Evolution and steady state of large-scale vortex structures

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We obtain equations that describe the relaxation of a large-scale vortex formation similar to a tropical hurricane, taking into account the variation of turbulent viscosity as a function of the vortex intensity. We calculate the Reynolds stresses by statistical averaging over small-scale turbulent fluctuations. When a mechanism for oscillation based on helical turbulence is included, our model can be used to describe the evolution and stationary states of large-scale structures. [S1063-7761(96)01005-0]

1. INTRODUCTION

The concept of helical turbulence[1,2] has been used with great success in astrophysics. For this reason, in the last decade this concept has become the basis for modeling the generation of large-scale structures in an incompressible fluid, and the resulting models have been used to describe tropical hurricanes[3–8]. When applied to the atmosphere, the concept of helical turbulence parametrizes such important factors in cyclogenesis as convection and the Coriolis force in a natural way, allowing systems of equations to be derived that contain the direct positive feedback between the solenoidal velocity fields that creates large-scale instability. Studies of this instability show that it leads to the formation of vortex structures with nontrivial topology, thereby providing a basis for a new generation of tropical hurricane models.

The equations for modeling large-scale velocity fields are derived using the method of statistical averaging to determine the Reynolds stresses that correspond to the helical part of the turbulence correlation tensor. The Reynolds stresses that correspond to the nonhelical part of the turbulence tensor, which give rise to turbulent viscosity, were obtained previously in Ref. 9. These expressions have also been used in studies of the large-scale equations, leading to the derivation of expressions for the growth rate of the instability, and also the conditions for its appearance.

It is natural to ask how such a vortex structure evolves to its steady state. The problem of deriving the steady-state structure requires the introduction of factors that limit the linear growth of the instability. Since the convective nonlinearities that are typical of hydrodynamic systems do not play a large role in the dynamics of vortex structures of the type under discussion, the choice of basic nonlinear factors that mediate the evolution of the system to its steady state within the framework of the original formulation becomes an extremely complicated problem. We might postulate that the intensity of the feedback will decrease in the nonlinear stage of evolution, and pose the problem in a form where nonlinear corrections to the helical instability mechanism are the limiting factor. However, this formulation is somewhat idealized, since it is well known (see, e.g., Refs. 7 and 8) that the intensity of the helical component of turbulence is considerably smaller than the nonhelical component, which is the source of turbulent viscosity[2]. Therefore, the most natural approach is to focus on the dissipative processes. In fact, tropical hurricanes, i.e., intense large-scale vortices with horizontal dimensions that greatly exceed their vertical dimensions, should experience considerable friction exerted by the underlying surface. As a result of the instability caused by the increase in shear of the horizontal component of the velocity due to friction, the large-scale flow should increase in intensity, leading to enhancement of the turbulence and, consequently, to an increase in the turbulent viscosity. This in turn should be especially effective in limiting the exponential growth of the large-scale structure. If so, we can limit our formulation of the problem (as in Ref. 9) to a prespecified turbulent distribution, which corresponds to the absence of feedback from the mean field to the latter. A model that takes into account these factors should contain nonlinear viscosity caused by turbulence.

The terms that describe the nonlinear viscosity, like the oscillatory terms associated with the helical component of the small-scale turbulence, can be computed by the method of statistical averaging, but now against the background of the more energetic nonhelical components of small-scale turbulence. In this case the turbulence must be treated as trapped by the external large-scale flow.

The task of this paper is to formulate and investigate a model of the evolution of a large-scale vortex structure arising from instability caused by the helical component of small-scale turbulence to a steady state due to enhancement of the turbulent dissipation. The steady state is determined by solving a boundary value problem that takes into account the finite amplitude of the hydrodynamic fields for a large-scale system with helical instability and nonlinear viscosity.

