Macroscopic Einstein equations for a system of self-gravitating particles: second-order accuracy in the interaction constant

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A method is developed for the ensemble averaging of the microscopic Einstein equations (Einstein equations whose right-hand side contains the energy–momentum tensors of individual particles) for a system of self-gravitating particles. This results in macroscopic Einstein equations for continuous media that are accurate to second order in the gravitation interaction constant. The equations differ from the classical Einstein equations by the presence of additional terms caused by particle interaction. The terms are proportional to the third power of Einstein's constant and can be expressed in terms of the two-particle correlation function of the particles. In addition, the relativistic kinetic equation for the one-particle distribution function of gravitating particles is derived by a method that is more compact and complete than the one used earlier by the same author (Sov. Phys. JETP **72**, 437 (1989)). © *1996 American Institute of Physics.* [S1063-7761(96)00107-2]

1. INTRODUCTION

As is known,[1] the macroscopic Maxwell equations for continuous media can be obtained from the microscopic Maxwell equations by ensemble averaging the latter.

The Einstein equations, whose right-hand side contains the energy–momentum tensor of matter, are phenomenological equations. It is natural to suppose that the Einstein equations (or their generalizations) for continuous media can also be obtained from the microscopic Einstein equations, i.e., Einstein equations whose right-hand side contains the sum of the energy–momentum tensors of individual particles. However, due to the nonlinearity of the left-hand side of Einstein equations, averaging the microscopic Einstein equations is much more complicated than averaging the microscopic Maxwell equations.[2,3]

The objective of the present paper is to develop a method for deriving the macroscopic Einstein equations by the ensemble averaging of microscopic equations that are accurately up to second-order terms in the interaction constant. Here we employ the ensemble averaging procedure introduced by Klimontovich[4,5] for deriving the relativistic kinetic equation for a plasma. The same procedure was used by the present author in Ref. 6 to derive a relativistic kinetic equation for gravitating particles accurate to second order in the interaction constant.

We write the system of microscopic Einstein equations for gravitating particles in terms of the random function $\widetilde{N}_a(q^i,\widetilde{p}_j)$ introduced by Klimontovich:[4]

$$\widetilde{N}_a(q^i,\widetilde{p}_j)=\sum_{l=1}^{n_a}\int d\widetilde{s}\,\delta^4(q^i-q^i_{(l)}(\widetilde{s}))\,\delta(\widetilde{p}_j-\widetilde{p}_j^{(l)}(\widetilde{s})). \tag{1}$$

Here n_a is the number of particles belonging to species a, $\widetilde{s}$ is the canonical parameter along the path, q^i and $\widetilde{p}_j$ are the coordinates in eight-dimensional phase space, and $q^i_{(l)}$ and $\widetilde{p}_i^{(l)}$ are the coordinates and momentum of the lth particle of the a species. The latter coordinates and momentum are found by solving the equations of motion

$$\frac{dq^i_{(l)}}{d\widetilde{s}}=\frac{1}{m_a c}\widetilde{p}^i_{(l)},\quad \frac{d\widetilde{p}_i^{(l)}}{d\widetilde{s}}=\frac{1}{m_a c}\widetilde{\Gamma}_{j,ik}\widetilde{p}^j_{(l)}\widetilde{p}^k_{(l)}. \tag{2}$$

Here $\widetilde{p}^i_{(l)}=\widetilde{g}^{ij}\widetilde{p}_j^{(l)}$, $\widetilde{g}^{ij}$ is the metric of the gravitational field generated by all the particles, $\widetilde{\Gamma}_{j,ik}$ are the Christoffel symbols of the first kind given by the metric $\widetilde{g}^{ij}$, m_a is the pass of a particle of species a, and c is the speed of light.

The microscopic energy–momentum tensor of a system of particles can be expressed in terms of $\widetilde{N}_q$ in the following way:

$$\widetilde{T}^{ij}=\sum_a c\int\frac{d^4\widetilde{p}_a}{\sqrt{-\widetilde{g}}}\widetilde{p}^i_a\widetilde{u}^j_a\widetilde{N}_a(q^i,\widetilde{p}_i), \tag{3}$$

where $\widetilde{g}$ is the determinant of $\widetilde{g}^{ij}$, $\widetilde{u}^i_a=(1/m_a c)\widetilde{p}^i_a$, and

$$\frac{d^4\widetilde{p}}{\sqrt{-g}}=\frac{d\widetilde{p}_1 d\widetilde{p}_2 d\widetilde{p}_3 d\widetilde{p}_4}{\sqrt{-g}}$$

is the invariant volume element in eight-dimensional space.[7] Actually (3) is the sum of the microscopic energy-momentum tensors of the individual particles.

We write the microscopic Einstein equations in the form

$$\widetilde{G}^{ij}=\chi\widetilde{T}^{ij}, \tag{4}$$

where $\widetilde{G}^{ij}$ is the Einstein tensor in a Riemannian space with metric $\widetilde{g}^{ij}$, the tensor $\widetilde{T}^{ij}$ is defined in (3), $\chi=8\pi k/c^4$ is Einstein's constant, and k is the gravitational constant.

In view of Eqs. (2) the random function (1) obeys the equation

$$\widetilde{p}^i\frac{\partial\widetilde{N}_a}{\partial q^i}+\widetilde{\Gamma}_{j,ik}\widetilde{p}^j\widetilde{p}^k\frac{\partial\widetilde{N}}{\partial\widetilde{p}_i}=0. \tag{5}$$

In the sections below we develop a procedure for obtaining the Einstein equations for a continuous medium by averaging the system of equations (3)–(5).

2. THE MACROSCOPIC EQUATIONS

Let us represent the metric $\tilde{g}_{ij}$ of the gravitational field generated by all particles as a sum of the averaged metric g_{ij} and a contribution h_{ij} due to particle interaction:

$$\tilde{g}_{ij}=g_{ij}+h_{ij}, \tag{6}$$

where $g_{ij}=\langle\tilde{g}_{ij}\rangle$ is the ensemble average[4] of $\tilde{g}_{ij}$. Note that $\langle h_{ij}\rangle\equiv 0$. In addition to the momenta $\tilde{p}_i=m_a c\, dq^i/d\tilde{s}$ we use the momenta p^i measured in the metric g_{ij}:

$$p^i=\frac{d\tilde{s}}{ds}\tilde{p}^i,\quad \frac{d\tilde{s}}{ds}\equiv\frac{1}{\alpha(q,p)}=\frac{(\tilde{g}_{ij}p^ip^j)^{1/2}}{(g_{ij}p^ip^j)^{1/2}}. \tag{7}$$

Here s is the canonical parameter introduced by g_{ij}.

