Functional identity of the spectra of Bose- and Fermi radiation of an accelerated mirror in 1+1 space and the spectra of electric and scalar charges in 3+1 space, and its relation to radiative reaction

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The discovery in Ref. 5 that the spectrum of scalar radiation by an accelerated mirror in 1+1 space is identical to that of electromagnetic radiation by a charge in 3+1 space is augmented by a demonstration that the spectrum of spinor radiation by a mirror in 1+1 space is identical to that of scalar radiation by a scalar charge in 3+1 space. No divergences are encountered in transforming the energy spectral representations of scalar and spinor radiations by a mirror into space-time representations, and the latter process leads to functionally identical distributions—the well known energy-momentum tensors of the radiation, which differ only by a factor of 1/2 and represent a local characteristic of the source path. This characteristic differs from the force of radiation reaction only by a Doppler factor. At the same time, transforming the spectral representations of the average numbers of bosons and fermions radiated by the mirror leads to two significantly different space-time distributions, which are functionals of the path. Their asymptotic behaviors are characterized by different functions of a relativistic invariant—the relative velocity of the source at the ends of the path. These functions determine the amplitude for preservation of the vacuum by the sources. It is shown that the correlation functions of the energy-momentum tensors of a scalar and a spinor field reflect the statistical properties of the fields in the spectral distribution of the energy of a pair between its constituent particles; however, it is also shown that they cease to disagree when interpreted in terms of the energy of one of the particles. It is argued that it is incorrect to consider spectra of sources that radiate infinite energy within a finite proper time.
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1. INTRODUCTION

The radiation of an accelerated mirror in vacuum has been actively discussed in recent years,[1–3] especially in connection with its analogy to the Hawking mechanism[4] for generation of particles by the gravitational field created by a black hole. The most detailed relativistically invariant characteristic of the radiation is the spectrum of the number of radiated particles. Since this is a functional of the path of the source, it depends nontrivially on the momentum and spin of the radiated quanta. In Ref. 5 it was shown that the spectrum of scalar quanta radiated by an accelerated mirror in 1+1 space coincides functionally (after a corresponding covariant mapping of variables) with the spectrum of photons radiated by an electric charge moving in 3+1 space along the same path as the mirror. Furthermore, the space-time distribution of the energy-momentum of a scalar field coincides (once more when the necessary covariances properties are included) with the bremsstrahlung force acting on a charge in electrodynamics. Analogies of this kind make possible a deeper understanding of the physical processes being compared.

In this paper it is shown that the spectrum of quanta of a spinor field radiated by an accelerated mirror in 1+1 space coincides functionally with the spectrum of scalar quanta radiated by a scalar charge (i.e., a source of the scalar field) moving in 3+1 space along the same path as the mirror.

In Sec. 3 the spectral integral for the total energy- momentum of the Bose radiation generated by the accelerated mirror is transformed into an integral over space-time (characteristic) variables of the energy-momentum tensor of the radiation field. This transformation recalls the theorem of Plancherel in the theory of Fourier integrals.[6] It can be regarded as a derivation of the energy-momentum tensor of the radiation field which in contrast to a direct derivation[7,8] does not encounter divergences and does not require a regularization procedure.

In Sec. 4 an analogous transformation is derived for the spectral representation of the total energy-momentum of Fermi radiation from an accelerated mirror. Despite the considerable difference between the spectra of boson and fermion radiation, the corresponding characteristic distributions—energy-momentum tensors—coincide functionally, disagreeing only by a factor 1/2. Unlike the spectra, these are not functionals of the path, but rather are represented by a local characteristic, namely the Schwartz derivative. Nevertheless, irreversible radiation of energy-momentum takes place not at a point, but rather within a space-time region outside of which the so-called Schott energy-momentum disappears.

In Sec. 5 the spectral integrals for the average numbers of bosons and fermions radiated by an accelerated mirror are transformed into integrals over the characteristic variable. The corresponding distributions with respect to the characteristic variable (the functions $K(u)$) possess the properties of path functionals, indicating that the generation of particles is significantly nonlocal. In this section the universal asymptotic behavior of the function $K(u)$, which determines the infrared behavior of the average number N of radiated quanta, is discussed for paths with nonzero relative velocity at their ends. The amplitude e^{iW} for preservation of the vacuum by the source of particles is completely recovered with the help of dispersion relations (with respect to the mass of the quanta) that relate the number of radiated particles $N = 2\,\mathrm{Im}\,W$ to the phase $\mathrm{Re}\,W$ of the vacuum amplitude.

Section 6 provides concrete examples of the spectra of bosons and fermions radiated by a mirror moving along a semihyperbola, and the corresponding space-time distributions for energy and number of particles are discussed.

In Sec. 7, where correlations are discussed between the emission and absorption of pairs that transfer energy- momentum from one point to another, the statistical properties of the field quanta are displayed. In the scalar case, the total energy of a pair is distributed for the most part evenly between the particles that make it up, whereas in the spinor case this distribution is never even. This latter fact is somewhat unexpected, since the fermion and antifermion are not identical due to the difference of their fermionic charges.

In the last section, the symmetry of the characteristic energy-momentum distributions of Bose and Fermi radiation from a mirror under the discrete transformation (96) is discussed. This symmetry excludes paths with nonzero Schott energy-momentum at their ends, which leads to additional considerations regarding the unphysical nature of these paths.

2. SPINOR FIELD AND BOGOLYUBOV COEFFICIENTS

The discovery in Ref. 5 that the spectra radiated by a scalar mirror in $1+1$ space coincide with those of an electric charge in $3+1$ space prompts the speculation that motion of such a mirror in $1+1$ space creates a spectrum of radiation similar to that of radiation of a scalar charge in $3+1$ space. It is well known that such a spectrum is described by the expression[1)]

$$d\bar{n}_{\mathbf{k}}^{sc}=|\rho(k)|^2\frac{d^3k}{16\pi^3k^0},\quad \rho(k)=e\int_{-\infty}^{\infty}d\tau\, e^{-ik_\alpha x^\alpha(\tau)},\qquad (1)$$

where $\rho(k)$ is the Fourier component of the scalar charge density, $\mathbf{k}$ and k_0 are the wave vector and frequency of the radiation, and $x^\alpha(\tau)$ is the path of the charge as a function of proper time τ.

For a charge moving along the 1 axis, transforming from x,t to the characteristic variables $u=t-x$, $v=t+x$, we obtain

$$d\bar{n}_{\mathbf{k}}^{sc}=\left|e\int_{-\infty}^{\infty}du\sqrt{f'(u)}\times\exp\left[\frac{i}{2}(k_+u+k_-f(u))\right]\right|^2\frac{dk_+dk_-}{16\pi^2}.\qquad (2)$$

Here, in place of τ we have chosen the variable $u=x_-(\tau)$, so that the v-coordinate of the charge, i.e., $x_+(\tau)$, is a function of u. As in Ref. 5, it is denoted by $f(u)=x_+(\tau)|_{\tau=\tau(u)}$.

It follows from Eq. (2) that in order for an accelerated mirror in $1+1$ space to reproduce the spectrum, it is necessary that the Bogolyubov coefficients have the form

$$\alpha_{\omega'\omega},\beta^*_{\omega'\omega}=\int_{-\infty}^{\infty}du\sqrt{f'(u)}\exp[\mp i\omega u+i\omega' f(u)]\qquad (3)$$

$$=\int_{-\infty}^{\infty}dv\sqrt{g'(v)}\exp[i\omega' v\mp i\omega g(v)].\qquad (4)$$

Here, as in Ref. 5, we denote the u coordinate of the mirror (or charge) by $g(v)$ to show that it is a function of the independent variable v. The functions $f(u)$ and $g(v)$ are mutual inverses: $f[g(v)]=v$. Their derivatives are positive. Representations (3) and (4) are conveniently referred to as the f- and g-representations, respectively.