2. LARGE-SCALE EQUATIONS

The procedure of statistical averaging against a background of small-scale helical turbulence, and the derivation of equations that contain terms responsible for the positive feedback between solenoidal components of the large-scale velocity, have been carried out many times, with the imposition of various conditions.[4–7] In Ref. 7, for example, a system of large-scale equations was derived under the assumption that small-scale convection develops, so that the Rayleigh number of the linear convection problem becomes equal to its critical value. On the one hand, this formulation of the problem turns out to be technically simple, since it leads to elimination of the equation for the large-scale temperature from the system. This is because the actual temperature profile coincides with the neutral profile by virtue of the intense small-scale motion. On the other hand, it physically corresponds to the fact that convection does not take place on the large scale. In this case, we can use the results of Ref. 7 and write the following equation for the large-scale velocity field V_i:

$$\frac{\partial V_i}{\partial t} - \nu_T \Delta V_i - \alpha P_{im} \nabla_k (e_m \varepsilon_{kra} + e_k \varepsilon_{mra}) e_r V_a = 0, \quad (1)$$

$$\alpha = \frac{H\lambda}{v_T^2} \frac{h}{\lambda} \frac{\mathrm{Re}}{8} (n-4) K_0^{-2} (1+\eta^2)^{-2} \times \ln[2K_0^2 \tau \nu \lambda^{-2} (1+\eta^2)].$$

In Ref. 7, Eq. (1) was used to describe the instability of large-scale flow in a layer of fluid with a thickness h that is small compared to its horizontal dimensions. The lower boundary of the fluid coincides with the xy plane; the z axis is directed vertically upward; τ and λ are the correlation time and correlation length of the large-scale turbulence; v_T is the velocity of turbulent fluctuations; ν_T is the coefficient of turbulent viscosity; α is the helicity coefficient of the instability; e_i is a unit vector directed along the z axis; ε_{kra} is the completely antisymmetric Levi-Civita unit tensor; $P_{im} = \delta_{im} - \nabla_i \nabla_m / \Delta$ is a projection operator that eliminates the irrotational part of the velocity field; the symbol ∇_k denotes differentiation with respect to coordinates; $H = \langle \mathbf{v}_T \cdot \nabla \times \mathbf{v}_T \rangle$ is the topological invariant of helical turbulence; Re is its Reynolds number; n is a parameter that characterizes the inertial subregion of convective turbulence; $K_0 = \mathrm{Ra}_c^{1/4}$, where Ra_c is the critical value of the flow Rayleigh number; and η is the aspect ratio of the small-scale convective cells.

In order to study the later stages of evolution of the large-scale structure, the nonlinear limitation of its growth and establishment of a steady state, we must include nonlinear terms in the equation of motion (1). Let us assume that the primary nonlinear effect that limits the linear growth of the structure is increased turbulent viscosity due to enhancement of the intensity of the shear hydrodynamic fields. This effect can be studied directly within the framework used to address the Reynolds stress problem, which gives rise to turbulent viscosity,[9] and the corresponding terms should be added to Eq. (1). These stresses can in turn be studied by formulating the problem of helical turbulence in an incompressible fluid under conditions of shear flow. We will prespecify the small-scale motion by adding a random external force $F_i(t,\mathbf{x})$ to the right side of the Navier–Stokes equation, which maintains a certain steady level of turbulence v_T in the liquid:

$$\left(\frac{\partial}{\partial t} - \nu \Delta \right) u_i + u_k \nabla_k u_i + \nabla_i \Pi = F_i(t,\mathbf{x}). \quad (2)$$

The quantity Π denotes the pressure divided by the density, which we assume to be constant.

Following Ref. 9, we will assume that the velocity field u_i consists of a sum of an average large-scale field V_i and a fluctuating part u_i':

$$u_i = V_i + u_i'$$

Statistical averaging of Eq. (2) over the small-scale fluctuations (denoted by angle brackets) leads to an equation for the average velocity, which contains the Reynolds stresses

$$\left(\frac{\partial}{\partial t} - \nu \Delta \right) V_i + V_k \nabla_k V_i + \langle u_k' \nabla_k u_i' \rangle + \nabla_i \langle \Pi \rangle = 0. \quad (3)$$

An equation that determines the fluctuating component as a function of the small-scale coordinates is obtained by eliminating the large-scale part from the original Eq. (2):

$$\left(\frac{\partial}{\partial t} - \nu \Delta \right) u_i' + V_k \nabla_k u_i' + u_k' \nabla_k V_i + (u_k' \nabla_k u_i' - \langle u_k' \nabla_k u_i' \rangle) + \nabla_i \Pi = F_i. \quad (4)$$