The transformation from $\tilde{p}_i$ to p_i is given by

$$\tilde{p}_j=\tilde{g}_{jk}\tilde{p}^k=\alpha\tilde{g}_{jk}g^{ki}p_i. \tag{8}$$

The Jacobian of transformation (8) is[6]

$$\left|\frac{\partial\tilde{p}_i}{\partial p_j}\right|=\alpha^4\tilde{g}g^{-1}, \tag{9}$$

where g is the determinant of g_{ij}.

Now we introduce the function $N_a(q^i,p_j)$ defined in the eight-dimensional phase space with coordinates (q,p) as

$$N_a(q^i,p_j)=\sum_{l=1}^{n_a}\int ds\,\delta^4(q^i-q^i_{(l)}(s))\,\delta(p_j-p_j^{(l)}(s)), \tag{10}$$

where $q^i_{(l)}(s)$ and $p_j^{(l)}(s)$ are found by solving equations obtained from (2) with the transformations (8) taken into account:

$$\frac{dq^i_{(l)}}{ds}=\frac{1}{m_ac}p^i_{(l)},$$

$$\frac{dp_i^{(l)}}{ds}=\frac{1}{m_ac}[\Gamma_{j,ik}p^j_{(l)}p^k_{(l)}-\Omega^m_{kj}\Delta_{mi}p^j_{(l)}p^k_{(l)}]. \tag{11}$$

Here

$$\Delta_{kj}=g_{kj}-u_ku_j,\quad u_k=\frac{1}{m_ac}p_k,$$

and $\Omega^m_{kj}=\tilde{\Gamma}^m_{kj}-\Gamma^m_{kj}$ is the difference of the Christoffel symbols of the second kind for the metrics $\tilde{g}_{ij}$ and g_{ij}. In view of Eqs. (11) the function $N_a(q^i,p_j)$ satisfies the following equation:[6]

$$p^i\frac{\partial N_a}{\partial q^i}+\Gamma_{j,ik}p^kp^j\frac{\partial N_a}{\partial p_i}=\frac{\partial}{\partial p^i}(\Omega^m_{jk}\Delta_{mi}p^jp^kN_a). \tag{12}$$

Note that the functions $\tilde{N}_a$ and N_a are related in the following manner:

$$\tilde{N}_a=\frac{g}{\tilde{g}\alpha^5}N_a. \tag{13}$$

Equation (12) can also be obtained directly from (5) by replacing the variables via (8) and (13).

If in (3) we go over to the variables (8) and (13), we get

$$\tilde{T}^{ij}=\sum_a c\int\frac{d^4p}{\sqrt{-g}}p^iu^j\alpha(q,p)\sqrt{\frac{g}{\tilde{g}}}N_a(q,p), \tag{14}$$

where $d^4p/\sqrt{-g}$ is the invariant element in the unperturbed momentum space.

For subsequent calculation is it convenient to write the Einstein equations as

$$\tilde{R}_{ij}=\chi\left(\tilde{g}_{ik}\tilde{g}_{jm}-\frac{1}{2}\tilde{g}_{ij}\tilde{g}_{km}\right)\tilde{T}^{km}$$

or, with allowance for (14) and the identity

$$\tilde{R}_{ij}=R_{ij}+\nabla_m\Omega^m_{ij}-\nabla_j\Omega^m_{im}+\Omega^m_{mn}\Omega^n_{ij}-\Omega^m_{jn}\Omega^n_{im}$$

(R_{ij} is the Ricci tensor of the Riemannian space with metric g_{ij}, and ∇_m is a covariant derivative in this space) as

$$R_{ij}+\nabla_m\Omega^m_{ij}-\nabla_j\Omega^m_{im}+\Omega^m_{mn}\Omega^n_{ij}-\Omega^m_{jn}\Omega^n_{im}$$
$$=\chi\sum_a\int\frac{d^4p}{\sqrt{-g}}\alpha\sqrt{\frac{g}{\tilde{g}}}\left[\tilde{g}_{ik}\tilde{g}_{jm}-\frac{1}{2}\tilde{g}_{ij}\tilde{g}_{km}\right]p^kp^mN_a(q,p). \tag{15}$$

We write the left-hand side of the Einstein equations (15) as

$$R_{ij}(g)+\overset{(1)}{R}_{ij}(h)+\overset{(2)}{R}_{ij}(h)+\dots,$$

where $\overset{(1)}{R}_{ij}(h)$ is the sum of all terms that are first-order in h, $\overset{(2)}{R}_{ij}(h)$ is the sum of all terms that are second-order in h, etc. In particular,

$$\overset{(1)}{R}_{ij}=\nabla_m\overset{(1)}{\Omega}{}^m_{ij}-\nabla_j\overset{(1)}{\Omega}{}^m_{im}, \tag{16a}$$

$$\overset{(2)}{R}_{ij}=\nabla_m\overset{(2)}{\Omega}{}^m_{ij}-\nabla_j\overset{(2)}{\Omega}{}^m_{im}+\overset{(1)}{\Omega}{}^m_{mn}\overset{(1)}{\Omega}{}^n_{ij}-\overset{(1)}{\Omega}{}^m_{jn}\overset{(1)}{\Omega}{}^n_{im}, \tag{16b}$$

where

$$\overset{(1)}{\Omega}{}^m_{ij}=\frac{1}{2}g^{ml}(-\nabla_lh_{ij}+\nabla_ih_{lj}+\nabla_jh_{li}),$$

$$\overset{(2)}{\Omega}{}^m_{ij}=-\frac{1}{2}h^{ml}(-\nabla_lh_{ij}+\nabla_ih_{lj}+\nabla_jh_{li})=-\frac{1}{2}h^m_l\overset{(1)}{\Omega}{}^l_{ij}. \tag{17}$$