It is not difficult to verify that the coefficients (3), (4) satisfy the conditions

$$\int_0^\infty\frac{d\omega''}{2\pi}(\alpha_{\omega''\omega}\alpha^*_{\omega''\omega'}\pm\beta_{\omega''\omega}\beta^*_{\omega''\omega'})=2\pi\delta(\omega-\omega'),$$
$$\int_0^\infty\frac{d\omega''}{2\pi}(\alpha_{\omega''\omega}\beta_{\omega''\omega'}\pm\beta_{\omega''\omega}\alpha_{\omega''\omega'})=0\qquad (5)$$

with the upper sign. This implies that in $1+1$ space we are dealing with a mirror interacting with a spinor field. The lower sign in (5) corresponds to interaction with a scalar field.[1,2]

In order to verify that coefficients (3), (4) apply to a spinor field, let us investigate the following in- and out-sets of solutions of the Dirac wave equation with spin projection $s=-1/2$:

$$\psi_{\mathrm{in}\;\omega'-1/2}=\begin{pmatrix}\xi_2\\0\end{pmatrix}\exp(-i\omega' v)+\begin{pmatrix}0\\\eta_2\end{pmatrix}\sqrt{f'(u)}\times\exp(-i\omega' f(u)),\qquad (6)$$

$$\psi_{\mathrm{out}\;\omega-1/2}=\begin{pmatrix}\xi_2\\0\end{pmatrix}\sqrt{g'(v)}\exp(-i\omega g(v))+\begin{pmatrix}0\\\eta_2\end{pmatrix}\times\exp(-i\omega u),\qquad (7)$$

and with spin projection $s=+1/2$:

$$\psi_{\mathrm{in}\;\omega' 1/2}=\begin{pmatrix}0\\\eta_1\end{pmatrix}\exp(-i\omega' v)+\begin{pmatrix}\xi_1\\0\end{pmatrix}\sqrt{f'(u)}\times\exp(-i\omega' f(u)),\qquad (8)$$

$$\psi_{\mathrm{out}\;\omega 1/2}=\begin{pmatrix}0\\\eta_1\end{pmatrix}\sqrt{g'(v)}\exp(-i\omega g(v))+\begin{pmatrix}\xi_1\\0\end{pmatrix}\times\exp(-i\omega u).\qquad (9)$$

Here

$$\xi_1=\eta_1=\begin{pmatrix}1\\0\end{pmatrix},\quad \xi_2=\eta_2=\begin{pmatrix}0\\1\end{pmatrix}$$

are the usual spinors, so that the bispinors $\psi_{\omega s}$ are eigenvectors of the matrix Σ_3, i.e., $\Sigma_3\psi_s = 2s\psi_s$.

Each of the in- (out-) solutions given here in the x,t plane to the right of the world line of the mirror describes a spinor field with definite projection of spin, consisting of a monochromatic (nonmonochromatic) wave incident on the mirror, and a wave that is nonmonochromatic (monochromatic) reflected from the mirror.

These solutions are determined unambiguously by specifying the monochromatic wave on the characteristic $u=u_R^- \to -\infty$ in the distant past (on the characteristic $v=v_R^+ \to +\infty$ in the distant future), matching the phases of the incident and reflected waves on the mirror, and the relation $j^1=\dot{\xi}(t)j^0$ between the spatial and temporal components of the current density, where $\dot{\xi}$ is the mirror velocity. These solutions satisfy the following orthogonality and normalization conditions:

$$\int dx\, \psi^+_{\text{in}\,\omega' s'}(x,t)\psi_{\text{in}\,\omega s}(x,t) = 2\pi\delta(\omega'-\omega)\delta_{s's}. \tag{10}$$

The condition for the out-solutions is analogous.

The Bogolyubov coefficients are determined by the scalar products

$$\alpha_{\omega' s', \omega s} = \int dx\, \psi^+_{\text{in}\,\omega' s'}(x,t)\psi_{\text{out}\,\omega s}(x,t), \tag{11}$$

$$\beta^*_{\omega' s', \omega s} = \int dx\, \psi^+_{\text{in}\,\omega' s'}(x,t)\psi^*_{\text{out}\,\omega s}(x,t). \tag{12}$$

It is not difficult to show that they actually reduce to Eqs. (3), (4), which are diagonal with respect to spin projections, and do not depend on the latter. Therefore, in the final representations (3), (4) the indices of spin projection are not written out. However, in order to emphasize the important difference between the Bogolyubov coefficients for Fermi and Bose fields, we will attach the indices F and B to them, respectively. We recall [5] that

$$\alpha^B_{\omega'\omega}, \beta^{B*}_{\omega'\omega} = \pm\sqrt{\frac{\omega}{\omega'}}\int_{-\infty}^{\infty} du\, \exp[\mp i\omega u + i\omega' f(u)] \tag{13}$$

$$= \sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{\infty} dv\, \exp[i\omega' v \mp i\omega g(v)]. \tag{14}$$

These are the f- and g-representations, respectively.

We note that along with the solutions $\psi_{\omega s}$, $s=\pm 1/2$ with definite spin projections we could also use solutions $\psi_{\omega\pm}$ with definite helicity, which are eigenfunctions of the matrix γ_5. These are related to the previous expressions as follows:

$$\psi_\pm = \frac{1}{2}(1\pm\gamma_5)(\psi_{+1/2}+\psi_{-1/2}), \tag{15}$$

which satisfy the same orthogonality and normalization conditions, and lead to the same Bogolyubov coefficients.

The spectra $d\bar{n}^{B,F}_\omega$ of the average numbers of bosons or fermions created by the accelerated mirror are determined from the expression

$$d\bar{n}^{B,F}_\omega = \left(\int_0^\infty \left|\beta^{B,F}_{\omega'\omega}\right|^2 \frac{d\omega'}{2\pi}\right)\frac{d\omega}{2\pi}. \tag{16}$$

The physical interpretation of $d\bar{n}^{B,F}_\omega$ follows from second quantization of the fields ϕ and ψ, in which the coefficients of the expansion of the fields in plane waves with positive and negative frequencies are assigned the meaning of operators for absorption of particles and creation of antiparticles. As is clear from Eqs. (2), (3) and (16), the spectra $d\bar{n}^F_\omega$ and $d\bar{n}^{sc}_\omega$ are functionally identical if we identify ω, ω' with $k_+/2$, $k_-/2$ and ignore the factor e^2 which determines the force $F=-e^2/4\pi r^2$ of interaction between two identical nonrelativistic scalar charges located at a distance r from one another.

3. TRANSFORMATION OF THE SPECTRAL INTEGRAL FOR THE ENERGY OF BOSON RADIATION

In Ref. 5 a relation was obtained between the T_{++} component of the energy-momentum tensor of a scalar field generated by an accelerated mirror and the g_+ component of the bremsstrahlung force in classical electrodynamics:

$$e^2T_{++}(u)du = -\frac{1}{2}g_+ d\tau. \tag{17}$$

More precisely, the component T_{++} appearing here is the component of the finite part of the average value of the energy-momentum tensor operator $\hat{T}_{\alpha\beta}$ in the vacuum state:

$$\langle 0|\hat{T}_{\alpha\beta}(x)|0\rangle = T^\varepsilon_{\alpha\beta}(x)+T_{\alpha\beta}(x),$$

$$T_{++}(u) = \frac{1}{12\pi}\left[\left(\frac{f''}{2f'}\right)^2 - \left(\frac{f''}{2f'}\right)'\right]. \tag{18}$$

This expression was found by Davies and Fulling [7,8] by accurately regularizing this average value and separating the divergent $T^\varepsilon_{\alpha\beta}$ and finite $T_{\alpha\beta}$ parts.

The total energy (more precisely, the total energy-momentum) of bosons or fermions radiated by an accelerated mirror at infinity is determined by the spectral integral

$$\mathscr{E}^R = \int_0^\infty \omega d\bar{n}_\omega = \int_0^\infty\int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}\omega|\beta_{\omega'\omega}|^2. \tag{19}$$

Let us try to transform the integral for the energy-momentum $\mathscr{E}^{RB}$ of the bosons into an integral over the characteristic variables u or v. For this we substitute the $f{*}g$ representation into (19):

$$|\beta^B_{\omega'\omega}|^2 = -\int\int_{-\infty}^{\infty} du\, dv\, \exp[i\omega u + i\omega' f(u) - i\omega' v - i\omega g(v)], \tag{20}$$

which follows from (13) and (14), or half the sum of this term and its complex-conjugate representation. In the latter case, after integrating over the frequencies ω, ω' we obtain

$$\mathscr{E}^{RB}=\frac{1}{8\pi^2 i}\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du \times\left[\frac{1}{(u-g(v)-i\delta)^2(f(u)-v-i\varepsilon)}-\text{c.c.}\right]. \quad (21)$$

