Quasidiffusion of a passive scalar

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The problems that arise in an "averaged" description of the dynamics of a passive scalar in a medium with convective flow are presented and solved in some simple examples. A regular method that takes into account the geometry of the problem is used to derive two one-dimensional equations with fractional derivatives with respect to the time that yield asymmetric quasidiffusion spreading and symmetric superdiffusion spreading of a cloud of the scalar. [S1063-7761(96)02004-5]

1. INTRODUCTION

In the most varied branches of physics, careful studies have recently been made of the transport of some scalar substance n (which could be material, a temperature, a single-component magnetic field, etc.) in a continuous medium with given convective flow and diffusion. In other words, these studies have investigated the general properties of the equation

$$\frac{\partial n}{\partial t}+\mathbf{v}\nabla n=D\Delta n, \tag{1}$$

where the velocity $\mathbf{v}(\mathbf{r})$ and the diffusion coefficient D=const do not depend on n, this justifying the widely employed term "passive scalar" (for the sake of variety, the term "admixture" is also used below). As a rule, the flow of the medium is assumed to be incompressible [this is already taken into account in (1)]:

$$\nabla\mathbf{v}=0$$

and fairly strong:

$$va\gg D$$

[here a is the characteristic spatial scale of $\mathbf{v}(\mathbf{r})$], i.e., the original diffusion in (1) has a "triggering effect."

In this field of enquiry, the so-called averaged or effective equations probably have the greatest interest. These describe the long-time evolution of n, when the triggering diffusion, despite its smallness, can smooth the sharp gradients generated by the inhomogeneous velocity field $\mathbf{v}$ (see, for example, Ref. 1). Naturally, the form of these equations depends on the topology of the given flow of the medium. Unfortunately, an attempt is usually made to reduce the problem (at least at the level of the macroscopic equations and not at the level of the law of displacement of the individual particles of the admixture) to an ordinary diffusion equation with "renormalized" diffusion coefficient D_{eff} appreciably greater than D.

In this paper, we analyze some special cases of convective transport in a medium when the corresponding "effective" equations differ from diffusion equations both in their appearance and in their properties. We are speaking here of two-dimensional (in the xy plane) "strip" [i.e., $\mathbf{v}=\{v(y),0\}$] flows with intensity (see above) described by the Péclet number

$$P\equiv\Psi_0/D\gg 1,$$

where Ψ_0 is the characteristic value of the flow function: $v(y)=d\Psi/dy$. This class of problems is fairly popular in the literature, first because it is often encountered in different practical situations and second because exact analytic results can be obtained (as the present paper also indicates), including answers to general questions that are important for any function $\mathbf{v}(\mathbf{r})$. The main attention is concentrated on the most rapid dynamics of the scalar n along the x axis, i.e., the corresponding effective equations are one dimensional (along the x axis, there is ordinary diffusion; see below).

It should be mentioned here that the term "effective" is by no means used in the sense of a coarse or qualitative description of the transport of n but in the sense of the greater adequacy of the derived equations for practical requirements as compared with the original (1). The corresponding transition is entirely rigorous and correct.

As a result, for the two cases considered in Secs. 2 and 4 we obtain transport equations that are combined in this paper under the general designation of "quasidiffusion" equations in order to emphasize, on the one hand, how they differ from ordinary diffusion and, on the other, their similarity to diffusion as regards the property of "information loss" in the process of evolution—the tendency with time of any initial profile $n_0(x)$ to approach a universal (for each given equation) self-similar profile. A further common property of the derived equations is the presence in them of fractional derivatives with respect to the time.[2] In the choice and study of these two forms of convective flows, the present paper relies heavily on the results of Refs. 3 and 4, while from the philosophical point of view it is a direct continuation of Ref. 5—an extension of the language of fractional derivatives to different problems of stochastic transport (i.e., the spreading of the initial profile of some physical quantity with "information loss").

At the same time, the physical systems described by the macroscopic equation (1) belong to a class that is different from those considered in Ref. 5. This can be seen even from comparison of the smooth microscopic motion of the individual particles of the admixture in (1) with the abrupt

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instantaneous displacements in the cited study. The macroscopic equations are also correspondingly different. Moreover, Eq. (1) quite often describes systems for which there does not exist at all any real "microscopic level" of the motion. This applies, for example, to electron magnetohydrodynamics[6,7]—the rapid evolution of the magnetic field in a plasma on a fixed ion background in the presence of a strong Hall effect: A magnetic field has no "particles." At the same time, this physical realization is very important, as can be seen just from the circumstance that both the original examples of Refs. 3 and 4 arose precisely in this field. The role of the flow function is here played by the ion concentration $\Psi \propto 1/n$, and the role of the Péclet number is played by the magnetization parameter $\omega_{Be}\tau_e$.

On the other hand, direct use of (1) is rather complicated from the technical point of view, and for not too rigorous semiqualitative arguments one rather often does make use of the notion of microscopic motion.[1,4] However, there is then the danger of losing sight of some fine details of the actual problem. A discussion of one of them, associated with the process of "averaging" in the derivation of effective evolution equations for a passive scalar in the cases when these equations are diffusion equations is the subject of Sec. 3, which relates Secs. 2 and 4.

Throughout this paper, we make wide use of the language of Laplace transformations with respect to the time. This is the most convenient tool for handling fractional derivatives with respect to t in physics problems.[5] However, all the results could, of course, have been obtained without this formalism.

2. QUASIDIFFUSION ALONG A NARROW FLOW

In contrast to the following sections, we first consider here the case with $\langle \mathbf{v}\rangle \neq 0$ (throughout the paper, the angular brackets denote averaging over a plane, which in the majority of cases is equivalent to averaging over y) but, nevertheless, with "forgetting" of the initial state, i.e., still applying to the class of stochastic transport processes.

Suppose that in a medium with triggering diffusion D at $y=0$ there is a flow in the positive x direction having a small width a but high velocity v_0 (for what follows, the precise distribution over the width is unimportant), so that $v_0 \gg D$. Then over times that exceed the time of diffusion of the admixture through the flow,

$$t \gg \frac{a^2}{D},$$

the flow can be represented in the form of a δ function:

$$v=\Psi_0\delta(y), \quad \Psi_0 \equiv v_0 a.$$

After integration of Eq. (1) with this v across the flow, we obtain the following balance of the convective and diffusion drift within the narrow flow:

$$P\left.\frac{\partial n}{\partial x}\right|_{y=0}=\left.\frac{\partial n}{\partial y}\right|_{y=+0}-\left.\frac{\partial n}{\partial y}\right|_{y=-0}.$$

If the distribution of the scalar is symmetric with respect to the streamline $y=0$, which is a rather natural and, more importantly, absolutely uncritical assumption (neither the type of the derived equation nor its properties depends on this assumption), then the entire influence of the flow reduces to the boundary condition

$$\left.\left(\frac{P}{2}\frac{\partial n}{\partial x}-\frac{\partial n}{\partial y}\right)\right|_{y=0}=0, \tag{2}$$

and the dynamics of n for $y>0$ can be described by the ordinary diffusion equation (since here $v\equiv 0$), in which, by virtue of the strong inequality $l_x \gg l_y$ of the instantaneous characteristic scales that follows from the large value of P (see below), we can omit the second derivative with respect to x:

$$\frac{\partial n}{\partial t}=D\frac{\partial^2 n}{\partial y^2}. \tag{3}$$

Equations (2) and (3) were used for the first time as basic equations in Ref. 3 (see also Ref. 7) to describe the rapid penetration of the magnetic field into a magnetized plasma along a highly conducting electrode, the condition (2) in this case corresponding to the absence on this electrode of a longitudinal component of the electric field (see the Introduction).

Laplace transformation with respect to time converts(3) into

$$pn_p=D\frac{d^2 n_p}{dy^2}+n_0(x,y),$$

where n_0 is the initial distribution of the scalar. The solution of this equation with allowance for the boundary condition $n_{p|y\to\infty}=0$ can be written in the form

$$n_p=C\exp\left(-\sqrt{\frac{p}{D}}y\right)+\exp\left(-\sqrt{\frac{p}{D}}y\right)\int_0^y\frac{n_0(x,y')}{2\sqrt{Dp}}\times\exp\left(\sqrt{\frac{p}{D}}y'\right)dy'+\exp\left(\sqrt{\frac{p}{D}}y\right)\int_y^\infty\frac{n_0(x,y')}{2\sqrt{Dp}}\times\exp\left(-\sqrt{\frac{p}{D}}y'\right)dy', \tag{4}$$

where $\sqrt{p}$, as everywhere below, takes positive real values for positive and real p, and C is an arbitrary constant determined by the boundary condition on $y=0$. The required effective equation now arises from substitution in the Laplace-transformed (2) of $dn_p/dy|_{y=0}$ from (4) expressed in terms of $n_1\equiv n(x,0)$:

$$\sqrt{\frac{p}{D}}n_{1p}+\frac{P}{2}\frac{dn_{1p}}{dx}=\frac{1}{D}\int_0^\infty n_0(x,y)\times\exp\left(-\sqrt{\frac{p}{D}}y\right)dy. \tag{5}$$

In the ordinary variables, it has the form

$$\frac{\partial}{\partial t}\int_0^t \frac{n_1(t')}{\sqrt{\pi D(t-t')}}\,dt' + \frac{P}{2}\frac{\partial n_1}{\partial x} = -\int_0^\infty \frac{n_0(x,y)}{\sqrt{\pi D t}}\frac{d}{dy}\exp\left(-\frac{y^2}{4Dt}\right)dy, \qquad (6)$$

where on the left-hand side we have the fractional derivative $\partial^{1/2}/\partial t^{1/2}$ (Refs. 2 and 5).

Equation (6) without the right-hand side arose for the first time in Ref. 3, which we have already cited more than once. However, the nonrigorous method of derivation used there made it impossible to choose formally the sign of the fractional derivative (it is easy to see that in the present derivation the correct sign of $\sqrt{p}$ in (5) is due to the presence in (4) of one decreasing exponential in the free term, i.e., it is due to the boundary condition at infinity with respect to y) and was suitable only for the case $n_0 \equiv 0$. This circumstance meant that it was only possible to consider in Ref. 3 boundary-value problems with respect to x, the solutions of which for the given "half" equation are identical to the solutions of the complete ordinary diffusion equation, admittedly with diffusion coefficient multiplied by $P^2/4$, i.e., in Ref. 3 the authors obtained the important result that the evolution is accelerated with respect to x by a factor $P/2$ compared with the evolution with respect to y (which justified the assumption $l_x \gg l_y$ made above).

The actual existing differences of (6) are manifested in the behavior of the solutions of the initial-value problem, i.e., in them the decisive role is played by its right-hand side, which has been found for the first time in this paper. It is easy to see that for functions $n_0(x,y)$ of "general form" it behaves at short times like $n_0(x,0)/\sqrt{\pi D t}$ and at large times like $1/\sqrt{\pi D t D t})\int_0^\infty n_0(x,y)y\,dy$, which in order of magnitude corresponds to multiplication of the initial asymptotic behavior by t_0/t, where $t_0 \sim y_0^2/D$ is the time of diffusion of the admixture through the initial profile n_0 (strictly speaking, the concepts of long and short times are defined precisely compared with t_0). The possible deviations from these laws in the specified cases can also be readily surveyed. For example, for $n_0\,(x,0)=0$, the right-hand side of (6) behaves in the initial stage as $\partial n_0(x,y)/\partial y|_{y=0}$.

The general solution of the effective equation (6) describing rapid quasidiffusion of the admixture along the narrow flow can be expressed in terms of its Green's function:

$$n_1 = \int_{-\infty}^{+\infty}\int_0^\infty n_0(x',y)G(x-x',y)\,dy\,dx' \qquad (7)$$

with

$$G_p = \frac{2\theta(x)}{PD}\exp\left[-\sqrt{\frac{p}{D}}\left(\frac{2x}{P}+y\right)\right],$$

$$G(x,y,t) = \frac{\theta(x)}{PDt\sqrt{\pi D t}}\left(\frac{2x}{P}+y\right)\exp\left(-\frac{(2x/P+y)^2}{4Dt}\right).$$

where $\theta(x)$ is the Heaviside function. As is always the case for equations of stochastic transport,[5,8] at times appreciably exceeding the time for the initial profile to spreading $x[t \gg x_0^2/(P^2D)]$—the time of "information loss"—the solution of (6) tends to a universal self-similar profile determined by the form of G. Indeed, when with the passage of time the width of the Green's function with respect to x begins to be significantly greater than x_0 it [the function in (7)] can be taken in front of the sign of the integral over x' [it is sometimes helpful[5,8] to retain as well the following terms in the expansion with respect to the parameter $x_0/l_x \sim x_0/(P\sqrt{Dt})$]. A specific feature of the given case is that it is here possible to distinguish two limiting cases of the self-similar regime. The first is when the change of the initial profile with respect to y is still small, i.e., we have the inequalities

$$\frac{x_0^2}{P^2D} \ll t \ll \frac{y_0^2}{D},$$

and the second is when the profile with respect to y has also become universal:

$$t \gg \frac{x_0^2}{P^2D},\quad \frac{y_0^2}{D}.$$

In the first case, the instantaneous values of the scales satisfy $l_x \ll P l_y \equiv P y_0$, and

$$n_1(x,t) \to \int_{-\infty}^{+\infty} n_0(x',0)\,dx'\,\frac{2\theta(\xi)}{P\sqrt{\pi D t}}\exp(-\xi^2),$$

$$\xi = \frac{x}{P\sqrt{Dt}},$$

i.e., the solution tends to the Gaussian profile multiplied by a factor 2 (in reality, its left-hand boundary is smeared to the extent of the initial width x_0). In the second case, $l_x \sim P l_y$, and

$$n_1(x,t) \to \int_{-\infty}^{+\infty}\int_0^\infty n_0(x',y)\,dy\,dx' \times \frac{2\theta(\xi)}{P\sqrt{\pi D t}}\,\xi\exp(-\xi^2).$$

It is possible that all this can be more readily seen from the Laplace-transformed equation (7) with replacement in the above inequalities of t by $1/p$.

Thus, although the self-similarity of Eq. (6) is identical with ordinary diffusion self-similarity (it is for this reason that the term "quasidiffusion" is most appropriately applied to it), the behavior of its solutions differs strongly from the classical equation: The spreading with respect to x occurs very asymmetrically—the reason being, of course, the fact that $\langle \mathbf{v} \rangle \neq 0$ in the original problem—and at large times there is also nonconservation of the integral $\int_{-\infty}^{+\infty} n_1(x,t)dx$, which decreases in accordance with a $1/\sqrt{t}$ law, this being naturally due to the broadening of the layer occupied by the admixture with respect to y. In other words, in this problem not only the appreciable acceleration of the evolution of n but also other effects are very important.

We can now somewhat generalize the problem by considering the influence in (5) and (6) of the opposite sign of $\sqrt{p}$ (the fractional derivative). This will make it possible, on the one hand, to analyze more deeply the details of the method of investigation proposed in the present section and, on the other, to relate it in the following section to some problems of the traditional averaged-diffusion description of the evolution of a passive scalar.[1] As we already noted above, the absence in (4) of a growing exponential is due to the boundary condition at infinity. If, however, at a sufficiently large distance $b \gg a \to 0$ from this flow there is another such flow (second electrode in a plasma with strong Hall effect) but with oppositely directed velocity (which, of course, already corresponds to the case with $\langle \mathbf{v} \rangle = 0$), then its effect must add to (5) a term with the opposite sign of $\sqrt{p}$. It is initially small, but for $t \gg b^2/D$ already has a very important effect on the structure of the effective equation (6) and the properties of its solution (7).

Let these two flows be situated on the lines $y=-b/2$ (the velocity along x is positive) and $y=+b/2$ (the velocity is negative). Then the evolution of the passive scalar in the strip between the flows will, as before, be described by (3) with the boundary conditions (2), specified now for $y=\pm b/2$. The arguments used in the derivation of (5) now give

$$\sqrt{\frac{p}{D}}\, n_{1p} \frac{\coth \eta + \tanh \eta}{2} - \sqrt{\frac{p}{D}}\, n_{2p} \frac{\coth \eta - \tanh \eta}{2} + \frac{P}{2} \frac{dn_{1p}}{dx} = n_0 \frac{\tanh \eta}{\sqrt{Dp}},$$

$$-\sqrt{\frac{p}{D}}\, n_{2p} \frac{\coth \eta + \tanh \eta}{2} + \sqrt{\frac{p}{D}}\, n_{1p} \frac{\coth \eta - \tanh \eta}{2} + \frac{P}{2} \frac{dn_{2p}}{dx} = -n_0 \frac{\tanh \eta}{\sqrt{Dp}},$$

where $\eta = \sqrt{p/D}b/2$, $n_{1,2} = n(x, \mp b/2, t)$. Here, to avoid excessive details, the initial profile $n_0(x)$ is assumed to be independent of y. For $t \ll b^2/D (\eta \gg 1)$, this system decomposes into exponentially weakly coupled quasidiffusion equations for the transport of the admixture along the flows in the opposite directions, going over in the opposite limit to the joint ordinary diffusion equation. Indeed, we can derive from it the following equation, which contains only n_1:

$$pn_{1p} - \frac{P^2 D}{4} \frac{d^2 n_{1p}}{dx^2} = n_0 - \frac{P}{2} \sqrt{\frac{D}{p}} \tanh \eta \frac{dn_0}{dx}, \qquad (8)$$

which in the limit $\tanh \eta \to 1$ corresponds to (5) [strictly, it is also (5) after application of the operator $(P/2)d/dx - \sqrt{p/D}$] while for $\eta \ll 1$, when the influence of the second term on the right-hand side on the solution becomes small, it is identical to the diffusion equation. In the intermediate region, (8) describes an interesting transition of the self-similar "half" Gaussian profile into a complete one (since by virtue of the chosen form of n_0 only the first possibility analyzed above can be realized).

The necessity for such a transformation of the effective equation follows from the simple circumstance that for $t \gg b^2/D$ the boundary condition (2) is transferred to the complete strip between the flows, and the level lines of the function n (for given t) become straight lines inclined at the small angle $2/P$ to the x axis (more precisely, at a small angle with this tangent). The evolution in this region is by pure diffusion, but it is, of course, at right angles to the level lines (along ∇n), and therefore the effective dynamics that arises from the double "projection" of the original equation onto the x axis remains a diffusion dynamics while still appreciably—to the extent that the angle is small—increasing its rate. The most important thing for what follows is that the value of the effective diffusion coefficient $D_{\mathrm{eff}} = (P^2/4)D$ does not depend on the distance between the flows b (although the time of establishment of this diffusion regime does depend on it).

3. CORRELATED MEAN FREE PATH IN THE AVERAGED DIFFUSION DESCRIPTION

It is easy to see that at the end of the previous section we have actually solved the problem of establishing the effective diffusion regime for a medium with periodic arrangement of strip convective flows in alternating directions—the assumed symmetry conditions of the function $n(y)$ with respect to each flow corresponds precisely to a periodically repeating picture. The regime established in this problem has been known for quite a long time, at least since the work of Zel'dovich.[9]. A simple generalization to the nonlinear regime $v \propto n$, i.e., to the case of a not completely passive scalar, which is extremely characteristic of electron magnetohydrodynamics (in which, as we have already noted, the component of the magnetic field perpendicular to the xy plane plays the role of the scalar), was implemented in Refs. 6 and 10 independently of Ref. 9.

It is a paradox that the exact solution to the problem [we see here the convenience of the case $\Psi(y)$; see the Introduction] is in apparent contradiction with the nonrigorous method based on investigation of the displacement of the individual particles of the admixture used to study the topologically more complicated convective flows with arbitrary functions $\Psi(x,y)$. The present section is devoted to establishing the reasons for this contradiction and to formulating the problems that arise in the semiqualitative "scaling" approach to the effective equations that was presented in the review of Ref. 1 (see also the original study of Ref. 11).

Thus, a very simple rigorous method of calculating the effective diffusion coefficient for flows with arbitrary periodic Ψ (i.e., with any a and b and any distribution laws of v within a, see Refs. 1, 6, 7, 9, and 10) yields the answer $\langle \Psi^2 \rangle / D$ (more precisely the nontrivial part of the averaged Ψ^2, i.e., $\langle (\Psi - \langle \Psi \rangle)^2 \rangle$). In the given case, the mean value of Ψ with respect to y is equal to the mean value over the period, and it is readily seen that for $a \ll b$, when in the greater part of the plane (outside the flows) Ψ takes the values $\pm v_0 a/2$, this answer agrees with the $P^2 D/4$ given above. The method of the previous section also makes it possible to investigate the transition in time to the diffusion regime (for its generalization to the case of arbitrary a and b, see the previous section).

It must be said that such a picture of the flow—the transport of particles of the admixture to large distances in narrow flows that are separated by a large distance compared with their width—is also characteristic of many non-one-dimensional functions $\Psi(x,y)$ (Ref. 1), including the "single-scale" function of Ref. 11, which has been studied in very great detail. Of course, here we do have $\mathbf{v}\neq 0$ between the flows, but this evidently is not important, since in the nonrigorous method of Ref. 1 presented below the motion of the particles in this region is not considered. Important differences appear only in the fact that in topologically complicated cases these flows are very sinuous and branched, forming so-called fractal clusters, and their "effective" width decreases with length. