Finite-size Anyons at low temperatures

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We consider a simplified model of finite-size anyons, that is, particles coupled to a gauge field the Lagrangian of which contains the Chern–Simons term. We analyze the two-particle problem, show that under certain conditions there is a strong effective attraction, and discuss possible consequences. © *1996 American Institute of Physics.* [S1063-7761(96)00205-3]

It is well known that particles coupled to a Chern–Simons gauge field in (2+1) dimensions undergo an effective change of statistics, i.e., become anyons.[1,2] Just as for bosons and fermions, it makes sense to speak of an "exchange interaction" of anyons, which tends to be attractive if the statistics is close enough to bosonic, and repulsive if it is close enough to fermionic, although the classical interaction force is equal to zero. A more general situation is that in which the gauge field Lagrangian is a sum of the Chern–Simons term and other term(s) of non-topological nature. In this case there is always a characteristic scale length (the size of the gauge field cloud created by a particle), and the behavior of such finite-size anyons depends on whether they are close to each other or far away on this scale. In the two limit cases they behave like ideal anyons, but with different values of the statistical parameter. It therefore makes sense to speak about the effective "distance dependence of statistics" for such particles.[3] This distance-dependent statistics emerges in various models, examples being Maxwell–Chern–Simons electrodynamics[4,5] and the Dorey–Mavromatos model for high-T_c superconductivity.[6] In this situation, as opposed to ideal anyons, the interaction force is always present.

In a previous work[3] the two-body problem for finite-size anyons—on particles with distance-dependent statistics—was considered, special attention being paid to the high-temperature behavior. It was shown that in the semiclassical approximation, a general formula for the second virial coefficient of such particles could be derived, which gives in limiting cases the result of Ref. 2 for ideal anyons. In this paper we study the same system in the low-temperature regime, emphasizing the ground state, and show in a simplified model that under certain conditions there is a strong attraction between particles.

Thus, we consider a conserved current j^{μ} coupled to a gauge field A^{μ}, the total Lagrangian being

$$\mathscr{L}=\mathscr{L}_0+\frac{1}{2}\alpha\epsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda}-A_{\mu}j^{\mu}, \qquad (1)$$

where $\mathscr{L}_0$ depends on A_{μ} and its derivatives. The corresponding field equations read

$$Q^{\mu}+\alpha\epsilon^{\mu\nu\lambda}\partial_{\nu}A_{\lambda}=j^{\mu}, \qquad (2)$$

where $Q^{\mu}=\Omega_{\nu}(\partial\mathscr{L}_0/\partial(\partial_{\nu}A_{\mu}))-\partial\mathscr{L}_0/\partial A_{\mu}$. In accordance with these equations, a static charge in the origin,

$$j^{\mu}=e\,\delta_0^{\mu}\delta^2(\mathbf{r}), \qquad (3)$$

gives rise, in general, to an electric field (due to Q^{μ}) as well as to a magnetic field (due to $\varepsilon^{\mu\nu\lambda}\partial_{\nu}A_{\lambda}$). If $\mathscr{L}_0$ does not depend explicitly upon time and polar angle φ, which is a natural assumption, then the solution of (2) with the right-hand side as in (3) depends only on r. In the Lorentz gauge one therefore has $A_r\equiv 0$. The temporal component A_0 corresponds to the usual charge–charge interaction, screened in the presence of the Chern–Simons term, and in any case irrelevant to statistics. The angular component can be always written

$$A_{\varphi}(\mathbf{r})=-\frac{\Delta(r)}{er}. \qquad (4)$$

The function $\Delta(r)$ possesses a clear physical meaning: $\Phi(r)=-(2\pi/e)\Delta(r)$ is the magnetic flux created by the charge through a circle of radius r. Therefore in any real model the above-mentioned function should be continuous, and its limiting values

$$\Delta_0=\Delta(0), \quad \Delta_{\infty}=\Delta(\infty) \qquad (5)$$

should be finite. If $\Delta(r)=\text{const}$ (this is the case if $\mathscr{L}_0=0$), Eq. (4) describes the conventional Aharonov-Bohm potential, making the particles effectively anyons. In the general case, there is always a characteristic distance scale d, which is essentially the size of the magnetic field cloud created by the charge, so that $\Delta(r)$ may be replaced by $\Delta_0(\Delta_{\infty})$ if $r\ll d(r\gg d)$.