Here ε and δ are infinitesimal quantities that must be set to zero in the final expression. If for fixed real v we introduce $u_1=g(v)+i\delta$, then ε, δ can be chosen so that we have $f(u_1)=v+i\varepsilon$. To first approximation, $\delta\approx g'(v)\varepsilon$. Then it is clear that the first term of the function under the integral sign has a third-order pole at the point $u=u_1=g(v)+i\delta$. Analogously, the second term also has a third-order pole at the point $u=u_2=g(v)-i\delta$. Therefore, as ε, $\delta\to 0$ the integral over u from the first term in square brackets of (21) can be written as a contour integral of the function

$$F(u,v)=\frac{1}{(u-g(v))^2(f(u)-v)}, \quad (22)$$

which is taken along the real u-axis and passes around the third-order pole at the point $u=u_0=g(v)$ from below, while the integral of the second term becomes a contour integral of the same function but taken along the u-axis in the opposite direction and passing over the same pole. In this case, the integral with respect to u in (21) reduces to the residue of the function F at the pole $u=u_0$:

$$\int_{-\infty}^{\infty} du[\ ]=2\pi i \text{ res } F(u,v)|_{u=g(v)} =2\pi i\frac{1}{f_0'}\left[\frac{1}{4}\left(\frac{f_0''}{f_0'}\right)^2-\frac{f_0'''}{6f_0'}\right]. \quad (23)$$

Here $f_0=f(u_0)$, $f_0'=f'(u_0)$, etc. Choosing the variable $u_0=g(v)$ in the final integral with respect to v, in place of v we obtain

$$\mathscr{E}^R=\int_{-\infty}^{\infty} du\ R(u),$$

$$R(u)=\frac{\hbar c}{12\pi}\left[\left(\frac{f''(u)}{2f'(u)}\right)^2-\left(\frac{f''(u)}{2f'(u)}\right)'\right]. \quad (24)$$

In the final expression, the label on the variable u_0 is omitted. The dimensional factor $\hbar c$ is written explicitly, in order to direct attention to the quantum nature of these quantities; u, v have the dimensions of length.

We emphasize that the pole of the function $F(u,v)$ is located on the path of the mirror, and that its residue or the function $R(u)$ is a local characteristic of the path. In the mathematical literature,[9,10] this characteristic of a function $f(u)$ is called its Schwartz derivative and is denoted

$$\{f,u\}\equiv\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2=-2\sqrt{f'}\left(\frac{1}{\sqrt{f'}}\right)'', \quad (25)$$

so that

$$R(u)=-\frac{1}{24\pi}\{f,u\}.$$

Let us now discuss the physical meaning of $R(u)$. As is clear from (24), the function $R(u)$ coincides with the $T_{++}(u)$ component of the energy-momentum tensor of the field created by the accelerated mirror. According to Eq. (17), we may also treat it as the plus-component of the quantum bremsstrahlung force acting on the mirror $(\hbar c/e^2)g_\alpha$. Finally, differentiating the proper acceleration of the mirror $a=f''/2f'^{3/2}$ with respect to the proper time τ $(d\tau=\sqrt{f'}du)$, we can represent $R(u)$ in terms of a Lorentz-invariant rate of change $da/d\tau$ of the acceleration in the proper-time system:

$$R(u)=\frac{1}{12\pi}\left[\left(\frac{f''}{2f'}\right)^2-\left(\frac{f''}{2f'}\right)'\right] =T_{++}(u)=-\frac{1}{2}\frac{\hbar c}{e^2}g_+\sqrt{f'}=-\frac{1}{12\pi}f'\frac{da}{d\tau}. \quad (26)$$

Generally speaking, the function $R(u)$ is not positive definite. This means we can no longer identify it with the power of irreversible radiation departing to infinity. In other words, the quantity $R(u)du\equiv R(u)d\tau/\sqrt{f'}$ cannot be treated as energy radiated to infinity by an element of the path $d\tau$ lying close to the point u, $v=f(u)$. However, we can introduce an average power $\Delta\mathscr{E}^R/\Delta u$ of irreversible radiation of energy:

$$\Delta\mathscr{E}^R=\frac{1}{12\pi}\int_{\Delta u} du\left(\frac{f''}{2f'}\right)^2 \quad (27)$$

with the finite interval Δu chosen such that on this interval the term with the perfect differential in (26) does not contribute, i.e., at the ends of which the value of the Schott energy-momentum [11,13]

$$\mathscr{E}^S=-\frac{1}{12\pi}\frac{f''}{2f'}=-\frac{1}{12\pi}a\sqrt{f'} \quad (28)$$

are the same. Of course, the size of the interval Δu and the portion of energy $\Delta\mathscr{E}^R$ radiated from it should satisfy the uncertainty relation $\Delta u\Delta\mathscr{E}^R>1$.

Note that if the uncertainty relation holds for some interval Δu and energy $\Delta\mathscr{E}^R$ radiated from it, then the mean-square value of the Schott energy on such an interval is small compared with $\Delta\mathscr{E}^R$. In fact, because

$$\Delta\mathscr{E}^R=\frac{1}{12\pi}\int_{\Delta u} du\left(\frac{f''}{2f'}\right)^2 =12\pi\int_{\Delta u} du(\mathscr{E}^S)^2=12\pi\overline{(\mathscr{E}^S)^2}\Delta u, \quad (29)$$

it follows from the relation $\Delta u\Delta\mathscr{E}^R>1$ that

$$\Delta u\Delta\mathscr{E}^R>\sqrt{\Delta u\Delta\mathscr{E}^R}=\sqrt{12\pi\overline{(\mathscr{E}^S)^2}}\Delta u, \quad (30)$$

from which

$$\sqrt{\overline{(\mathscr{E}^S)^2}}<\frac{\Delta\mathscr{E}^R}{\sqrt{12\pi}}. \quad (31)$$

The Schott energy characterizes the magnitude of the change in field energy carried along by the mirror and located in the region where the true radiation is generated. Its mean square and average radiated power are determined by each other:

$$\overline{(\mathscr{E}^S)^2}=\frac{\hbar c}{12\pi}\frac{\Delta \mathscr{E}^R}{\Delta u}. \quad (32)$$

This is the distinctive relation between the fluctuations and energy dissipation.

If β denotes the velocity of the mirror, then its Doppler factor is

$$D(\beta)\equiv\sqrt{\frac{1+\beta}{1-\beta}}=\sqrt{f'(u)}. \quad (33)$$

Therefore, in the proper-time system of the mirror-source, the Schott energy equals

$$\mathscr{E}^S_{\text{self}}=-\frac{a}{12\pi} \quad (34)$$

and characterizes the change in mass of the field carried along by the source.

We now turn to an important fact. Equation (24) is a new integral representation of the total radiated energy $\mathscr{E}^R$, which originally was given by its spectral representation (19). In deriving (24), we have implicitly assumed the existence of all the integrals encountered in the course of making the transformation (19). In particular, the existence of the integrals over u of the two functions in the square brackets (21) does not impose any requirements on the asymptotic behavior of $f(u)$ as $u\to\pm\infty$. However, if we change the order of integration in the integral (21), and first carry out the v-integration, then the convergence of the v-integrals of the two functions in square brackets (21) is assured only when the asymptotic increase in the absolute value of $g(v)$ is not too weak, implying that the increase of $f(u)$ is not too strong:

$$|g(v)|\gtrsim[\ln(\pm v)]^\alpha, \quad v\to\pm\infty,$$
$$|f(u)|\lesssim\exp(\pm u)^\beta, \quad u\to\pm\infty, \quad (35)$$

$$0<\beta=\frac{1}{\alpha}<2. \quad (36)$$

In this case, the integration with respect to v reduces to the residue of the function $-F(u,v)$ at the point $v=v_0=f(u)$:

$$\int_{-\infty}^{\infty} dv[\ ]=-2\pi i \text{ res } F(u,v)|_{v=f(u)}$$
$$=2\pi i\frac{1}{g_0'^2}\left[\frac{3}{4}\left(\frac{g_0''}{g_0'}\right)^2-\frac{g_0'''}{3g_0'}\right]. \quad (37)$$

Differentiation of the identity $g(v_0)=u$ with respect to u gives the derivatives

$$g_0'\equiv g'(v_0)=\frac{1}{f'(u)}, \quad g_0''\equiv g''(v_0)=-\frac{f''(u)}{f'^3(u)},\dots \quad (38)$$

As a result, the following representation for $\mathscr{E}^R$ is obtained:

$$\mathscr{E}^R=\int_{-\infty}^{\infty} du\, \tilde{R}(u), \quad \tilde{R}(u)=\frac{1}{12\pi}\left[\left(\frac{f''}{2f'}\right)^2+\left(\frac{f''}{f'}\right)'\right], \quad (39)$$

in which the function $\tilde{R}(u)$ differs from $R(u)$ in having a different coefficient in the term with the perfect differential. However, the derivation of this representation requires that the asymptotic conditions (35), (36) be satisfied, for which the Schott energy-momentum behaves like

$$\mathscr{E}^S\propto-(\beta-1)u^{-1}\mp\beta(\pm u)^{\beta-1}, \quad (40)$$

as $u\to\pm\infty$, and asymptotically vanishes if the exponents α, β are bounded by the condition

$$0<\beta=\frac{1}{\alpha}<1, \quad (41)$$

which is a stricter condition than (36). In this case, the terms with perfect differentials both in $\tilde{R}(u)$ and in $R(u)$ give no contribution to the integral for $\mathscr{E}^R$, and even if the energy-momentum $\mathscr{E}^R$ itself increases as the interval of integration, it does so more slowly than linearly. Thus, both representations (24), (39) are equivalent in the integrated sense; however, the function $\tilde{R}(u)$ no longer has the physical meaning that $R(u)$ had, according to Eq. (26).