Let us represent the average field V_k in the form of an expansion with respect to the coordinates

$$V_k \simeq V_k^0 + x_s \nabla_s V_k + \ldots \quad (5)$$

Direct substitution of Eq. (5) into Eq. (4) would lead to the appearance of a convective term $V_k^0 \nabla_k u_i'$ in the latter, which we can eliminate by a Galilean transformation if the source of the external force F_i moves along with the constant average flow V_k^0. Sources of turbulence are ordinarily trapped in the surrounding flow under natural conditions; therefore, if our statement of the problem assumes that the random external force is natural, we should eliminate this convective term from consideration. Then the equation for the field u_i' takes the form

$$\left(\frac{\partial}{\partial t} - \nu \Delta \right) u_i' + x_s \nabla_s V_k \nabla_k u_i' + u_k' \nabla_k V_i + (u_k' \nabla_k u_i' - \langle u_k' \nabla_k u_i' \rangle) + \nabla_i \Pi = F_i. \quad (6)$$

The fluctuating component u_i' is made up of the turbulent velocity field u_i^T caused by the random external force $F_i(t,\mathbf{x})$ and a correction $\tilde{u}_i$ corresponding to the presence in the turbulent medium of the large-scale average field V_i:

$$u_i' = u_i^T + \tilde{u}_i. \quad (7)$$

If we collect all the nonzero terms that depend only on the turbulent field u_i^T in Eq. (6) for the fluctuating component u_i', we obtain the following equation for the latter:

$$\left(\frac{\partial}{\partial t} - \nu \Delta \right) u_i^T + u_k^T \nabla_k u_i^T + \nabla_i \Pi^T = F_i. \quad (8)$$

Equation (8) is a complicated nonlinear system that describes the establishment of a steady-state level of turbulent fluctuations maintained by the external force F_i. As a digression from the study of the processes that generate and maintain the turbulent force F_i, let us specify the turbulent field u_i^T itself, assuming it is uniform, isotropic, and steady:

$$\langle u_i^T(t,\mathbf{x}) u_k^T(s,\mathbf{y}) \rangle = Q_{ik}^T(t-s, \mathbf{x}-\mathbf{y}). \quad (9)$$

The remaining terms in Eq. (6) make up an equation that determines the field $\tilde{u}_i$, which in lowest approximation takes the following form:

$$\left(\frac{\partial}{\partial t}-\nu\Delta\right)\tilde{u}_i+x_s\nabla_s V_k\nabla_k\tilde{u}_i+\tilde{u}_k\nabla_k V_i+\nabla_i\tilde{\Pi}$$
$$=-(x_s\nabla_s V_k\nabla_k u_i^T+u_k^T\nabla_k V_i). \tag{10}$$

Thus, the question of evolution of small-scale structure reduces to study of Eq. (3), in which the Reynolds stresses must be expressed in terms of the large-scale velocity using representation (7) for the fluctuating velocity component u_i', Eq. (10) for the field $\tilde{u}_i$, and the information about the properties of the turbulence of the surrounding medium contained in (9).

3. REYNOLDS STRESSES

For convenience, let us introduce the diffusion operator $D=(\partial/\partial t-\nu\Delta)$, and rewrite Eq. (10) using this notation:

$$D\tilde{u}_i+P_{im}(x_s\nabla_s V_k\nabla_k\tilde{u}_m+\tilde{u}_k\nabla_k V_m)$$
$$=-P_{im}(x_s\nabla_s V_k\nabla_k u_m^T+u_k^T\nabla_k V_m). \tag{11}$$