We also expand the expression

$$\alpha\sqrt{\frac{g}{\tilde{g}}}\left(\tilde{g}_{ik}\tilde{g}_{jm}-\frac{1}{2}\tilde{g}_{ij}\tilde{g}_{km}\right),$$

in the integrand on the right-hand side of Eq. (15) in powers of h:

$$\alpha\sqrt{\frac{g}{\tilde{g}}}\left(\tilde{g}_{ik}\tilde{g}_{jm}-\frac{1}{2}\tilde{g}_{ij}\tilde{g}_{km}\right)$$
$$=g_{ik}g_{jm}-\frac{1}{2}g_{ij}g_{km}+\overset{(1)}{L}_{ijkm}(h)+\overset{(2)}{L}_{ijkm}(h)+\cdots. \tag{18}$$

In particular,

$$\overset{(1)}{L}_{ijkm}=-\frac{1}{2}\left(\frac{h_{st}p^sp^t}{g_{st}p^sp^t}+h_{st}g^{st}\right)\left(g_{ik}g_{jm}-\frac{1}{2}g_{ij}g_{km}\right)+h_{ik}g_{jm}+g_{ik}h_{jm}-\frac{1}{2}h_{ij}g_{km}-\frac{1}{2}g_{ij}h_{km}. \quad (19)$$

We average (15) over the paths:

$$R_{ij}(g)+\left\langle\overset{(1)}{R}_{ij}(h)\right\rangle+\left\langle\overset{(2)}{R}_{ij}(h)\right\rangle+\cdots=\sum_a\chi\int\frac{d^4p}{\sqrt{-g}}\langle N_a\rangle\left(g_{ik}g_{jm}-\frac{1}{2}g_{ij}g_{km}\right)p^kp^m+\sum_a\chi\int\frac{d^4p}{\sqrt{-g}}\left\langle N_a\overset{(1)}{L}_{ijkm}(h)\right\rangle p^kp^m+\cdots. \quad (20)$$

We introduce the one-particle distribution function[4–6]

$$f_a(q,p)=\left\langle\int ds\,\delta^4(q^i-q^i_{(l)}(s))\,\delta(p_j-p_j^{(l)}(s))\right\rangle,$$
$$\langle N_a\rangle=n_af_a, \quad (21)$$

and write (20) in the form

$$R_{ij}(g)+\left\langle\overset{(1)}{R}_{ij}(h)\right\rangle+\left\langle\overset{(2)}{R}_{ij}(h)\right\rangle+\cdots=\sum_a\chi\int\frac{d^4p}{\sqrt{-g}}n_af_a\left(g_{ik}g_{jm}-\frac{1}{2}g_{ij}g_{km}\right)p^kp^m+\sum_a\chi\int\frac{d^4p}{\sqrt{-g}}\left\langle N_a\overset{(1)}{L}_{ijkm}(h)\right\rangle p^kp^m+\cdots. \quad (22)$$

Since $\overset{(1)}{R}_{ij}$ is linear in h, we have $\langle\overset{(1)}{R}_{ij}(h)\rangle\equiv 0$. As a result the approximate macroscopic equations for the averaged metric g_{ij} that allow for the second-order terms in the interaction constant assume the form

$$R_{ij}+\Lambda_{ij}=\chi(T_{ij}-(1/2)\,Tg_{ij}), \quad (23)$$

where

$$T^{ij}=\sum_a n_ac\int\frac{d^4p}{\sqrt{-g}}p^iu^jf_a(q,p)$$

is the macroscopic energy–momentum tensor, and

$$\Lambda_{ij}=\left\langle\overset{(2)}{R}_{ij}\right\rangle-\sum_a\chi\int\frac{d^4p}{\sqrt{-g}}\left\langle N_a\overset{(1)}{L}_{ijkm}\right\rangle p^mp^k. \quad (24)$$

The expressions for $\overset{(2)}{R}_{ij}$ and Λ_{ij} can be obtained from Eqs. (16b), (18), and (19).

Let us now calculate h_{ij} inside the region determined by the correlation radius and the corresponding correlation time. We assume the average gravitational field generated by the particles, and the correlation function as well, to be constant within the correlation region. In this case we can interpret g_{ij} within the correlation region to be the Minkowski metric.

To obtain the macroscopic equations (23) to the required accuracy it is enough to substitute into (24) the value of h_{ij} obtained from the Einstein equations linearized with respect to g_{ij}. By employing the gauge $\nabla_i\gamma^{ij}=0$, where

$$\gamma_{ij}=h_{ij}-\frac{1}{2}hg_{ij},\quad h=g^{ij}h_{ij},$$

we arrive at the following form of the linearized Einstein equations:[8]

$$\Box\gamma^{ij}=-\sum_b 2\chi m_bc^2\int d^4p_b'^{\,i}\Phi_b(q,p_b'^{\,i})u_b'^{\,i}u_b'^{\,j}, \quad (25)$$

where $\Phi_b=N_b-n_bf_b$, $\Box=g^{ij}\nabla_i\nabla_j$, and indices are raised and lowered using the Minkowski metric g_{ij}.

Subsequent calculations do not have a covariant form, but they are all done for the purpose of determining the components of the tensor Λ_{ij} at some point (q) in a selected coordinate system, where $g_{ij}(q)=\eta_{ij}$ is the Minkowski tensor. No difficulties are encountered in writing the final result in covariant form.

We use the identity

$$\Phi_b(\eta,\mathbf{q},p_i^b)=\frac{1}{(2\pi)^3}\times\int d^3\mathbf{q}'\int d^3\mathbf{k}e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\Phi_b(\eta,\mathbf{q}',\mathbf{p}_b). \quad (26)$$

Here η is the temporal coordinate, and $\mathbf{q}=(q^1,q^2,q^3)$ are the spatial coordinates. We substitute (26) into (25) and seek a solution of Eq. (25) in the form

$$\gamma_{ij}(\eta,\mathbf{q})=\frac{1}{(2\pi)^3}\sum_b\int d^4p_b'\int d^3\mathbf{q}'\int d^3\mathbf{k}\times e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\gamma_b^{ij}(\eta,\mathbf{q}',p',\mathbf{k}). \quad (27)$$

For γ^{ij} we obtain the equation

$$(\gamma_b^{ij})''+k^2\gamma_b^{ij}=-2\chi m_bc^2u_b'^{\,i}u_b'^{\,j}\Phi_b(\eta,\mathbf{q}',p_b'), \quad (28)$$

where the prime on the right-hand side stands for a derivative with respect to η, and $k=\|\mathbf{k}\|$. We write the solution of Eq. (28) in the form

$$\gamma_b^{ij}=\frac{2\chi m_bc^2}{k}\int_{-\infty}^{\eta}d\eta'\,\sin k(\eta'-\eta)u_b'^{\,i}u_b'^{\,j}\Phi_b(\eta',\mathbf{q}',p_b'). \quad (29)$$

As a result, for γ^{ij} we obtain

$$\gamma^{ij}(\eta,\mathbf{q})=\sum_b\int d^4p_b'\int d^3\mathbf{q}'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'\times e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\gamma_b^{ij}(\eta,\eta',p_b',\mathbf{k})\Phi_b(\eta',\mathbf{q}',p_b'), \quad (30)$$

where

$$\gamma_b^{ij}(\eta,\eta',p_b',\mathbf{k})=\frac{2\chi m_bc^2}{(2\pi)^3k}u_b'^{\,i}u_b'^{\,j}\sin k(\eta'-\eta). \quad (31)$$

For

$$h_{ij}(\eta,\mathbf{q})=\gamma_{ij}-\frac{1}{2}\gamma\eta_{ij},$$

where $\gamma=g^{ij}\gamma_{ij}$, we have

$$\eta^{ij}(\eta,\mathbf{q})=\sum_b\int d^4p'_b\int d^3\mathbf{q}'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'$$
$$\times e^{-i\mathbf{k}(\mathbf{q}-\mathbf{q}')}h_{ij}^{(b)}(\eta,\eta',p'_b,\mathbf{k})\Phi_b(\eta',\mathbf{q}',p'_b),$$
$$h_{ij}^{(b)}=\frac{2\chi m_bc^2}{(2\pi)^3k}\left(u_b'^iu_b'^j-\frac{1}{2}\eta^{ij}\right)\sin k(\eta'-\eta). \qquad (32)$$

Let us also calculate the quantities

$$\overset{(1)}{\Omega}{}^i_{jk}=\frac{1}{2}g^{il}(-\partial_lh_{jk}+\partial_jh_{lk}+\partial_kh_{li}).$$

To this end we write $h_{ij}^{(b)}(\eta,\eta',p'_b,\mathbf{k})$ in the form

$$h_{ij}^{(b)}(\eta,\eta',p'_b,\mathbf{k})=-\frac{i\chi m_bc^2}{(2\pi)^3k}\left(u_i'^bu_j'^b-\frac{1}{2}h_{ij}\right)$$
$$\times(e^{ik(\eta'-\eta)}-e^{-ik(\eta'-\eta)}) \qquad (33)$$

and then introduce the 4-vectors $k_l^+=(k,\mathbf{k})$ and $k_i^-=(-k,\mathbf{k})$, where obviously

$$k_i^-=-k_i^+(-\mathbf{k}). \qquad (34)$$

Calculating $\overset{(1)}{\Omega}{}^i_{jk}$, we get

$$\Omega^i_{jk}(\eta,\mathbf{q})=\sum_b\int d^4p'_b\int d^3\mathbf{q}'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'$$
$$\times e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\Omega^{i(b)}_{jk}$$
$$\times(\eta,\eta',p'_b,\mathbf{k})\Phi_b(\eta',p'_b,\mathbf{k}), \qquad (35)$$

where

$$\Omega^{i(b)}_{jk}=\frac{\chi m_bc^2}{2(2\pi)^3k}\left\{\left[\left(u'_ju'_k-\frac{1}{2}\eta_{jk}\right)k^i_+-\left(u'_ju'^i-\frac{1}{2}\delta^i_j\right)k^+_k\right.\right.$$
$$\left.-\left(u'_ku'^i-\frac{1}{2}\delta^i_k\right)k^+_j\right]e^{ik(\eta'-\eta)}-\left[\left(u'_ju'_k-\frac{1}{2}\eta_{jk}\right)k^i_-\right.$$
$$\left.-\left(u'_ju'^i-\frac{1}{2}\delta^i_j\right)k^-_k-\left(u'_ku'^i-\frac{1}{2}\delta^i_k\right)k^-_j\right]$$
$$\left.\times e^{-ik(\eta'-\eta)}\right\}. \qquad (36)$$

The expressions (32) and (35) must be substituted into (12) and (24), with the latter becoming

$$\Lambda_{ij}=-\frac{1}{2}\nabla_m\left(\left\langle h^m_l\overset{(1)}{\Omega}{}^l_{ij}\right\rangle\right)+\frac{1}{2}\nabla_j\left(\left\langle h^m_l\overset{(1)}{\Omega}{}^l_{im}\right\rangle\right)+\left\langle\overset{(1)}{\Omega}{}^m_{mn}\overset{(1)}{\Omega}{}^n_{ij}\right\rangle$$
$$-\left\langle\overset{(1)}{\Omega}{}^m_{jn}\overset{(1)}{\Omega}{}^n_{im}\right\rangle-\chi\sum_a\int\frac{d^4p}{\sqrt{-g}}\left\{-\frac{1}{2}p_ip_ju^ku^m\right.$$
$$-\frac{1}{4}g_{ij}p^kp^m-\frac{1}{2}p_ip_jg^{km}+\frac{1}{4}m_a^2c^2g_{ij}g^{km}+p_ip^k\delta^m_j$$
$$\left.+p_jp^k\delta^m_i-\frac{1}{2}m_a^2c^2\delta^k_i\delta^m_j\right\}\langle N_ah_{km}\rangle\frac{1}{m_a}. \qquad (37)$$

After averaging in (37) we arrive at the final macroscopic Einstein equations, accurate second order in the interaction constant. But first we must derive a kinetic equation for the one-particle distribution function to the same accuracy.