Note that if we use the gg^* representation for $|\beta_{\omega'\omega}|^2$ in (19)

$$|\beta_{\omega'\omega}|^2=\frac{\omega'}{\omega}\int\int_{-\infty}^{\infty} dv\, dv'\exp[i\omega'v+i\omega g(v)-i\omega'v'$$
$$-i\omega g(v')] \quad (42)$$

or half the sum of it and its complex-conjugate representation, then in place of (21) the following expression appears

$$\mathscr{E}^{RB}=-\frac{1}{8\pi^2 i}\int_{-\infty}^{\infty} dv\int_{-\infty}^{\infty} dv'$$
$$\times\left[\frac{1}{(v'-v-i\varepsilon)^2(g(v')-g(v)-i\delta)}-\text{c.c.}\right]. \quad (43)$$

The inner integral exists without requirements on $g(v')$, and reduces to the residue of the function

$$G(v',v)=\frac{1}{(v'-v)^2(g(v')-g(v))} \quad (44)$$

at the third-order pole $v=v'$. As a result, we obtain for $\mathscr{E}^R$ the representation

$$\mathscr{E}^R=\int_{-\infty}^{\infty} dv\, I(v), \quad I(v)=-\frac{1}{12\pi g'(v)}$$
$$\times\left[\left(\frac{g''}{2g'}\right)^2-\left(\frac{g''}{2g'}\right)'\right], \quad (45)$$

which coincides with the representation (24). Specifically, if we replace the variable of integration v by the variable $u=g(v)$, so that $v=f(u)$, the expression $I(v)dv$ under the integral sign becomes equal to $R(u)du$, and

$$-\frac{1}{g'}\left[\left(\frac{g''}{2g'}\right)^2-\left(\frac{g''}{2g'}\right)'\right]=\frac{1}{f'}\left[\left(\frac{f''}{2f'}\right)^2-\left(\frac{f''}{2f'}\right)'\right]=-\frac{da}{d\tau}. \quad (46)$$

The first equation is the well-known relation between the Schwartz derivatives of mutually inverse functions. Physically, however, these are different ways of writing the Lorentz-invariant quantity $-da/d\tau$, the rate of change of the acceleration in the proper system, so that

$$I(v)=-\frac{1}{12\pi}\frac{da}{d\tau}=\frac{1}{12\pi}\left[a^2+\left(\frac{g''}{2g'^2}\right)'\right]. \tag{47}$$

The latter equation represents $I(v)$ in terms of the square of the intrinsic acceleration and the derivative of the Schott energy with respect to v:

$$a=-\frac{g''}{2g'^{3/2}},\quad \mathcal{E}^S=\frac{1}{12\pi}\frac{g''}{2g'^2}. \tag{48}$$

On the interval Δv where the total change of Schott energy is $\Delta\mathcal{E}^R=0$, the average value of $I(v)$ may be considered to be the average radiated power:

$$\bar{I}=\frac{\Delta\mathcal{E}^R}{\Delta v}=\frac{1}{12\pi}\overline{a^2} \tag{49}$$

(compare with classical electrodynamics[14]).

With regard to the covariance properties of these quantities, we note that under a Lorentz transformation with velocity V along the x-axis the frequencies ω, ω' become $\tilde{\omega}$, $\tilde{\omega}'$ according to the expressions

$$\omega=D\tilde{\omega},\quad \omega'=D^{-1}\tilde{\omega}',\quad D=\sqrt{\frac{1+V}{1-V}}. \tag{50}$$

The quantities v, $f(u)$, F, $\mathcal{E}^R$ transform like ω, while u, $g(v)$, G, transform like ω'. Finally, the functions $R(u)$, $f(u)$ transform like ω^2 and the functions $I(v)$, $da/d\tau$, the left side of the uncertainty relation $\Delta u\Delta\mathcal{E}^R$, the Bogolyubov coefficients, and the spectrum $d\bar{n}_\omega$, are all invariants. Thus, the choice of variables of integration u, v, or τ in the representations (24), (45) determines the covariance properties of the functions T_{++}, $da/d\tau$, or g_+ under the integral sign, and their physical meaning.

4. TRANSFORMATION OF THE SPECTRAL INTEGRAL FOR THE ENERGY OF FERMION RADIATION

Let us turn now to a fermion field radiated by an accelerated mirror. We transform the spectral integral (19) for the radiated energy-momentum $\mathcal{E}^{RF}$ of fermions into an integral over the coordinates u or v. We emphasize that although each fermion is radiated as a pair with an antifermion, Eq. (19) for $\mathcal{E}^{RF}$ is the energy carried away only by the fermions or the antifermions, which are experimentally different depending on their fermionic charge.

According to (3), (4), the f^*g representation for $|\beta^F_{\omega'\omega}|^2$ differs from (20) by an additional factor, i.e., $-\sqrt{f'(u)g'(v)}$, under the integral. Then this factor $-\sqrt{f'(u)g'(v)}$ appears in Eqs. (21) or (22) for the energy $\mathcal{E}^{RF}$ and the function $F^F(u,v)$, so that

$$F^F(u,v)=-\frac{\sqrt{f'(u)g'(v)}}{(u-g(v))^2(f(u)-v)}, \tag{51}$$

the integral over u, as in (23), now reduces to the residue of the function $F^F(u,v)$ at the third-order pole $u=g(v)$. It is not difficult to show that this residue equals half the residue of the function $F^B(u,v)$. As a result, we obtain for $\mathcal{E}^{RF}$

$$\mathcal{E}^{RF}=\int_{-\infty}^{\infty}du\, R^F(u),$$

$$R^F(u)=\frac{\hbar c}{24\pi}\left[\left(\frac{f''}{2f'}\right)^2-\left(\frac{f''}{2f'}\right)'\right]. \tag{52}$$

Note that the expression given here consists of the energy of radiated fermions with definite charge and projection of the spin or helicity. If the mirror radiates quanta of a spinor field with two spin projections or helicities, then this result should be doubled. It is also necessary to double it when taking into account the energy of the antifermions.

It is interesting to note that if we change the order of integration with respect to u, v in the double integral under discussion for $\mathcal{E}^{RF}$, and first carry out the integration with respect to variable v, then in contrast to the boson case (37) the integral over v reduces to the Schwartz derivative

$$\int_{-\infty}^{\infty}dv[\;]^F=-2\pi i\ \text{res}\ F^F(u,v)|_{v=f(u)}$$

$$=\frac{2\pi i}{12g_0'^2}\left[\frac{g_0'''}{g_0'}-\frac{3}{2}\left(\frac{g_0''}{g_0'}\right)^2\right]$$

$$=\frac{2\pi i}{6}\left[\left(\frac{f''}{2f'}\right)^2-\left(\frac{f''}{2f'}\right)'\right]. \tag{53}$$

Therefore we obtain the same representation (52) for $\mathcal{E}^{RF}$. This stability of the representation (52) is a consequence of the factor $\sqrt{f'(u)g'(v)}$ in the function $F^F(u,v)$, which improves the convergence of the integrals of this function with respect to both u and v. Therefore, the asymptotic conditions (35), (36) are no longer needed. In particular, if $g(v)\rightarrow$ const as $v\rightarrow+\infty$, then the integral with respect to v of $F^B(u,v)$ diverges, while that of $F^F(u,v)$ converges since $\sqrt{g'(v)}\rightarrow\infty$.

The transformation of the spectral integral for $\mathcal{E}^{RF}$ using the gg^* or ff^* representations for $|\beta^F_{\omega'\omega}|^2$ now reduces to Eq. (52). In this case, the residue of the function

$$G^F(v',v)=\frac{\sqrt{g'(v')g'(v)}}{(v'-v)(g(v')-g(v))^2}, \tag{54}$$

which is half the residue of the function $G^B(v',v)$ from (44), works for, e.g., the gg^* representation.

Let us now discuss the physical meaning of the function $R^F(u)$. In the paper by Davies and Unruh[15], these authors found the tensor operator for the energy-momentum of a neutrino field with definite helicity averaged over the vacuum. As in the case of a scalar field, it consisted of a sum of divergent and finite parts. The single nonzero component of the finite part equals

$$T^F_{++}(u)=\frac{1}{12\pi}\left[\left(\frac{f''}{2f'}\right)^2-\left(\frac{f''}{2f'}\right)'\right], \tag{55}$$

i.e., it coincides with $T^B_{++}(u)$. Thus, the function $R^F(u)$ we have found equals half of $T^F_{++}(u)$. The difference of a factor of 2 is related to the fact that $\mathcal{E}^{RF}$ consists of the energy-momentum of only the fermions or only the antifermions, whereas the energy-momentum tensor of the quantized fermion field contains contributions from both of these fields.[16] Consequently, by analogy with Eq. (26) for $R^F(u)$ we have

$$R^F(u)=\frac{1}{24\pi}\left[\left(\frac{f''}{2f'}\right)^2-\left(\frac{f''}{2f'}\right)'\right]=\frac{1}{2}T^F_{++}(u)$$
$$=-\frac{1}{2}\frac{\hbar c}{e^2}g^{sc}_+\sqrt{f'}=-\frac{1}{24\pi}f'\frac{da}{d\tau}. \tag{56}$$

Here g^{sc}_+ is the plus component of the bremsstrahlung force g^{sc}_α acting on a scalar charge, and $g^{sc}_\alpha=(1/2)g^{el}_\alpha$.

To conclude Secs. 3 and 4, we note that by multiplying by e^2 the expressions $\mathcal{E}^F$, $\mathcal{E}^S$ obtained here for the Bose and Fermi radiation by the mirror, we obtain the plus components of the radiative and Schott momenta $(1/2)G^R_+$, $(1/2)G^S_+$ for electromagnetic and scalar radiation of charges in 3+1 space (see the comments below Eq. (16) and Section 6 in Ref. 5).

5. TRANSFORMATION OF THE SPECTRAL INTEGRALS FOR THE AVERAGE NUMBERS OF RADIATED BOSONS AND FERMIONS

Our next task is to transform the spectral integral for the average number of radiated particles

$$N=\int_0^\infty d\bar{n}_\omega=\int\int_0^\infty\frac{d\omega\, d\omega'}{(2\pi)^2}|\beta_{\omega'\omega}|^2 \tag{57}$$

into an integral over the characteristic variables u or v. This makes it possible to say something about the space–time region where the particles radiated by the mirror are generated.

Using the half-sum of the $f*g$ representation (20) and its complex conjugate for $|\beta^B_{\omega'\omega}|^2$, and integrating over frequencies, we obtain

$$N^B=\frac{1}{8\pi^2}\int_{-\infty}^\infty du\int_{-\infty}^\infty dv$$
$$\times\left[\frac{1}{(v-f(u)-i\varepsilon)(g(v)-u-i\delta)}+\text{c.c}\right]. \tag{58}$$

We will use the notation $v_1=f(u)+i\varepsilon$, and functionally relate the infinitesimal parameters ε, δ so that $g(v_1)=u+i\delta$. Let us subtract from the integral the following zero-valued integral:

$$\int_{-\infty}^\infty dv\left[\frac{1}{g'(v_1)(v-v_1)^2}+\text{c.c.}\right]=0. \tag{59}$$

It is not difficult to see that this subtracted integral differs from the original integral over v by the replacement in the latter of the function $g(v)$ by the linear part $g_1(v)=g(v_1)+g'(v_1)(v-v_1)$ of its expansion around the point $v=v_1$. Then in the limit ε, $\delta\to 0$ we obtain

$$N^B=\frac{1}{4\pi^2}\int_{-\infty}^\infty du\, K^B(u),$$
$$K^B(u)=\int_{-\infty}^\infty\frac{dv}{v-f(u)}\left[\frac{1}{g(v)-u}-\frac{1}{g_1(v)-u}\right], \tag{60}$$

where now the function

$$g_1(v)=u+\frac{v-f(u)}{f'(u)}, \tag{61}$$

and the integral over v is taken in the sense of a principal value.

In contrast to the function $R^B(u)$, the function $K^B(u)$ is a functional of the path; its value at the point u is determined by the portion of the path around the point with coordinates $u, v=f(u)$ that is important in the integral. Furthermore, the integral for $K^B(u)$ diverges if $g(v)$ increases more slowly than $[\ln(\pm v)]^\alpha$, $\alpha>1$ as $v\to\pm\infty$. In this case the representation (60) does not exist. Therefore we will assume that $g(v)$ increases more rapidly than $\ln(\pm v)$ as $v\to\pm\infty$. This condition coincides with the condition that the Schott energy-momentum vanish at infinity. The vanishing of $\mathcal{E}^S$ at infinity in turn implies that the radiation with the largest power takes place within a finite region of space-time.

In order to obtain a similar representation for the total number of radiated fermions N^F, it is necessary to multiply the function under the integral sign in (58) by $-\sqrt{f'(u)g'(v)}$. Then after the subtraction procedure, passing to the limit ε, $\delta\to 0$ and using (61), we obtain

$$N^F=\frac{1}{4\pi^2}\int_{-\infty}^\infty du\, K^F(u),$$
$$K^F(u)=-\sqrt{f'(u)}\int_{-\infty}^\infty\frac{dv}{v-f(u)}\left[\frac{\sqrt{g'(v)}}{g(v)-u}-\frac{\sqrt{f'(u)}}{v-f(u)}\right]. \tag{62}$$

In contrast to the boson case, the function under the integral sign in the integral for $K^F(u)$ is finite at $v=f(u)$, and therefore the principal value symbol is not essential. Furthermore, the function $K^F(u)$ exists even if $g(v)\to$ const as $v\to\pm\infty$, because in this case $\sqrt{g'(v)}\to 0$. In other words, the condition that $g(v)$ increases as $v\to\pm\infty$ more rapidly than $\ln(\pm v)$ is not necessary for the existence of $K^F(u)$.

If at the initial and final ends of the path the source possesses asymptotically constant velocities β_1, β_2, and a relative velocity

$$\beta_{12}=\frac{\beta_1-\beta_2}{1-\beta_1\beta_2} \tag{63}$$

that is nonzero, then the average number of radiated zero-mass quanta turns out to be infinite (the infrared divergence). In fact, in this case it follows from Eqs. (60)–(62) that as $u\to\infty$ (more precisely, for $|u|\gg\kappa^{-1}$, i.e., outside the region where the source is subject to the characteristic acceleration κ) the functions $K^{B,F}(u)$ have the universal asymptotic behavior:

$$K^B(u)\approx\pm\frac{1}{u}\left(\frac{\theta e^{\pm\theta}}{\sinh\theta}-1\right), \tag{64}$$

$$K^F(u)\approx\pm\frac{1}{u}\left(1-\frac{\theta}{\sinh\theta}\right),\quad \theta=\text{Arth}\,\beta_{12}. \tag{65}$$

Since the transformation properties and dimensions of the function $K(u)$ and variable u^{-1} coincide, the coefficients for u^{-1} are Lorentz-invariant and dimensionless, and are determined by a single Lorentz-invariant parameter — the relative velocity of the ends β_{12}. Dimensional parameters similar to the characteristic acceleration κ enter into the next terms of the expansion in inverse powers of u. According to (64), (65), at the ends of the path the signs of K^B are opposite, while $K^F > 0$. As a result, the average number of quanta radiated by a portion of the path enclosing the region of acceleration increases logarithmically as the length $2L$ of this portion increases:

$$N^B=\frac{1}{2\pi^2}\left(\frac{\theta}{\tanh\theta}-1\right)\ln L\kappa+2b^B(\theta), \tag{66}$$

$$N^F=\frac{1}{2\pi^2}\left(1-\frac{\theta}{\sinh\theta}\right)\ln L\kappa+2b^F(\theta),\quad L\kappa\gg 1. \tag{67}$$

The terms $2b^{B,F}$ do not depend on L for $L\kappa\gg 1$. When multiplied by $e^2=4\pi\alpha$, these expressions coincide with the average numbers N_1 and N_0 of photons and scalar quanta radiated by electric and scalar charges as they move along analogous paths in $3+1$ space.[12,17]