We can write Eq. (11) in a form that is convenient for iteration by assuming that the turbulent field is weak:

$$\tilde{u}_i=-\frac{1}{D}P_{im}(x_s\nabla_s V_k\nabla_k u_m^T+u_k^T\nabla_k V_m)$$
$$+\frac{1}{D}P_{im}\Bigg(x_s\nabla_s V_k\nabla_k\Bigg[\frac{1}{D}P_{mm'}(x_{s'}\nabla_{s'}V_{k'}\nabla_{k'}\tilde{u}_{m'}$$
$$+\tilde{u}_{k'}\nabla_{k'}V_{m'})\Bigg]+\frac{1}{D}P_{km'}(x_{s'}\nabla_{s'}V_{k'}\nabla_{k'}\tilde{u}_{m'}$$
$$+\tilde{u}_{k'}\nabla_{k'}V_m)\nabla_k V_m\Bigg). \tag{12}$$

The first two terms in parentheses, which correspond to the Reynolds stresses studied in Ref. 9, describe the turbulent viscosity. In calculating the leading corrections with respect to the average field V_i contained in the remaining terms on the right side of Eq. (12), we should keep in mind that terms that are quadratic in the average field give no contribution to the turbulent viscosity; otherwise, the parameter for turbulent viscosity would depend linearly on the average velocity, and its sign would change when the direction of the velocity is changed. Thus, in the course of the iterative process, corrections that are quadratic in the average field drop out; hence, the lowest-order corrections we should include are cubic in the average velocity field.

If we proceed to iterate and discard terms that are quadratic in the average field, we obtain to lowest approximation an expression that specifies the field:

$$\tilde{u}_i=\tilde{u}_i\{\mathbf{V},\mathbf{u}^T\} \tag{13}$$

as a functional that depends on the fields V_i and u_i^T.

The term $\langle u_k'\nabla_k u_i'\rangle$, which describes the main stresses in Eq. (3) for the average velocity field V_i, can be expressed with the help of Eq. (7) in terms of the fields u_i^T and $\tilde{u}_i$:

$$S_i\equiv\langle u_k'\nabla_k u_i'\rangle=\langle u_k^T\nabla_k u_i^T\rangle+\langle u_k^T\nabla_k\tilde{u}_i\rangle+\langle\tilde{u}_k\nabla_k u_i^T\rangle$$
$$+\langle\tilde{u}_k\nabla_k\tilde{u}_i\rangle. \tag{14}$$

By virtue of the incompressibility of the fluid, the first term on the right side of Eq. (14) reduces to zero because it is the total derivative of the average, which does not depend on the large-scale fields. The next term yields the leading correction in the field $\tilde{u}_i$. Thus, within the framework of this approximation we can interpret the following symmetric combination of fields u_i^T and $\tilde{u}_i$ as the Reynolds stresses S_i:

$$S_i\simeq\langle u_k^T\nabla_k\tilde{u}_i\rangle+\langle\tilde{u}_k\nabla_k u_i^T\rangle. \tag{15}$$

The Reynolds stresses will be computed from the Furutsu–Novikov formula (see, e.g., Ref. 10):

$$\langle u_i^T(t_1,\mathbf{x}_1)\tilde{u}_k(t_2,\mathbf{x}_2)\rangle=\int ds\int d\mathbf{y}\, Q_{kr}^T(t_1-s,\mathbf{x}_1-\mathbf{y})$$
$$\times\left\langle\frac{\delta\tilde{u}_i(t_2,\mathbf{x}_2)}{\delta u_r^T(s,\mathbf{y})}\right\rangle. \tag{16}$$

Because the field $\tilde{u}_i$ depends linearly on the turbulent field u_i^T given by (12), the variational derivative in the Furutsu–Novikov expression can be computed without difficulty:

$$\left\langle\frac{\delta\tilde{u}_i\{u_f^T(t_2,\mathbf{x}_2)\}}{\delta u_r^T(s,\mathbf{y})}\right\rangle=u_{ir}\{\delta_{fr}\delta(\mathbf{x}_2-\mathbf{y})\delta(t_2-s)\}. \tag{17}$$