3. RELATIVISTIC KINETIC EQUATION: SECOND-ORDER ACCURACY IN THE INTERACTION CONSTANT

We substitute (35) into (12) and get

$$p^i\frac{\partial N_a}{\partial q^i}+\Gamma_{i,jk}p^ip^k\frac{\partial N_a}{\partial p_j}$$
$$=\frac{\partial}{\partial p_j}\sum_b\int d^4p'_b\int d^3\mathbf{q}'\int d^3\mathbf{k}$$
$$\times\int_{-\infty}^{\eta}d\eta'e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\Omega^{j(b)}_{lm}$$
$$\times(\eta,\eta',p'_b,\mathbf{k})p^l_ap^m_b\Delta^a_{ji}N_a(\eta,\mathbf{q},p_a)\Phi_b(\eta',\mathbf{q}',p'_b). \qquad (38)$$

Below we denote the set of all variables $(\eta,\mathbf{q},p_a)$ by x, the set $(\eta,\mathbf{q}',p'_b)$ by x', while the momenta p'_b are denoted simply by p' and the p''_c by p''.

We average (38) over the set of systems:[4]

$$p^i\frac{\partial\langle N_a\rangle}{\partial q^i}+\Gamma_{i,jk}p^ip^k\frac{\partial\langle N_a\rangle}{\partial p_j}$$
$$=\frac{\partial}{\partial p_j}\sum_b\int d^4p'\int d^3\mathbf{q}'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'$$
$$\times e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')}\Omega^{i(b)}_{lm}$$
$$\times(\eta,\eta',p',\mathbf{k}')p^lp^m\Delta_{ji}N_a(x)\Phi_b(x'). \qquad (39)$$

Multiplying (38) by $\Phi'_b(x')$ and averaging yields

$$\frac{\partial\langle N_a(x)\Phi_b(x')\rangle}{\partial q^i}+\Gamma_{i,jk}p^ip^k\frac{\partial\langle N_a(x)\Phi_b(x')\rangle}{\partial p_j}$$
$$=\frac{\partial}{\partial p_j}\sum_c\int d^4p''\int d^3\mathbf{q}''\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta''$$
$$\times e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}'')}\Omega^{i(c)}_{lm}$$
$$\times(\eta,\eta'',p'',\mathbf{k})p^lp^m\Delta_{ji}\langle N_a(x)\Phi_b(x')\Phi_c(x'')\rangle. \qquad (40)$$

Equation (40) is an equation for the second moment $\langle N_a(x)\Phi_b(x')\rangle$. The second equation for this moment can be obtained from Eq. (40) by the substitutions $a\leftrightarrow b$ and $x\leftrightarrow x'$.

We introduce two-particle, three-particle, etc. distribution functions:

$$\left\langle\int ds\,\delta(x-x_a(s))\right\rangle=f_a(x),$$
$$\left\langle\int ds\,\delta(x-x_a(s))\int ds'\,\delta(x'-x_b(s'))\right\rangle=f_{ab}(x,x'),$$

$$\left\langle \int ds\,\delta(x-x_a(s)) \int ds'\,\delta(x'-x_b(s'))\right.$$

$$\left. \times \int ds''\,\delta(x''-x_a(s''))\right\rangle = f_{abc}(x,x',x''). \tag{41}$$

Here

$$\delta(x-x_a(s)) \equiv \delta^4(q^i - q_a(s))\,\delta^4(p_j - p_j^a(s)).$$

For the moments of random functions we have the following formulas:[4]

$$\langle N_a(x)\rangle = n_a f_a, \tag{41a}$$

$$\langle N_a(x) N_b(x')\rangle = (n_a n_b - n_a \delta_{ab}) f_{ab}(x,x') + n_a \delta_{ab} f_a(x) \int ds'\,\delta(x'-x_a(s'/x)), \tag{41b}$$

$$\begin{aligned}\langle N_a(x) N_b(x') N_c(x'')\rangle &= (n_a n_b n_c - n_a n_b \delta_{ac} - n_a n_b \delta_{bc} - n_a n_c \delta_{ac} \\ &\quad + 2 n_a \delta_{ab} \delta_{bc}) f_{abc}(x,x',x'') \\ &\quad + (n_a n_c - n_a \delta_{ac})\,\delta_{ab} f_{ac}(x,x'') \int ds'\,\delta(x'-x_a(s'/x)) \\ &\quad + (n_a n_b - n_a \delta_{ab})\,\delta_{ac} f_{ab}(x,x') \int ds''\,\delta(x''-x_a(s''/x)) \\ &\quad + (n_a n_b - n_b \delta_{ab})\,\delta_{bc} f_{ab}(x,x') \int ds''\,\delta(x'' \\ &\quad - x_b(s''/x')) + n_a \delta_{ab} \delta_{bc} f_a(x) \int ds'\,\delta(x' \\ &\quad - x_a(s'/x)) \int ds''\,\delta(x''-x_a(s''/x)).\end{aligned} \tag{41c}$$

Here $x_a(s/x)$ stands for the particle path through point x of the phase space. Bearing in mind that $\Phi_a = N_a - n_a f_a$ and that f_a is not a random function, we can easily obtain expressions for the averages $\langle N_a(x)\Phi_b(x')\rangle$ and $\langle N_a(x)\Phi_b(x')\Phi(x'')\rangle$.