As is well known,[18] classical radiation sources are characterized by an amplitude $\exp(i\Delta W_s)$ for maintaining the vacuum, or by the change in self-action ΔW_s whose doubled imaginary part coincides with the average number of radiated particles with spin s:

$$2\,\text{Im}\,\Delta W_s=N_s.$$

The real part of ΔW_s appears automatically if we use for $\Delta W_s(\mu^2)$ the well-known expression containing a causal function for propagation of quanta with nonzero mass μ.[18,19] In this case the very small mass of the quanta ($\mu\ll\kappa$) may be considered as a convenient relativistically invariant parameter that allows us to avoid the infrared divergence of N_s. We can also recover $\text{Re}\Delta W_s(\mu^2)$ from the imaginary part of $\Delta W_s(\mu^2)$ by using the dispersion relations obtained in Ref. 20 for the latter:

$$\Delta W(\mu^2)=-\frac{2\mu}{\pi}\int_0^\infty\frac{dx\,\text{Im}\,\Delta W(x^2)}{x^2-\mu^2},$$

$$\Delta W(\mu^2)=\frac{2i}{\pi}\int_0^\infty\frac{dx\,x\,\text{Re}\,\Delta W(x^2)}{x^2-\mu^2}. \tag{68}$$

Here $\text{Im}\,\mu\to-0$. Exactly these relations for ΔW appear in problems that involve a charge whose massless field satisfies the boundary conditions on a mirror.[21] In this case, the modification of the field of a charge by a mirror located at large but finite distances L from the charge avoids the infrared divergence, and the role of the quantum mass is played by the realistic parameter L^{-1}.

It is natural to assume that the amplitude e^{iW} for preservation of the vacuum will also enter in problems of radiation by an accelerated mirror in $1+1$ space if we set $2\,\text{Im}\,W=N$, while Re W is found from the dispersion relations (68). Then if in accordance with (66), (67), and Refs. 22, 23, we write

$$\text{Im}\,W=2a\,\ln L\kappa+b,$$

$$a^B=\frac{1}{8\pi^2}\left(\frac{\theta}{\tanh\theta}-1\right),\quad a^F=\frac{1}{8\pi^2}\left(1-\frac{\theta}{\sinh\theta}\right), \tag{69}$$

we find that

$$\text{Re}\,W=\pi a. \tag{70}$$

Note that the functions $a^{B,F}(\theta)$ have the following interesting integral representations due to Legendre:

$$a^{B,F}(\theta)=\frac{1}{4\pi^2}\int_0^\infty\frac{dx\,\sin x}{e^{\pi x/\theta}\mp 1}. \tag{71}$$

It is difficult to interpret these physically, even if we assume the path of the source on the accelerated segment between velocities β_1, β_2 is hyperbolic. In this case the parameter $\theta=\kappa(\tau_2-\tau_1)$, where $2\tau=\tau_2-\tau_1$ is the proper time of the acceleration, and if we set $x=\omega\tau$, then (71) can perhaps be regarded as spectral representations of the functions $a^{B,F}(2\kappa\tau)$ that depend on the duration of the acceleration. The contribution of frequency ω is given by the Bose- or Fermi-distribution with a "temperature" $2\kappa/\pi$.

However, in $3+1$ space the same functions multiplied by e^2 determine the change in self-energy of the electric and scalar charges-sources only for Bose fields with spins 1 and 0, respectively. The appearance of a Fermi distribution in the latter case is difficult to explain.

6. RADIATION OF BOSONS AND FERMIONS BY A MIRROR DURING SEMIHYPERBOLIC MOTION

The motion of a mirror is said to be semihyperbolic[5] if before time $t=0$ it is at rest (at the point $x=0$ for example), and then proceeds to move along the hyperbolic path $x=\xi(t)=B-\sqrt{B^2+t^2}$. In u, v variables, this path is specified by the function $f(u)=u$ for $u\leqslant 0$ and $f(u)=Bu(B+u)^{-1}$ for $u\geqslant 0$, or by $g(v)=v$ for $v\leqslant 0$ and $g(v)=Bv(B-v)^{-1}$ for $0\leqslant v<B$.

The spectrum of the average number of radiated bosons was found in Ref. 5 and is given by the expression

$$d\bar{n}^B_\omega=-\text{Re}\left\{1+e^{iz}\text{Ei}(-iz)+ize^{2iz}\int_0^1 dt\,e^{-iz(t+1/t)}\right.$$
$$\left.\times\text{Ei}(-iz(1-t))\right\}\frac{dz}{2\pi^2 z}, \tag{72}$$

where $z=\omega B$, and $\text{Ei}(x)$ is the exponential integral function.[24] Its behavior in the infrared and ultraviolet regions is as follows:

$$d\bar{n}^B_\omega=\frac{dz}{2\pi^2 z}\begin{cases}\ln\dfrac{1}{\gamma z}-1+\pi z+\ldots, & z\ll 1,\ \gamma=1.78\ldots, & (73)\\[2mm] \dfrac{1}{10z^2}-\dfrac{1}{21z^4}+\ldots, & z\gg 1. & (74)\end{cases}$$

It is interesting that the functions $R^B(u)$ and $I^B(u)$ in the representations (24), (45) reduce to δ-functions:

$$R^B(u)=\frac{\kappa}{12\pi}\delta(u),\quad I^B(v)=\frac{\kappa}{12\pi}\delta(v),\quad \kappa=B^{-1}. \tag{75}$$

which reflects the discontinuous jump in the proper acceleration $a=f''/2f'^{3/2}$ from a value $a=0$ for $u<0$ to a value $a=-\kappa<0$ for $u>0$. The Schott energy also changes discontinuously, jumping from zero for $u<0$ to $\mathscr{E}^S=(\kappa/12\pi)(1+\kappa u)^{-1}$ for $u>0$. The energy radiated by the mirror according to Eqs. (24), (75) is finite, equals $\mathscr{E}^S=\kappa/12\pi$, and coincides with the jump in $\mathscr{E}^S$ at zero. This value was obtained in Ref. 5 by a roundabout path.

As we already discussed above, the function $R(u)$ is not the irreversible radiated power. Therefore, despite the δ-like behavior of R^B in this example, we cannot assume that all the radiated energy $\mathscr{E}^R$ comes from an arbitrarily small interval around $u=0$. In reality this energy is generated on that interval Δu which satisfies the uncertainty relation $\Delta u\Delta\mathscr{E}^R>1$ and for which the change in Schott energy is small compared to its characteristic value inside Δu. In our case

$$\Delta\mathscr{E}^R=\frac{1}{12\pi}\int_{\Delta u}du\frac{1}{(B+u)^2}=\mathscr{E}^R\frac{\Delta u}{B+\Delta u} \tag{76}$$

and the uncertainty relation is fulfilled for $\Delta u>12\pi B$.

Since the radiated energy is carried by quanta, it is interesting to trace how the considerations discussed above regarding the length over which the energy is created agree with the picture of particle generation.

For paths with a horizon $v=B$, the function $K^B(u)$ is written in the form

$$K^B(u)=\int_{-\infty}^{B}\frac{dv}{v-f(u)}\left[\frac{1}{g(v)-u}-\frac{1}{g_1(v)-u}\right]-\int_B^{\infty}\frac{dv}{(v-f(u))(g_1(v)-u)} \tag{77}$$

which for the semihyperbolic path under discussion equals

$$K^B(u)=\begin{cases}\frac{1}{B+u}\ln\frac{u}{B}-\frac{1}{B+u}, & u>0,\\ \frac{1}{u}-\frac{1}{B+u}\ln\left(1-\frac{B}{u}\right)+\frac{B^2}{u^2(B+u)}\ln\left(1-\frac{u}{B}\right), & u<0.\end{cases} \tag{78}$$

It is not difficult to see that this function is negative for $u<eB\equiv 2.718B$, and positive for $u>eB$, and at the point $u=0$ has an integrable logarithmic singularity:

$$K^B(u)=-\frac{1}{B}\ln\frac{B}{|u|}-\frac{1\pm3}{4B}+\ldots,\quad u\to\pm0.$$

The integral for the average number of radiated particles diverges, because the nonzero change in the velocity of the mirror over an arbitrarily large interval of the path gives rise to the radiation of an arbitrarily large number of quanta. However, for a finite interval $-L<u<L$ the integral

$$N^B(\nu)=\frac{1}{4\pi^2}\int_{-L}^{L}du\,K^B(u)=\frac{1}{4\pi^2}\left[\ln\nu\ln(1+\nu)+L_2(1-\nu)-2\ln(1+\nu)-\frac{\pi^2}{6}+1-\frac{1}{\nu}\ln(1+\nu)\right] \tag{79}$$

gives a representation of the number of bosons radiated from this interval and having a wavelength smaller than L. In this expression we have written $\nu=L/B$, and $L_2(x)$ is the Euler dilogarithm.[9] In accordance with the arguments given for the energy, this interval should be large compared to the value of the inverse acceleration B. In this case

$$N^B(\nu)=\frac{1}{4\pi^2}\left(\frac{1}{2}\ln^2\nu-2\ln\nu-\frac{\pi^2}{3}+1-\ldots\right),\quad \nu\gg1. \tag{80}$$

The spectrum of the average number of fermions radiated by the mirror for a semihyperbolic path is found in this paper and given by the expression

$$d\bar{n}^F_\omega=\left\{1-\mathrm{Im}\,z\exp(2iz)\int_0^1\frac{dt}{t}\exp\left[-iz\left(t+\frac{1}{t}\right)\right]\times\mathrm{Ei}(-iz(1-t))\right\}\frac{dz}{2\pi^2 z}. \tag{81}$$

It possesses the following behavior in the infrared and ultraviolet regions:

$$d\bar{n}^F_\omega=\frac{dz}{2\pi^2 z}\times\begin{cases}1-\pi z\ln\gamma z+z^2\left(2\ln^2\gamma z-\frac{\pi^2}{6}+\frac{7}{2}\right)+\ldots, & z\ll1, \quad (82)\\ \frac{1}{15z^2}-\frac{4}{105z^4}+\ldots, & z\gg1. \quad (83)\end{cases}$$

It is clear that the fermion and boson spectra differ significantly, and that they have a common behavior only in the ultraviolet region. The spectral density of the average number of bosons is always larger than the analogous density of fermions:

$$\frac{d\bar{n}^B}{d\omega}>\frac{d\bar{n}^F}{d\omega}, \tag{84}$$

and this inequality is strengthened as the frequency decreases.

On the other hand, the functions R^F and I^F in the representations for the energies $\mathscr{E}^{RF}$ functionally coincide with the boson functions (75), differing from them only by the additional factor of 1/2.

Conditions are otherwise for the function $K^F(u)$ in the representation for the number of radiated fermions. For the path under discussion here it equals

$$K^F(u)=\begin{cases}\frac{1}{B+u}, & u>0,\\ -\frac{1}{u}-\frac{B}{u^2}\ln\left(1-\frac{u}{B}\right), & u<0.\end{cases} \tag{85}$$

In contrast to $K^B(u)$ this function is everywhere positive and finite at $u=0$, although here too there is a jump:

$$K^F(u)=\frac{3\pm 1}{4B}+\ldots, \quad u\to\pm 0.$$

The average number of radiated fermions N^F is infinite for the same reason as the number N^B is. However, the integral

$$N^F(\nu)=\frac{1}{4\pi^2}\int_{-L}^{L}du\, K^F(u)$$
$$=\frac{1}{4\pi^2}\left[2\ln(1+\nu)-1+\frac{1}{\nu}\ln(1+\nu)\right] \tag{86}$$

for $\nu=L/B\gg 1$ may be regarded as the average number of fermions emitted on the interval $-L<u<L$ and having a wavelength smaller than L.

To conclude this section we give the functions

$$R^B(u)=\frac{1}{12\pi}\left[\frac{\kappa^2}{4}+\frac{\kappa}{2}\delta(u)\right], \quad u\geqslant 0, \tag{87}$$

$$I^B(v)=\frac{1}{12\pi}\left[\frac{\kappa^2}{4(1-\kappa v)}+\frac{\kappa}{2}\delta(v)\right], \quad 0\leqslant v<\kappa^{-1}, \tag{88}$$

for a mirror path that is semiexponential, for which $f(u)=u$ for $u\leqslant 0$ and $f(u)=\kappa^{-1}-\kappa^{-1}e^{-\kappa u}$ for $u\geqslant 0$, or $g(v)=v$ for $v\leqslant 0$ and $g(v)=-\kappa^{-1}\ln(1-\kappa v)$ for $0\leqslant v<\kappa^{-1}$. This path, like the semihyperbolic one, has a horizon, but is characterized by a proper acceleration that is not constant but rather exponentially growing: $a=-(\kappa/2)e^{\kappa u/2}$ for $u\geqslant 0$, which leads to infinite radiated energy. For a finite interval Δu

$$\Delta\mathscr{E}^{RB}=\int_0^{\Delta u}du\, R^B(u)=\frac{1}{12\pi}\left(\frac{\kappa^2}{4}\Delta u+\frac{\kappa}{2}\right). \tag{89}$$

The uncertainty relation is satisfied for $\Delta u>2\sqrt{12\pi}\kappa^{-1}$. In this case we can introduce the average radiated power $\Delta\mathscr{E}^R/\Delta u=\kappa^2/48\pi$, and the Schott energy $\mathscr{E}^S=\kappa/24\pi$ is determined by extrapolating the linear law (89) to zero. Note that for $t\gg\kappa^{-1}$ the coordinate of the mirror $u\approx 2t-\kappa^{-1}$. On the other hand, we may write this coordinate as a function of the proper time τ in the form

$$u=\frac{2}{\kappa}\ln\frac{2}{2-\kappa\tau}, \quad 0\leqslant\tau<2\kappa^{-1}. \tag{90}$$

From this it is clear that infinite energy is radiated over a finite proper time. Because infinite energy and power are meaningless, the exponentially growing acceleration should be replaced by something more moderate. Paths on which a source radiates inifinite energy within a finite proper time are physically unrealizable, since they extend beyond any finite region of space-time, and the theory of the spectra of such sources conceals a contradiction.

7. CORRELATION FUNCTIONS OF BOSON AND FERMION FIELDS

Let us define the correlation function by the relation

$$C_{\lambda\sigma,\mu\nu}(x_2,x_1)=\langle 0|\hat{T}_{\lambda\sigma}(x_2)\hat{T}_{\mu\nu}(x_1)|0\rangle -\langle 0|\hat{T}_{\lambda\sigma}(x_2)|0\rangle\langle 0|\hat{T}_{\mu\nu}(x_1)|0\rangle, \tag{91}$$

where $\hat{T}_{\mu\nu}$ is the operator for the energy-momentum tensor of the field. Then for boson and fermion fields that are in contact with the accelerated mirror we obtain

$$C^{B,F}_{\lambda\sigma,\mu\nu}(x_2,x_1)=\int\int_0^\infty\frac{d\omega\, d\omega'}{(2\pi)^2}C_{\lambda\sigma}(x_2,\Omega)\times C^*_{\mu\nu}(x_1,\Omega)w^{B,F}(\omega,\omega'), \tag{92}$$

where

$$C_{\mu\nu}(x_1,\Omega)=(-1)^{\mu+\nu}f_1'^2e^{-i\Omega f_1}+e^{-i\Omega v_1},$$
$$\Omega=\omega+\omega', \tag{93}$$

$$w^B(\omega,\omega')=\frac{1}{2}\omega\omega', \quad w^F(\omega,\omega')=\frac{1}{4}(\omega-\omega')^2. \tag{94}$$

For brevity we use the notations $f_1=f(u_1)$, $f_1'=f'(u_1)$, etc., and by x_1 and x_2 we mean pairs of characteristic coordinates u_1, v_1 and u_2, v_2.

Let us turn our attention to the following fact: the spectral correlation function $C_2C_1^*w$ depends on the spin of the field only through the function w, which does not depend on coordinates or Lorentz-transform labels, and therefore does not feel the effect of the mirror. The function $C_2C_1^*$ that depends on these variables is factorized and is the same for scalar and spinor fields. Furthermore, it depends on the sum $\omega+\omega'$, and not on ω and ω' individually. This is related to the fact that it describes a correlation between the energy (and momentum) emitted at point x_1 and absorbed at a point x_2, and the particle + antiparticle pair that is transferred. The function w describes the distribution of total energy of the pair Ω between particles that make it up. It is clear that in the scalar case this energy is distributed on the whole evenly, while in the spinor case the distribution is always distinctly uneven. The latter fact implies that the fermion and antifermion, regardless of their opposite fermionic charges, here appear as identical particles that are subject to Fermi statistics. This is a consequence of charge symmetry of the energy-momentum tensor (see §19 in Ref. 17).