According to common quantum mechanical rules, the Hamiltonian of the relative motion of two particles under consideration is

$$\widetilde{\mathscr{H}}_2=\frac{p_r^2}{m}+\frac{[p_{\varphi}+\Delta(r)]^2}{mr^2}+V(\mathbf{r}), \qquad (6)$$

$V(\mathbf{r})$ being the mechanical interaction potential. Assuming the latter to be central, the relative wave function is to be searched for in the usual form,

$$\widetilde{\Psi}(\mathbf{r})=\frac{\chi(r)}{\sqrt{r}}\exp(i\ell\varphi), \qquad (7)$$

and the levels E_n^{ℓ} (ℓ is the angular momentum, n the radial quantum number) are determined from the radial equation for the partial wave

$$\tilde{\mathscr{H}}_2^{\ell}\chi = E_n^{\ell}\chi, \tag{8}$$

where

$$\tilde{\mathscr{H}}_2^{\ell} = -\frac{1}{m}\frac{d^2}{dr^2} + U^{\ell}(r), \tag{9}$$

the effective potential energy being, as usual, a sum of the centrifugal energy and the interaction potential,

$$U^{\ell}(r) = W^{\ell}(r) + V(r), \tag{10}$$

$$U^{\ell}(r) = \frac{[\ell + \Delta(r)]^2 + 1/4}{mr^2} + V(r), \tag{11}$$

and ℓ being even (odd) for bosons (fermions). Equation (8), of course, cannot be solved exactly for arbitrary $\Delta(r)$ and $V(r)$; however, if they are such that $U^{\ell}(r)$ takes the oscillator form,

$$U^{\ell}(r) = U_0 + \frac{M^2 + 1/4}{mr^2} + \frac{m\Omega^2 r^2}{4}, \tag{12}$$

then the exact solution is available; in particular, the partial wave ground-state energy is

$$E_0^{\ell} = U_0 + (|M| + 1)\Omega\ , \tag{13}$$

and the mean square radius in this state is

$$\langle r^2 \rangle = \frac{2(|M| + 1)}{m\Omega}. \tag{14}$$

With regard for this let us assume the external potential to be harmonic:

$$V(r) = \frac{m\omega^2 r^2}{4}. \tag{15}$$

We will investigate the behavior of the system with ω changing. As a measure of interaction we take the pressure, which we define as

$$p = -\frac{\partial E_0}{\partial A} = -m\omega^2 \frac{\partial E_0}{\partial \omega}, \tag{16}$$

where the "area" is $A = 1/m\omega$.

If within some approximation a constant Δ may be substituted for $\Delta(r)$, then $U^{\ell}(r)$ takes just the form (12) with $U_0 = 0$, $M = \ell + \Delta$, $\Omega = \omega$, so that

$$E_0^{\ell} = (|\ell + \Delta| + 1)\omega; \tag{17}$$

the true ground state is achieved for that ℓ which gives minimal $|\ell + \Delta|$, and the result then corresponds to the well-known formula for anyons:[1]

$$E_0 = (|\delta| + 1)\omega, \tag{18}$$

where δ is the statistical parameter ($|\delta| \leq 1$); the mean square radius is

$$\langle r^2 \rangle = \frac{2(|\delta| + 1)}{m\omega}, \tag{19}$$

and the pressure is

$$p = (|\delta| + 1)m\omega^2. \tag{20}$$

Excluding the trivial case, in which $\Delta(r) \simeq \text{const}$ for all $0 \leq r < \infty$, substitution of constant for Δ is legal in two cases only: when $\sqrt{\langle r^2 \rangle} \ll d$ (in which case $\Delta(r) \simeq \Delta_0$ over the whole region in which, the wave function is concentrated, and when $\sqrt{\langle r^2 \rangle} \gg d(\Delta(r) \simeq \Delta_\infty$ in that region). Thus, at distances much less than the cloud size the particles behave like anyons with statistical parameter Δ_0 mod 2, and at distances much more than that size—like anyons with statistical parameter Δ_∞ mod 2 (Ref. 3) (the "bare" particles ar assumed to be bosons; the changes $\Delta_0 \to \Delta_0 + 1$ and $\Delta_\infty \to \Delta_\infty + 1$ are to be made if they are fermions).