Using Eq. (17) and Fourier transforming the coordinate dependence ($\partial/\partial x_i=ik_i$), we obtain the following expression for the one-point average $\langle u_p^T\tilde{u}_i\rangle$ appearing in the definition of the Reynolds stresses (15):

$$\langle u_k^T(t,\mathbf{x})\tilde{u}_i(t,\mathbf{x})\rangle=\lim_{\mathbf{x}_2\to\mathbf{x}}\int ds\int\frac{d\mathbf{k}}{(2\pi)^3}\exp(-i\mathbf{k}\mathbf{x})\hat{Q}_{kr}^T(t$$
$$-s,\mathbf{k})\hat{u}_{ir}\{\delta_{fr}\exp(i\mathbf{k}\mathbf{x}_2)\delta(t-s)\}. \tag{18}$$

Let us pick the simplest form for the Fourier transform of the correlation tensor in Eq. (18):

$$\hat{Q}_{kr}^T(t-s,k)=U_T^2\lambda^4\delta(k-K_0)\exp\left(-\frac{|t-s|}{\tau}\right)$$
$$\times\left(\delta_{kr}-\frac{k_k k_r}{k^2}\right), \tag{19}$$

where τ and λ are the correlation time and length.

Calculation of the one-point average $\langle u_p^T\tilde{u}_i\rangle$ based on Eq. (18) is very laborious. Without going outside the framework of the problem as posed, we will calculate the terms that are nonlinear with respect to the average field by assuming that the average large-scale velocity field V_i consists of a horizontally oriented vortex whose horizontal dimensions greatly exceed its vertical dimensions, and we will neglect derivatives with respect to the horizontal coordinates in the calculations.

As a result we obtain a final expression for the Reynolds stresses:

$$S_i=-\frac{\tau}{3\pi^2}v_T^2\Delta V_i-\frac{11\tau^3}{70\pi^2}v_T^2\nabla_z((\nabla_z V_m)^2\nabla_z V_i). \quad (20)$$

The first term in Eq. (20) gives the Reynolds stresses that correspond to those obtained in Ref. 9 in this model. The second term is a nonlinear correction corresponding to a large-scale horizontally oriented vortex. The Reynolds stresses (20) should be substituted into the equation for the large-scale velocity (3). If we are studying the evolution of a vortex with this type of geometry, the nonlinear term $\langle V_k\nabla_k V_i\rangle$ in Eq. (3) will disappear, and the equation for the large-scale vortex (1) will take the form

$$\frac{\partial V_i}{\partial t}-\nu_T\Delta V_i-\beta P_{im}\nabla_z((\nabla_z V_n)^2\nabla_z V_m)$$
$$=\alpha P_{im}\nabla_k(e_m\varepsilon_{kra}+e_k\varepsilon_{mra})e_r V_a,$$
$$\nu_T=\frac{\tau}{3\pi^2}v_T^2,\quad \beta=\frac{11\tau^3}{70\pi^2}v_T^2. \quad (21)$$

Equation (21) describes instability and nonlinear dissipation of large-scale motion in an incompressible fluid due to small-scale turbulent fluctuations. The existence of nonlinear stabilization against a background of instability should lead to formation of stationary vortex structures of finite amplitude.

Note that these results were obtained by assuming that the small-scale turbulence is weak, and that the amplitudes of the large-scale fields are small. This is associated more with the methodology of the calculations than with the physical nature of the problem. Specifically, the ability of helical turbulence to lead to oscillation, which exists for small Reynolds numbers, cannot disappear for Reynolds numbers larger than unity without some definite topological reasons for doing so, since the oscillation effect is connected with the topologically nontrivial nature of helical turbulence and does not depend on the intensity of the latter. However, the numerical coefficients in the corresponding terms should, of course, depend both on the turbulent intensity and on the method of calculation. For exactly the same reason, although it would be unnatural to expect that the stabilizing influence of nonlinear dissipation arising from turbulence could be replaced by a destabilizing influence as the amplitude of the field increases, there is no way to determine the required coefficients without the use of some computational method. Thus, it should be understood that the results we have obtained within the framework of this model have been derived under the assumption of small Reynolds numbers and for large amplitudes will be primarily qualitative in character.

4. VERTICAL DISTRIBUTION OF HYDRODYNAMIC FIELDS

It is convenient to write the large-scale velocity field V_i as a sum of poloidal V_i^φ and toroidal V_i^ψ components:

$$V_i=V_i^\varphi+V_i^\psi,\quad \mathbf{V}^\varphi=\nabla\times\nabla\times\mathbf{e}_z\varphi,$$
$$\mathbf{V}^\psi=\nabla\times\mathbf{e}_z\psi. \quad (22)$$

In this case Eq. (21) for the large-scale velocity field V_i must be written in the form of a system of equations for the potentials $\varphi(t,\mathbf{x})$ and $\psi(t,\mathbf{x})$. For this we let the operators $\mathbf{e}_z\cdot\nabla\times\nabla\times$ and $\mathbf{e}_z\cdot\nabla\times$ act on Eq. (21), and isolate the corresponding poloidal and toroidal parts. Limiting the discussion to axially symmetric problems in cylindrical coordinates (r,θ,z) and neglecting the leading derivatives with respect to horizontal coordinates, we obtain the following system for the potentials $\varphi(t,r,z)$ and $\psi(t,r,z)$:

$$\left(\frac{\partial}{\partial t}-\nu_T\Delta\right)\Delta_\perp\psi-\beta\nabla_z\frac{1}{r}\frac{\partial}{\partial r}\left(r(\nabla_z V_n)^2\nabla_z\frac{\partial\psi}{\partial r}\right)$$
$$=-\alpha\Delta_z\Delta_\perp\varphi,$$
$$\left(\frac{\partial}{\partial t}-\nu_T\Delta\right)\Delta\Delta_\perp\varphi-\beta\nabla_z^2\frac{1}{r}\frac{\partial}{\partial r}\left(r(\nabla_z V_n)^2\nabla_z^2\frac{\partial\varphi}{\partial r}\right)$$
$$=\alpha(\Delta_z-\Delta_\perp)\Delta_\perp\psi. \quad (23)$$

The vertical boundary value problem for the corresponding linear operator with various boundary conditions has been solved many times in investigations of large-scale instability.[4,5,7] In this paper we discuss a simplified approach to the boundary value problem starting with the fact that the vertical dimensions of the vortex structure under study are small compared to its horizontal dimensions an approach that, despite its simplicity, will allow us to investigate qualitatively the basic characteristics of the large-scale vortex:

$$\nu_T\psi=\alpha\varphi,\quad \nu_T^2\nabla_z^2\varphi-\alpha^2\varphi=0. \quad (24)$$

In this formulation the boundary value problem can be solved only for the simplest boundary conditions along the vertical direction:

$$\varphi(t,r,0)=\varphi(t,r,h)=0,\quad \psi(t,r,0)=\psi(t,r,h)=0, \quad (25)$$

resulting in a solution with the very simple form

$$\varphi=\varphi(t,r)\sin\kappa z,\quad \psi=\psi(t,r)\sin\kappa z,$$
$$\kappa^2=(\alpha/\nu_T)^2=(\pi/h)^2. \quad (26)$$

When we take Eq. (22) into account, it is easy to see that this solution corresponds to free-surface boundary conditions on the upper and lower boundaries of the layer for the poloidal component, conditions that are natural for this problem since under real conditions the profile of the horizontal flow usually corresponds to the presence of free boundaries in both regions except for a narrow boundary layer where the velocity reduces abruptly to zero. Thus, the formulation of free boundary conditions for the poloidal component implies a certain idealization that leads to definite simplifications without changing the problem in any essential way.

For the toroidal field the situation is fundamentally different. Its structure, which is uniform with the height of the flow, is such that it can reach considerable intensity when nothing constrains it. In fact, under real tropical-hurricane conditions, the toroidal component of the velocity field considerably exceeds the poloidal component in intensity. Within the framework of our model, this formally indicates that in the limit of a thin horizontal layer the linear operator for the vertical boundary value problem (23) allows us to determine the toroidal field only to accuracy up to an arbitrary constant; thus, a degeneracy in the flow occurs which can only be lifted if we take into account factors that lie outside our simple formulation. In this paper we limit ourselves to the simplest formulation of the boundary value problem (24), for which the toroidal field is fixed by boundary conditions (25).

