ) a circular ring can be diamagnetic, and an elliptical ring can be paramagnetic (see Fig. 3b). Physically, this is attributed to the fact that for specified values of μ and Φ (and at zero temperature) the number of particles depends on the eccentricity of the ring.

The picture obtained of smooth teeth on the plots of $J(\Phi)$ outwardly resembles the variation of the persistent current in a circular ring as a result of the effects of randomly arranged impurities and/or a finite temperature. We note, however, a qualitative difference in our situation. In an ideal elliptical ring the gaps in the spectrum and, accordingly, the smoothed teeth are observed only at the center of the Brillouin zone. This is due to the invariance of the effective potential under the transformation $v \rightarrow v + \pi$. Since this potential is determined purely geometrically, the gap width can be controlled and varied by external forces (for example, uniaxial compression, see Sec. 1). In an elliptical ring with impurities these gaps can differ from the impurity gaps, which also appear along the zone edges. Thus, an experimental investigation of the persistent current in nonideal elliptical rings can provide additional information on the character of the action of the impurity potential.

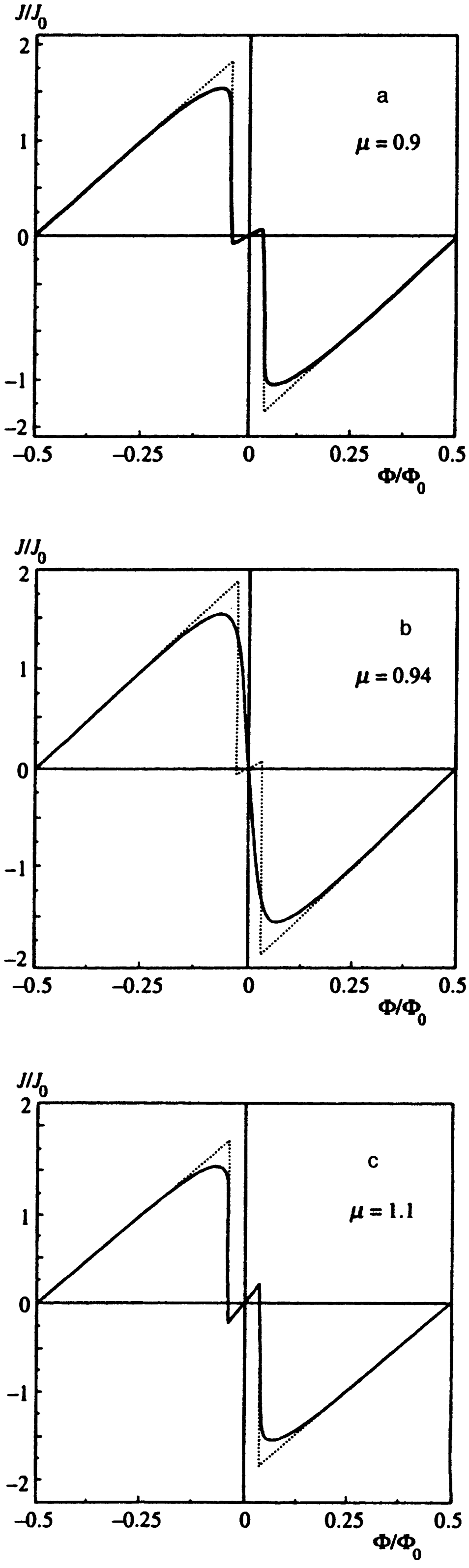

FIG. 3. Persistent current in an elliptical ring of constant width as a function of the magnetic flux for various positions of the Fermi level ($J_0 = e\hbar/2\pi ma^2$). The dotted lines correspond to the situation of a circular ring ($\varepsilon = 0$).

6. CONCLUSIONS

In the present work we have considered the quantum-mechanical properties of quasi-one-dimensional curvilinear electronic systems. From the standpoint of pure mathematics a curve does not have an internal geometry, and in this sense any curved line is equivalent to a straight line. However, it is physically clear that we are dealing with narrow waveguides in the ultraquantum limit: the ground state for transverse motion. In this case only a circle is equivalent to a straight line, since its curvature is constant. The variable curvature of an ellipse (and, obviously, any other curve) leads to a Schrödinger equation with variable coefficients, which describes the motion of a particle with a mass that depends on the coordinates in a certain potential. Of course, this causes significant changes in the energy spectrum and, consequently, in the magnetic-field dependence of the persistent current (for the case of closed curves). The periodicity (with period Φ_0) of the persistent current as a function of the magnetic flux through the contour, however, remains universal. Only in the quasiclassical limit $\lambda_e \ll L$ (where the electron wavelength is much smaller than the distance over which the radius of curvature varies significantly) does the influence of the variable curvature vanish, and under quantization conditions it is simply necessary to replace $2\pi a$ (a is the radius of the circle) by the length of the contour. The error committed here is equivalent to the neglect of superbarrier reflection in quasiclassical scattering, i.e., in the case of smooth curves it corresponds to ignoring the exponentially narrow gaps in the energy spectrum. If there are singularities on the contour, the effects not taken into account by the quasiclassical approximation are small as a certain power of λ_e/L.