At intermediate distances, a special consideration is required. Since the qualitative picture should not depend crucially on the shape of $\Delta(r)$, we introduce a simplified model (called "extended anyons" in Ref. 7) which allows one to carry on the analytic treatment. In this model

$$\Delta(r) = \begin{cases} \Delta \dfrac{r^2}{d^2}, & 0 \leq r \leq d, \\ \Delta, & r \geq d. \end{cases} \tag{21}$$

The physical meaning is obvious: a magnetic field is associated with each particle, uniformly distributed within a circle of radius d centered on the particle. The results for ideal anyons are to be recovered in the limit $d \to 0$. The function $U^{\ell}(r)$ for $0 \leq r \leq d$ has the form (12) with

$$U_0 = 2\Delta\xi\ell, \quad M = \ell, \quad \Omega = \sqrt{4\Delta^2\xi^2 + \omega^2}, \tag{22}$$

where

$$\xi = \frac{1}{md^2}, \tag{23}$$

and for $r \geq d$, still the same form but with

$$U_0 = 0, \quad M = \ell + \Delta, \quad \Omega = \omega. \tag{24}$$

Consider the behavior of the ground state under the following assumptions: 1) the "bare" particles are bosons; 2) $\Delta \gg 1$; 3) $\delta \equiv \Delta$ mod $2 < 1$ (the latter is exclusively for simplifying the formulas). Let us determine where the minima of $U^{\ell}(r)$ can be located. The derivative of the centrifugal energy is

$$W^{\ell\prime}(r) = \frac{2[\ell + \Delta(r)]}{mr^2}\left\{\Delta'(r) - \frac{\ell + \Delta(r)}{r}\right\} - \frac{1}{2mr^3} \tag{25}$$

$$= \begin{cases} \dfrac{2\Delta^2 r}{md^4} - \dfrac{2\ell^2 + 1/2}{mr^3}, & 0 \leq r < d, \\ -\dfrac{2(\ell + \Delta)^2 + 1/2}{mr^3}, & r > d \end{cases} \tag{26}$$

(at $r = d$ it is of course discontinuous), and the equation for a minimum reads

$$W^{\ell\prime}(r) + \frac{m\omega^2 r}{2} = 0. \tag{27}$$

A priori there are three possibilities: a minimum can exist at $r > d$, at $0 \leq r < d$ or at the cusp point $r = d$. At $r > d$, the second line of (26) being substituted in (27) yields

$$r_0^2=\frac{2\sqrt{(\ell+\Delta)^2+1/4}}{m\omega}. \tag{28}$$

For arbitrary value of ω one can choose ℓ in such a way that $r_0>d$. However, for large ω the needed values of ℓ are also large, and then neither of the relevant states with energies (17) will be the ground state. One of them certainly will be if $\omega\ll\xi$, in which case $r_0\gg d$ even for $|l+\Delta|\sim 1$. In this case the wave function is concentrated in the region $r\gg d$, so the partial wave ground-state energy and the mean square radius are indeed given by Eqs. (17) and (14) with M and Ω as in (24), respectively. If D denotes the entire part of Δ (so that $\Delta=D+\delta$, D being an even number) then the true ground states corresponds to $l=-D$, and Eqs. (18)–(20) take place.

On the other hand, taking the first line in (26), we have for the minimum point

$$r_0^2=\frac{2\sqrt{\ell^2+1/4}}{m\Omega} \tag{29}$$

with Ω as in (22), and assuming the wave function to be concentrated at $r\ll d$, Eq. (13) yields

$$E_0^{\ell}=2\Delta\xi\ell+\{|\ell|+1\}\sqrt{4\Delta^2\xi^2+\omega^2}; \tag{30}$$

this is minimal for $\ell=0$, and in that case the above assumption is true because the mean square radius equals $\sqrt{2/m\Omega}$, and $\Omega\gg 1/md^2$ always, as is seen from (22) and (23). We conclude that a state with $\ell=0$ and

$$E_0=\sqrt{4\Delta^2\xi^2+\omega^2} \tag{31}$$

does *always* exist. The relevant mean square radius is

$$\langle r^2\rangle=\frac{2}{m\sqrt{4\Delta^2\xi^2+\omega^2}}\ll d^2, \tag{32}$$

and the pressure is

$$p=\frac{m\omega^3}{\sqrt{4\Delta^2\xi^2+\omega^2}}; \tag{33}$$

this corresponds essentially to bosons in a harmonic potential with the "effective" frequency $\sqrt{4\Delta^2\xi^2+\omega^2}$.

Let us now assume that $\omega\gg\xi$. As long as ℓ is such that r_0 from (28) is greater than d, the energy of the partial wave ground state decreases with decreasing $|\ell+\Delta|$.But it does not go this way beyond $r_0=d$ since the potential energy has another form for $r<d$.