5. AMPLITUDE EQUATIONS

In the present problem, the quadratic combination of velocities in the nonlinear terms of Eqs. (23) can be written thus:

$$(\nabla_z V_m)^2 \simeq \kappa[\kappa^2(\varphi'(t,r))^2 \sin^2 \kappa z + (\psi'(t,r))^2 \cos^2 \kappa z].$$

Taking this into account, let us isolate the secular terms from system (23) by imposing conditions on its solubility:

$$[(\partial/\partial t - 2\nu_T\Delta_\perp) + \nu_T\kappa^2]\varphi' + \tfrac{1}{4}\beta\kappa^4(3(\varphi')^3\kappa^2 + \varphi'(\psi')^2) = \alpha(1+\Delta_\perp/\kappa^2)\psi',$$
$$[\kappa^{-2}(\partial/\partial t - \nu_T\Delta_\perp) + \nu_T]\psi' + \tfrac{1}{4}\beta\kappa^2(\kappa^2(\varphi')^2\psi' + 3(\psi')^3) = \alpha\varphi'. \quad (27)$$

Here the dashes denote differentiation with respect to the horizontal coordinate r: $\varphi' = \partial\varphi/\partial\tau$, $\psi' = \partial\psi/\partial\tau$.

Condition (27) contains the first (lowest) approximation to the boundary value problem, which is satisfied identically by virtue of condition (26). In order to eliminate it, it is necessary to keep in mind that the derivatives with respect to time t and horizontal coordinate r are slowly varying, and thus, like the nonlinear terms, represent a small correction compared to the first approximation. Eliminating the first approximation from the system (27), we write it in the form of a single equation for the amplitude of the poloidal field $\varphi(t,r)$:

$$(\partial/\partial t - 2\nu_T\Delta_\perp)\varphi + \beta\kappa^6 \int_0^r (\varphi'(t,\rho))^3 d\rho = \nu_T\kappa^2\alpha_1\varphi. \quad (28)$$

In lowest approximation, the amplitude of the toroidal field is found to be

$$\psi = \frac{\alpha}{\nu_T}\varphi. \quad (29)$$

The parameter α_1 is the increase in the absolute value of the helicity coefficient $|\alpha|$ above its neutral value $\pi h^{-1}\nu_T$:

$$|\alpha| = \pi h^{-1}\nu_T(1+\alpha_1),$$

which controls the instability of the linear problem.

It should be noted that Eq. (28), which describes the evolution of the poloidal field amplitude $\varphi(t,r)$, contains information about the absolute value of the helicity coefficient $|\alpha|$, but not about its sign. This implies that the sign of the helicity of the original small-scale turbulence, which in investigations of planetary atmospheres may be regarded as a result of the Coriolis force,[1] has no effect on the behavior of the vertical and radial flows in the large-scale structure. In contrast, Eq. (29) implies that the sign of the helicity coefficient controls the direction of tangential flow in the structure, which corresponds to the fact that the direction of the horizontal rotation of air in a real tropical hurricane depends on the hemisphere in which it is formed, and persists throughout the entire period of its development.

Equation (28) is the nonlinear Schroedinger equation, which describes the behavior of the amplitude of the large-scale structure as a function of external conditions described by the parameter α_1. In Refs. 5–8, the analogous linear Schroedinger equation was investigated, in which the shape of the potential well was specified by a power law; this makes it possible to describe the dependence of the large-scale structure on the horizontal coordinates in terms of Laguerre polynomials. In this paper we investigate a simpler formulation, in which the r-dependence of the parameter $\alpha_1(\tau)$ mimics an axially symmetric well with vertical walls of radius R:

$$\alpha_1(r) = \begin{cases} \alpha_1, & 0<r<R, \\ -\infty, & R<r<\infty. \end{cases}$$