After integrating over one of the frequencies ω, ω' for fixed Ω the dependence on the spin of the field disappears:

$$C^{B,F}_{\lambda\sigma,\mu\nu}(x_2,x_1)=\frac{1}{48\pi^2}\int_0^\infty d\Omega\,\Omega^3C_{\lambda\sigma}(x_2,\Omega)C^*_{\mu\nu}(x_1,\Omega)$$
$$=\frac{1}{8\pi^2}\left\{(-1)^{\lambda+\sigma+\mu+\nu}\frac{f_1'^2f_2'^2}{(f_1-f_2)^4}+(-1)^{\lambda+\sigma}\frac{f_2'^2}{(v_1-f_2)^4}+(-1)^{\mu+\nu}\frac{f_1'^2}{(f_1-v_2)^4}+\frac{1}{(v_1-v_2)^4}\right\}. \tag{95}$$

This latter expression is in agreement with the correlation function obtained in Ref. 25 for a scalar field. We note that if we regard the spinor field as having not a single spin projection or helicity but rather two, the result (95) must be doubled.

8. DISCUSSION

Although the representations (24), (39) are equivalent if the Schott energy-momentum at the ends of the path disappears, it is interesting to understand the cause of the difference between the functions R and $\tilde{R}$. For this we turn to the double integral in (21) and denote the function in square brackets under the integral sign by $A(u,v)$.

Let us transform from the integration variables u, v to variables $\tilde{u}$, $\tilde{v}$ given by the relation

$$u=g(\tilde{v}), \quad v=f(\tilde{u}) \text{ or } \tilde{v}=f(u), \quad \tilde{u}=g(v). \tag{96}$$

It is not difficult to see that the function A is form-invariant with respect to the transformation (96), i.e.,

$$A(u,v)=A(\tilde{u},\tilde{v}). \tag{97}$$

Therefore,

$$\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du\, A(u,v) = \int_{-\infty}^{\infty} d\tilde{u} f'(\tilde{u}) \int_{-\infty}^{\infty} d\tilde{v} g'(\tilde{v}) A(\tilde{u},\tilde{v}), \tag{98}$$

where the tilde on the variables on the right side can be dropped. The change in order of integration we have made is accompanied by appearance in the function A of a factor $f'(u)g'(v)$, which in general does not equal unity. The inner integral on the right side of (98) reduces to the residue of the function $-g'(v)F(u,v)$ at the pole $v=v_0=f(u)$:

$$\int_{-\infty}^{\infty} dv\, g'(v)A(u,v) = -2\pi i \operatorname{res} g'(v)F(u,v)|_{v=v_0}. \tag{99}$$

If this pole were first-order, then the function $g'(v)$ could be taken outside the integration sign in the form of its value at the pole $v=v_0$, and (99) would acquire the form

$$g'(v_0)\int_{-\infty}^{\infty} dv\, A(u,v) = -2\pi i g'(v_0) \operatorname{res} F(u,v)|_{v=v_0}.$$

In this case the right side of (98) would differ from the left only by changing the order of integration, because $f'(u)g'(v_0)=1$ (see (38)). However, the functions A and F have poles that are not first but rather third order, and we are not allowed to carry out this procedure.

The difference between the original integral and the integral with changed order of integration, according to (98), can be written in the form

$$\left(\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du - \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\right) A(u,v) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv [f'(u)g'(v)-1]\, A(u,v). \tag{100}$$

Once more using the transformation (96), it is not difficult to verify that the right side of (100), like the left side, changes sign when the order of integration is changed. Its inner integral reduces to the residue

$$-\operatorname{res}\, [f'(u)g'(v)-1]F(u,v)|_{v=v_0\equiv f(u)} = -\left(\frac{f''}{2f'}\right)'$$

at the second-order pole $v=v_0$. As a result, taking into account the factor $(1/8\pi^2 i)$ we obtain in accordance with (24), (39) that the difference between the two representations for the energy $\mathscr{E}^R$, which differ by the order of integration with respect to u and v, equals three times the difference of the Schott energies at the ends of the path:

$$3[\mathscr{E}^S(\infty)-\mathscr{E}^S(-\infty)], \quad \mathscr{E}^S(u) = -\frac{1}{12\pi}(\ln\sqrt{f'(u)})'. \tag{101}$$

The natural requirement that such an ambiguity be absent selects paths with zero Schott energy (i.e., constant velocity) at their ends.

Paths with nonzero Schott energy at infinity are equivalent sources that radiate infinite energy outside any finite region of space-time. Furthermore, in the case of a semiexponential path with proper acceleration $a=-\kappa(2-\kappa\tau)^{-1}$, $0 \leqslant \tau < 2\kappa^{-1}$, infinite energy is radiated within a finite proper time. Similar paths arise in the classical electrodynamics of a charge when radiation reaction is taken into account, and are considered unphysical.[26–28] In contrast to these paths, the physical paths are described by those solutions to the Abraham–Lorentz–Dirac equation for which a charge possesses zero acceleration far from the region where the external force acts. The equation of motion of a charge with such boundary conditions is equivalent to an integrodifferential equation[29–32]

$$m\frac{du_\alpha}{d\tau} = \int_0^{\infty} K_\alpha(\tau+\tau_0\xi)e^{-\xi}d\xi, \quad \tau_0=\frac{2}{3}r_0, \tag{102}$$

in which the Schott term disappears from the expression for the force

$$K_\alpha = K_\alpha^{\text{ext}} - \frac{e^2}{6\pi}a^2 u_\alpha,$$

but its action becomes washed out over a region determined by the classical electron radius r_0.

In this case, as long as the force of radiation reaction in the proper-time system of the charge is small compared with the external force, i.e., $e^2 a/mc^4 \ll 1$, the radiation can be treated as a perturbation. Summing the corresponding series, we can find an exact solution that is analytic in e^2 near $e^2=0$. However, for accelerations $a \gtrsim mc^3/\hbar$ the motion of the charge ceases to be classical due to quantum effects, although the parameter mentioned above can nevertheless be small.

Analogously, for very large accelerations of the mirror we cannot describe the quantum nature of the radiation mechanism using classical boundary conditions.

For hyperbolic paths the Schott energy is

$$\mathscr{E}^S=\frac{\kappa}{12\pi}(1+\kappa u)^{-1}\rightarrow\infty \quad \text{as} \quad u\rightarrow-\kappa^{-1},$$

i.e., at the end in the past. In this case the representation (24) is found to be inconsistent, because its left side is $\mathscr{E}^R=+\infty$, according to (19), while the right side is at the very least undetermined, if not equal to zero, since $R(u)=0$. The contradiction disappears if we treat the hyperbolic path as the limit of a quasihyperbolic path at whose end the source has subluminous velocities $\pm\beta$ and zero Schott energy, and which reduces to hyperbolic as $\beta\rightarrow 1$ over every large portion around the turning point. Similar discussions were given in Ref. 5.

These arguments show that sources of radiation with nonzero Schott energy at infinity are in reality unrealizable, because they consume infinite energy in the course of radiation; the spectra of this radiation, at least in the long-wavelength region, do not allow us to distinguish real from virtual quanta, since this can be done only outside the region where the radiation is generated.

Turning to the fermion case, we note that the function $A^F(u,v)$ differs from the boson function by an additional factor $-\sqrt{f'(u)g'(v)}$, which when multiplied by $dvdu$ forms a surface element that is invariant with respect to the transformation (96):

$$dv\ du\sqrt{f'(u)g'(v)}=d\tilde{u}d\tilde{v}\sqrt{f'(\tilde{u})g'(\tilde{v})}. \tag{103}$$

Therefore, the integral over the u, v variables of the function $A^F(u,v)$ becomes an integral of the same function with reversed order of integration under the transformation (96). Thus, for $\mathscr{E}^{RF}$ only one representation (52) appears with a function $R^F=(1/2)T^F_{++}$, and a representation similar to (39) does not arise.

In this connection, the appearance in the boson case (and consequently in electrodynamics as well) of an anomalous representation of $\mathscr{E}^{RB}$ in addition to the normal one, with function $\tilde{R}^B\neq T^B_{++}$, once more compels us to examine the physical meaning of the Schott energy-momentum. Does the acceleration of a source in the field carried along by it create other modes of excitation?

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1)The natural system of units is used here, with Heaviside units for charge and a metric with trace +2, so that $\hbar=c=1$, $e^2/4\pi=1/137$, $k_\alpha x^\alpha=\mathbf{kx}-k^0x^0$, $k_\pm=k^0\pm k^1$, $t=x^0$, $x=x^1$.

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