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APPENDIX A:

We find the matrix elements of the perturbation operator

$$\delta\hat{H} = -\frac{\beta\cos(2v)}{1-\beta\cos(2v)}\frac{d^2}{dv^2} + \frac{2\beta\sin(2v)}{[1-\beta\cos(2v)]^2}\frac{d}{dv} + \frac{2\beta\cos(2v)}{[1-\beta\cos(2v)]^2} + (\xi-3)\frac{\beta^2\sin^2(2v)}{(1-\beta\cos(2v))^3} \quad \text{(A1)}$$

between the wave functions (24). We note, first of all, that for any two functions $\chi_1(v)$ and $\chi_2(v)$

$$\int_0^{2\pi} dv\,\chi_2^*\left[-\frac{\beta\cos(2v)}{1-\beta\cos(2v)}\frac{d^2}{dv^2} + \frac{2\beta\sin(2v)}{[1-\beta\cos(2v)]^2}\frac{d}{dv}\right]\chi_1 = \int_0^{2\pi} dv\,\frac{\beta\cos(2v)}{1-\beta\cos(2v)}\frac{d\chi_2^*}{dv}\frac{d\chi_1}{dv}. \quad \text{(A2)}$$

Therefore, the deired matrix element is written in the form

$$(\delta H)_{n'n} = \frac{1}{2\pi}\int_0^{2\pi} dv\,\exp[i(n-n')v] \times\left[\frac{\beta\cos(2v)(n'+\tilde{\Phi})(n+\tilde{\Phi})}{1-\beta\cos(2v)} + \frac{2\beta\cos(2v)}{[1-\beta\cos(2v)]^2} + (\xi-3)\frac{\beta^2\sin^2(2v)}{[1-\beta\cos(2v)]^3}\right]. \quad \text{(A3)}$$

After integration by parts and some other relatively simple transformations, it can be reduced to the form

$$(\delta H)_{n'n} = \frac{1}{2\pi}\left\{-(n+\tilde{\Phi})^2\delta_{n,n'} + \left[(n'+\tilde{\Phi})(n+\tilde{\Phi}) - \frac{\xi-3}{8}(n-n')^2 + \frac{\xi+1}{2}\beta\frac{\partial}{\partial\beta}\right]\right\} \times\int_0^{2\pi} dv\,\frac{\exp[i(n-n')v]}{1-\beta\cos(2v)}. \quad \text{(A4)}$$

This expression is nonzero, only if $n-n'$ is an even number and, using the notation

$$m=(n-n')/2, \quad s=(n+n')/2, \quad \text{(A5)}$$

can be written as

$$(\delta H)_{n'n} = \left(\frac{\beta}{2}\right)^{|m|}\left\{(1-\delta_{m,0})(s+\Phi)^2 + \frac{|m|}{2}(\xi+1 - |m|(\xi-1))\sum_{k=1}^{\infty}\left[(s+\Phi)^2 - \frac{\xi-1}{2}m^2 + \frac{\xi+1}{2}(|m|+2k)\right]\left(\frac{\beta}{2}\right)^{2k} C_{|m|+2k}^{k}\right\}. \quad \text{(A6)}$$

This yields Eq. (25) to leading order in β.

[1] M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. A **96**, 365 (1983).
[2] R. Landauer and M. Büttiker, Phys. Rev. Lett. **54**, 2049 (1985).
[3] H. Müller-Groeling, H. A. Weidemüller, and C. H. Lewenkopf, Europhys. Lett. **22**, 193 (1993).
[4] A. O. Govorov, V. Chaplik, L. Vendler, and V. M. Fomin, JETP Lett. **60**, 643 (1994).
[5] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill, New York, 1961) [Russ. transl., Nauka, Moscow, 1978].
[6] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed. (Cambridge University Press, Cambridge, 1927) [Russ. transl., Part 2, Fizmatgiz, Moscow, 1963].
[7] *Bateman Manuscript Project. Higher Transcendental Functions, Vol. 3*, edited by A. Erdélyi (McGraw-Hill, New York, 1955) [Russ. transl., Vol. 3, Nauka, Moscow, 1967].
[8] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, 3rd ed. (Pergamon Press, Oxford, 1977).
[9] A. M. Dykhne, Zh. Eksp. Teor. Fiz. **40**, 1423 (1961) [Sov. Phys. JETP **13**, 999 (1961)].
[10] A. O. Gogolin and N. V. Prokof'ev, Phys. Rev. B **50**, 4921 (1994).

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