Substituting (41) into (39), (40), and similar equations, we arrive at an infinite chain of kinetic equations for the distribution functions f_a, f_{ac}, f_{abc}, etc. To obtain a kinetic equation for the one-particle distribution function f_a to second order in the interaction constant, we truncate the chain and assume that

$$f_{ab}(x) = f_a(x) f_b(x') + g_{ab}(x,x'),$$

$$f_{abc}(x) \approx f_a(x) f_b(x') f_c(x''). \tag{42}$$

As a result we get an approximate system of equations for the functions $f_a(x)$ and $g_{ab}(x,x')$. For $n_a \gg 1$,

$$p^i \frac{\partial f_a}{\partial q^i} + \Gamma_{i,jk} p^i p^k \frac{\partial f_a}{\partial p_j} = \sum_b n_b \frac{\partial}{\partial p_i} \int d^4p' \int d^3\mathbf{q}' \int d^3\mathbf{k} \int_{-\infty}^{\eta} d\eta' \times e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')} \Omega_{lm}^{j(b)}(\eta,\eta',p_b',\mathbf{k}) p_a^l p_a^m \Delta_{ji} g_{ab}(x,x'), \tag{43}$$

$$\begin{aligned}p^i \frac{\partial g_{ab}(x,x')}{\partial q^i} + \Gamma_{i,jk} p^i p^k \frac{\partial g_{ab}(x,x')}{\partial p_j} &= \frac{\partial}{\partial p_i} \int d^4p'' \int d^3\mathbf{q}'' \int d^3\mathbf{k} \int_{-\infty}^{\eta} d\eta'' e^{-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}'')} \Omega_{lm}^{j(c)} \\ &\quad \times (\eta,\eta'',p_b'',\mathbf{k}) p_a^l p_a^m \Delta_{ji} f_a(x) f_b(x') \\ &\quad \times \int ds''\,\delta(x''-x_b(s''/x')).\end{aligned} \tag{44}$$

In deriving (43) and (44) we assumed that $x' \neq x_a(s/x)$, i.e., point x' is not on the path of particles of species a passing through the point x of the phase space.

In view of the weakness of the interaction, the path of particles of species b can be thought of as a geodesic in the Minkowski world:

$$p_i^b(s'',x') = p_i' = \text{const},$$

$$\mathbf{q}_b(s''/x') = \mathbf{q}' + \frac{\mathbf{v}'}{c}(\eta''-\eta'), \quad q_b^0(s''/x) = \eta''.$$

Here $\mathbf{v}' = c\mathbf{u}_b'/u_b'^0$, with $\mathbf{u}' = (u'^1, u'^2, u'^3)$.

Integrating in (44) with respect to s'', $\mathbf{q}''$, and p'', we obtain

$$\begin{aligned}p^0 \frac{\partial g_{ab}}{\partial \eta} + p^a \frac{\partial g_{ab}}{\partial q^a} &= \frac{\partial}{\partial p_i} \int d^3\mathbf{k} \int_{-\infty}^{\eta} d\eta'' \Omega_{lm}^{j(b)} \\ &\quad \times (\eta,\eta'',p',\mathbf{k}) p^l p^m \Delta_{ij} \frac{1}{u'^0} f_a(x) f_b(x') \\ &\quad \times \exp\left[-i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}') + \frac{i}{c}(\mathbf{k}\cdot\mathbf{v}')(\eta''-\eta')\right].\end{aligned} \tag{45}$$

In this equation for g_{ab} we have put $\Gamma_{i,jk} = 0$ on the left-hand side, since we assume that within the correlation region the metric coefficients g_{ij} are constant. The solution of this equation has the form

$$\begin{aligned}g_{ab}(x,x') &= \int d^3\mathbf{k}' \int_{-\infty}^{\eta} \frac{d\tau}{p^0} \left[\frac{\partial(p^l p^m \Delta_{ij} f_a(x))}{\partial p_i}\right]_\tau \\ &\quad \times \int_{-\infty}^{s} \frac{d\tau'}{u'^0} f_b(x') \Omega_{lm}^{j(b)}(\tau,\tau',\mathbf{p},k) \\ &\quad \times \exp\left[-i\mathbf{k}'\cdot(\mathbf{q}-\mathbf{q}') + \frac{i}{c}(\mathbf{k}'\cdot\mathbf{v})(\eta-\tau) \right. \\ &\quad \left. + \frac{i}{c}(\mathbf{k}'\cdot\mathbf{v}')(\tau'-\eta')\right].\end{aligned} \tag{46}$$

Here the subscript τ indicates that after calculating the derivatives with respect to p we must replace the arguments η and $\mathbf{q}$ by τ and $\mathbf{q} + (\mathbf{v}/c)(\tau-\eta)$, respectively. The solution (46) allows for the effect of the path of a particle of species

b on a particle of species a. The reciprocal effect is taken into account by the solutions of the equation obtained from (45) via the substitutions $a \leftrightarrow b$ and $x \leftrightarrow x'$. Such a solution can also be obtained from (46) via the same substitutions. The right-hand side of Eq. (43) must include the sum of these solutions. As a result we arrive at the desired relativistic equation to second order in the interaction constant. Here we restrict our discussion to space-time variation of the distribution function that is so slow that it can be considered constant in the region determined by the correlation radius and correlation time. Then in calculating the integrals with respect to $\mathbf{q}'$, η', τ and τ' in Eqs. (43) and (46) we can ignore the dependence of f_a and f_b on coordinates and time. After integrating with respect to $\mathbf{q}'$ and $\mathbf{k}$ we arrive at the following equation for $f_a(f_a=f_a(q^i,p_j)$ and $f_b' = f_b(q^i,p_j'))$:

$$p^i \frac{\partial f_a}{\partial q^i}+\Gamma_{i,jk}p^i p^k \frac{\partial f_a}{\partial p_j}=\frac{\partial}{\partial p_i}\sum_b J_i^{ab}, \tag{47}$$

where

$$\begin{aligned}J_i^{ab}=&(2\pi)^3 n_b\int d^4p'\int d^3\mathbf{k}\int_{-\infty}^{\eta} d\eta'\Omega_{lm}^{j(b)}\\&\times(\eta,\eta',p',\mathbf{k})p^l p^m\Delta_{ij}\left\{\int_{-\infty}^{\eta}\frac{d\tau}{p^0}\left[\frac{\partial(p^s p^t\Delta_{rk}f_a)}{\partial p_k}\right]_\tau\right.\\&\times f_b'\int_{-\infty}^{\tau}\frac{d\tau'}{u'^0}\Omega_{st}^{r(b)}(\tau,\tau',p',-\mathbf{k})\\&\times\exp\left[-\frac{i}{c}(\mathbf{k}\cdot\mathbf{v})(\eta-\tau)-\frac{i}{c}(\mathbf{k}\cdot\mathbf{v}')(\tau'-\eta')\right]\\&+\int_{-\infty}^{\eta'}\frac{d\tau'}{p^0}\left[\frac{\partial(p'^s p'^t\Delta'_{rk}f_a')}{\partial p_k'}\right]_{\tau'} f_a\int_{-\infty}^{\tau'}\frac{d\tau}{u^0}\Omega_{st}^{r(a)}\\&\times(\tau',\tau,p,\mathbf{k})\exp\left[-\frac{i}{c}(\mathbf{k}\cdot\mathbf{v})(\eta-\tau)\right.\\&\left.\left.-\frac{i}{c}(\mathbf{k}\cdot\mathbf{v}')(\tau'-\eta')\right]\right\}.\end{aligned}\tag{48}$$