Consider the situation in more detail. One has

$$U^l(d)=\left[(\ell+\Delta)^2+\frac{1}{4}\right]\xi+\frac{\omega^2}{4\xi}, \tag{34}$$

$$U_-^{l'}\equiv U^{l'}(d-0)=\frac{1}{d}\left\{2\xi\left[\Delta^2-\ell^2-\frac{1}{4}\right]+\frac{\omega^2}{2\xi}\right\}, \tag{35}$$

$$U_+^{l'}\equiv U^{l'}(d+0)=\frac{1}{d}\left\{-2\xi\left[(\ell+\Delta)^2+\frac{1}{4}\right]+\frac{\omega^2}{2\xi}\right\}. \tag{36}$$

Since $\omega\gg\xi$, one may neglect 1/4 everywhere. Let $\ell=-\Delta-k$. The minimum is shifted to the point $r=d$ when $k=\omega/2\xi$. (In fact, k can only take discrete values since ℓ is quantized, but for $\Delta\gg 1$ one may pay no attention to this.) The energy of the relevant state is certainly not less than $U^l(d)$; therefore it can be the ground state only if $\omega\lesssim\sqrt{\Delta}\xi$, otherwise the energy (31) is less than this. The last relation implies $\omega\ll\Delta\xi$, which means $k\ll\Delta$; in such an approximation $U_-^{\ell'}=-2\Delta\omega/d$ (the term with ω^2 is much less). The first derivative on the right vanishes, while the second one equals

$$U_+^{\ell''}=\frac{6k^2}{md^4}+\frac{m\omega^2}{2}=2m\omega^2.$$

This means that in the adopted approximation

$$U^{\ell}(r)=\begin{cases}\infty, & r<d,\\ \dfrac{\omega^2}{2\xi}+m\omega^2(r-d)^2, & r\geq d.\end{cases} \tag{37}$$

The lowest level of such a "one-sided oscillator" obviously coincides with the first excited level of the relevant "normal" oscillator:

$$E_0=\frac{\omega^2}{2\xi}+3\omega \tag{38}$$

(remember that the mass is $m/2$). From this answer, one sees that the first line in Eq. (37) is justified, since the right classical turning point is at $r=d(1+\sqrt{3\xi/\omega})$ while the left one is at $r=d(1-3/2\Delta)$, and $1/\Delta\ll\sqrt{\xi/\omega}$. Now, the mean square radius in this state can be written as

$$\langle r^2\rangle=\langle(d+r-d)^2\rangle\simeq d^2+2d\langle r-d\rangle=d^2+\frac{4d}{\sqrt{\pi m\omega}}, \tag{39}$$

and the pressure equals

$$p=\frac{m\omega^3}{\xi} \tag{40}$$

(here and in what follows one may neglect the second term in (38)). The energies given by (31) and (38) become equal for

$$\omega_{cr}\simeq 2\sqrt{\Delta}\xi. \tag{41}$$

For $\omega<\omega_{cr}$, the energy in (31) is less than in (31), for $\omega>\omega_{cr}$, vice versa. Consequently, the ground state changes its structure as ω passes through ω_{cr}. Let $\omega^{(1)}\sim\Delta^{3/8}\xi$. Then, from (38), $E_0^{(1)}\sim\Delta^{3/4}\xi$ and, from (40), $p^{(1)}\sim m\Delta^{9/8}\xi^2$. Now let $\omega^{(2)}\sim\Delta^{5/8}\xi$; then, from (31) and (33), $E_0^{(2)}\sim\Delta\xi$ and $p^{(2)}\sim m\Delta^{7/8}\xi^2$. Thus, $\omega^{(2)}/\omega^{(1)}\sim\Delta^{1/4}$ but at the same time $p^{(2)}/p^{(1)}\sim\Delta^{-1/4}$. Now, the mean distance between the particles is $\langle r^2\rangle^{(1)}\sim d\gg 1/\sqrt{m\omega}$ and $\langle r^2\rangle^{(2)}\sim d/\sqrt{\Delta}\ll 1/\sqrt{m\omega}$, respectively.

This is a "van der Waals-like" picture (for the two-particle system): in a certain region, decreasing the area ($\sim 1/\omega$) results in a decrease of pressure. This means that the interaction of the particles takes the character of a strong attraction. A possible consequence of this would be a van der Waals-like phase transition in a gas of particles under consideration. To see whether this is indeed the case, one has to consider, at least qualitatively, the many-body problem, a question that we hope to address in a future publication.

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