In this case, the solution to this equation is cell-like, with fluxes rising upward in the center portion and a radial velocity component that vanishes at the boundary. It has the form

$$\varphi(t,r) = \varphi_0(t) J_0(k_R r), \quad k_R = \mu/R, \quad (30)$$

where $J_0(k_R r)$ is a Bessel function of zero order, and the parameter μ is the first (nonzero) root of the first-order Bessel function. The equation for the amplitude $\phi_0(t)$ is determined once we separate out the secular terms in the form of conditions for solvability of Eq. (28):

$$\frac{\partial\varphi_0(t)}{\partial t} = \gamma\left[1 - \frac{\varphi_0^2(t)}{\varphi_m^2}\right]\varphi_0(t),$$
$$\gamma = \left(\frac{\pi}{h}\right)^2 \nu_T\left[\alpha_1 - 2\left[\frac{\mu}{\pi}\frac{h}{R}\right]^2\right],$$
$$\varphi_m^2 = \gamma\left[M\beta\left(\frac{\pi}{h}\right)^6\left(\frac{\mu}{R}\right)^2\right]^{-1}. \quad (31)$$

Here the parameters γ and φ_m denote the linear growth rate of the instability and steady-state amplitude of the structure. The coefficient M in the expression for the steady-state amplitude φ_m is determined by the expression

$$M = -2\mu J_0^{-2}(\mu)\int_0^1 xJ_0(\mu x)\int_0^x J_1^3(\mu x')dx'dx.$$

The numerical value of this coefficient is approximately $M \simeq 0.252$.

Equation (31) for the amplitude $\varphi_0(t)$ is an ordinary nonlinear differential equation of first order, and is easily integrated, which determines the dependence of the amplitude $\varphi_0(t)$ on time through the expression

$$\varphi_0(t) = \frac{\varphi_m \exp(\gamma t)}{\sqrt{1+\exp(2\gamma t)}}. \quad (32)$$

Thus, the potentials of the poloidal and toroidal components of the large- scale velocity field take the final forms:

$$\varphi(t,r,z) = \varphi_0(t)\sin(\kappa z)J_0(k_R r),$$

$$\psi(t,r,z)=\frac{\alpha}{\nu_T}\,\varphi_0(t)\sin(\kappa z)J_0(k_R r). \tag{33}$$

Relation (33) allows us to determine the vertical and horizontal components of the large-scale velocity field

$$V_z^\varphi(t,r,z)=-\frac{1}{r}\frac{\partial}{\partial r}\left[r\,\frac{\partial\varphi}{\partial r}\right]=k_R^2\varphi_0(t)\sin(\kappa z)J_0(k_R r),$$

$$V_r^\varphi(t,r,z)=\frac{\partial^2\varphi}{\partial r\partial z}=-\kappa k_R\varphi_0(t)\cos(\kappa z)J_1(k_R r),$$

$$V_\theta^\psi(t,r,z)=\mathrm{curl}_\theta\ \mathbf{e}\psi=-\frac{\partial\psi}{\partial r}$$

$$=\frac{\alpha}{\nu_T}\,k_R\varphi_0(t)\sin(\kappa z)J_1(k_R r). \tag{34}$$

The solution (34) describes the evolution of a large-scale vortex structure in which the flow lines of the poloidal and toroidal components of the velocity field interpenetrate. The structure starts from a nucleation stage and ultimately arrives to a stationary level with flow amplitude $(\pi\mu/hR)\varphi_m$.

Direct substitution shows that the convective nonlinear term $\langle V_k\nabla_k V_i\rangle$ vanishes within the framework of this approach; consequently, this nonlinearity plays no role in the development of the structure and formation of its stationary state. We should emphasize, however, that this assertion relates to the large-scale convective nonlinearity, whereas the oscillation term and linear and nonlinear viscosities in the equation for the large-scale velocity (1) are in essence different forms of the Reynolds stresses and are the result of averaging the same convective nonlinearity in the original small-scale equations while taking into account various parameters.

6. CONCLUSION

In this paper we have treated the problem of evolution and steady state of large-scale vortex structures of the tropical hurricane type, which form as the result of the development of an instability caused by the helical component of small-scale turbulence of the surrounding medium. Since the convective large-scale nonlinearity does not make itself felt in this problem, the system is driven to its steady state by the nonlinearity connected with enhancement of the turbulent dissipation, which in turn is caused by an increase in the amplitude of the hydrodynamic fields.

The system of large-scale velocity fields in this problem consists of linked cells. The dimensions of the vortex structure are a result of the size of the region of small-scale turbulence that contains a helical component with sufficient intensity.

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