Now let us examine the seven-dimensional distribution function, which depends on the coordinates and the spatial components of momentum p_α (here and in what follows Greek letters are used to denote the spatial components of vectors and tensors):

$$n_a f_a(x)=F_a(q^i,p_\alpha)\delta(\sqrt{g^{ij}p_i p_j}-m_a c). \tag{49}$$

The equation for F can be obtained from (47) by integrating both sides with respect to p_0. We must also integrate with respect to p_0' in (48). Let us take into account the identity

$$\frac{\partial(p^s p^t\Delta_{rk}\Omega_{st}^r F)}{\partial p_k}=p^0\frac{\partial}{\partial p_\alpha}\left(\frac{1}{p^0}p^s p^t\Delta_{r\alpha}\Omega_{st}^r F\right). \tag{50}$$

Here and in what follows we assume that there is summation over repeated Greek indices. In calculating the partial derivative with respect to p_k on the left-hand side of (50) we assume all the components p_k independent and only allow for the fact that $g^{ij}p_i p_j=m^2c^2$. On the right-hand side of Eq. (50) this fact is taken into account in calculating the derivatives with respect to the spatial components of the momentum p_a.

Using (50), we arrive at the following kinetic equation for F_a:

$$p^i\frac{\partial F_a}{\partial q^i}+\Gamma_{i,\alpha k}p^i p^k\frac{\partial F_a}{\partial p_\alpha}=u^0\frac{\partial}{\partial p_\alpha}\sum_b J_\alpha^{ab}, \tag{51}$$

where

$$\begin{aligned}J_\alpha^{ab}=&(2\pi)^3\int d^3p'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'\Omega_{lm}^{j(b)}\\&\times(\eta,\eta',p',\mathbf{k})\Delta_{\alpha j}(u^0u'^0)^{-1}\\&\times\left\{\int_{-\infty}^{\eta}d\tau\left[\frac{\partial}{\partial p_\beta}\left(\frac{p^s p^t\Delta_{r\beta}F_a}{p^0}\right)\right]_\tau\right.\\&\times F_b'\int_{-\infty}^{\eta}\frac{d\tau'}{u^0}\Omega_{st}^{r(b)}(\tau,\tau',p',-\mathbf{k})\\&\times\exp\left[\frac{i}{c}(\mathbf{k}\cdot\mathbf{v})(\tau-\eta)+\frac{i}{c}(\mathbf{k}\cdot\mathbf{v}')(\eta'-\tau')\right]\\&+\int_{-\infty}^{\eta'}d\tau'\left[\frac{\partial}{\partial p_\beta'}\left(\frac{p'^s p'^t\Delta_{r\beta}'F_b'}{p^0}\right)\right]_{\tau'}\\&\times F_a\int_{-\infty}^{\tau'}\frac{d\tau}{u^0}\Omega_{st}^{r(a)}(\tau',\tau,p,\mathbf{k})\\&\left.\times\exp\left[\frac{i}{c}(\mathbf{k}\cdot\mathbf{v})(\tau-\eta)+\frac{i}{c}(\mathbf{k}\cdot\mathbf{v}')(\eta'-\tau')\right]\right\}.\end{aligned}\tag{52}$$

Now we only have to substitute the expression (36) for $\Omega_{jk}^{i(b)}$ into (52) and integrate with respect to η', τ, and τ'.

The kinetic equation assumes the form

$$\begin{aligned}&\frac{c}{p^0}\left(p^i\frac{\partial F_a}{\partial q^i}+\Gamma_{j,\alpha k}p^j p^k\frac{\partial F_a}{\partial p_\alpha}\right)\\&=\frac{\partial}{\partial p_\alpha}\sum_b\int d^3p' E_{\alpha\beta}\left(\frac{\partial F_a}{\partial p_\beta}F_b'-\frac{\partial F_b'}{\partial p_\alpha}F_a\right),\end{aligned}\tag{53}$$

where

$$\begin{aligned}E_{\alpha\beta}=&2G^2(m_a c)^2(m_b c)^2\frac{[2(u^l u_l')^2-1]^2}{(u^0u'^0)^2}\\&\times\int d^3\mathbf{k}\frac{\mathbf{k}_\alpha\cdot\mathbf{k}_\beta\delta(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}')}{[k^2c^2-(\mathbf{k}\cdot\mathbf{v})^2]^2}.\end{aligned}\tag{54}$$

After integrating with respect to k in (54), we arrive at the following expression for $E_{\alpha\beta}$:

$$\begin{aligned}E_{\alpha\beta}=&\frac{2\pi G^2L[2(u^k u_k')(p^i p_i')-(u^k p_k)(u'^i p_i')]^2}{c^5u^0u'^0[(u_i'u^i)^2-1]^{3/2}}\\&\times\{-g_{\alpha\beta}[(u^i u_i')^2-1]-u_\alpha u_\beta-u_\alpha'u_\beta'+(u_i u'^i)\\&\times(u_\alpha u_\beta'+u_\alpha'u_\beta)\},\end{aligned}\tag{55}$$

where

$$L=\int \frac{dk}{k}$$

is an analog of the Coulomb logarithm.

Now we can easily write the covariant kinetic equation for the function f of eight variables q^i and p_i:

$$u^i \frac{\partial f_a}{\partial q^i}+\Gamma_{j,ik}p^j u^k \frac{\partial f_a}{\partial p_i}=\frac{\partial}{\partial p_i}\sum_b \int \frac{d^4p'}{\sqrt{-g}} E_{ij}\left(\frac{\partial f_a}{\partial p_j} f'_b - \frac{\partial f'_b}{\partial p_j} f_a\right), \tag{56}$$

where

$$E_{\alpha\beta}=\frac{2\pi G^2 L n_b[2(u^k u'_k)(p^i p'_i)-(u^k p_k)(u'^i p'_i)]^2}{c^6[(u'_i u)^2-1]^{3/2}} \times\{-g_{ij}[(u^i u'_i)^2-1]-u_i u_j-u'_i u'_j +(u'_k u^k)(u_i u'_j+u'_i u_j)\}. \tag{57}$$

The collision integral thus obtained is logarithmically divergent. Just as in plasma theory this difficulty must be resolved by introducing a cutoff in the expression for L. We set the upper limit in the integral

$$L=\int_{k_0}^{k_\infty} \frac{dk}{k}$$

to $1/r_{\min}$, where $r_{\min}$ is the distance at which the kinetic energy of colliding particles becomes equal to their potential energy. The lower limit k_0 is set to $1/R$, where the distance R depends on the nature of the averaged metric g_{ij}. For instance, if g_{ij} is the Friedmann metric, then $R=\langle v^2\rangle^{1/2}t$, where $\langle v^2\rangle^{1/2}$ is the average thermal velocity of the particles, and t is the cosmological time. As shown in Ref. 9, allowing for an expanding universe removes the divergences as $k\rightarrow 0$, with the contribution to L of the region with $k<1/R$ becoming negligible.

The right-hand side of the resulting kinetic equation vanishes if instead of the function f_a we substitute the relativistic Maxwell distribution

$$f_a(q^i,p_j)=A_a \exp\left(-\frac{c\bar{u}_i p^i}{k_B T}\right)\delta(\sqrt{g^{ij}p_i p_j}-m_a c).$$

Here A_a is a normalization constant, $\bar{u}^i$ is the vector of the average velocity in an equilibrium state, T is the temperature, and k_B is the Boltzmann constant.

Earlier the kinetic equation (56) was derived in Ref. 6, but in a somewhat questionable manner. An additional metric was introduced in the process of deriving the kinetic equation, and the metric did not coincide with the averaged metric. As a result the left-hand side of the kinetic equation acquired a term interpreted as the action of a self-consistent gravitational field. If the variables in the corresponding equation of Ref. 6 calculated by this additional metric are replaced by variables calculated by the averaged metric according to a scheme similar to (7) and (13), we immediately arrive at the kinetic equation (56) with the kernel (57).

4. EXPRESSING THE ADDITIONAL TERM IN THE MACROSCOPIC EINSTEIN EQUATIONS IN TERMS OF THE CORRELATION FUNCTION

We substitute (32) and (35) into (37) and use (41). As a result the additional term Λ_{ij} in the macroscopic Einstein equations can be expressed in terms of the correlation function $g_{ab}(x,x')$ as follows:

$$\Lambda_{ij}=(\delta^k_n\delta^s_j-\delta^k_j\delta^s_n)\left[-\frac{1}{2}\nabla_k P^n_{is}+Q^n_{kis}\right]+\lambda_{ij}, \tag{58}$$

where

$$P^n_{is}=\langle h^n_l \Omega^l_{is}\rangle=\sum_{b,c}\int d^4p'_b\int d^4p''_c\int d^3\mathbf{q}'\int d^3\mathbf{q}'' \times\int d^3\mathbf{k}'\int d^3\mathbf{k}''\int_\infty^\eta d\eta'\int_\infty^\eta d\eta''\exp[-i\mathbf{k}'(\mathbf{q}-\mathbf{q}')-i\mathbf{k}''(\mathbf{q}-\mathbf{q}'')]h^{n(b)}_l(\eta,\eta',p',\mathbf{k}')\Omega^{l(c)}_{is} \times(\eta,\eta'',p'',\mathbf{k}'')n_b n_c g_{bc}(x',x''), \tag{59a}$$

$$Q^n_{kis}=\sum_{b,c}\int d^4p'_b\int d^4p''_c\int d^3\mathbf{q}'\int d^3\mathbf{q}''\int d^3\mathbf{k}' \times\int d^3\mathbf{k}''\int_\infty^\eta d\eta'\int_\infty^\eta d\eta''\exp[-i\mathbf{k}'(\mathbf{q}-\mathbf{q}')-i\mathbf{k}''(\mathbf{q}-\mathbf{q}'')]\Omega^{n(b)}_{kl}(\eta,\eta',p',\mathbf{k}')\Omega^{l(c)}_{is} \times(\eta,\eta'',p'',\mathbf{k}'')n_b n_c g_{bc}(x',x''), \tag{59b}$$

$$\lambda_{ij}=-\chi\sum_{a,b}\int d^4p_a\int d^4p'_b\int d^3\mathbf{q}'\int d^3\mathbf{k}\int_{-\infty}^\eta d\eta' \times\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}'')]\left\{-\frac{1}{2}p_i p_j u^k u^m-\frac{1}{4}g_{ij}p^k p^m -\frac{1}{2}p_i p_j g^{km}+\frac{1}{4}m_a^2c^2 g_{ij}g^{km}+p_i p^k\delta^m_j+p_j p^k\delta^m_i -\frac{1}{2}m_a^2c^2\delta^k_i\delta^m_j\right\}h^{(b)}_{km}(\eta,\eta',p',\mathbf{k})n_a n_b g_{ab}(x,x'). \tag{59c}$$

Here the quantities $h^{(b)}_{ij}$ and $\Omega^{i(b)}_{jk}$ are determined by (32) and (36), and the expression for the correlation function $g_{ab}(x,x')$ is obtained by adding to (46) a similar term obtained via the substitutions $a\leftrightarrow b$ and $x\leftrightarrow x'$.

Further simplification of the macroscopic equations can be achieved by substituting the explicit form of g_{ab} into (59) and calculating the integrals in (59) with respect to the variables $\mathbf{k}'$, $\mathbf{k}''$, q', q'', η', and η''. This procedure requires separate study.

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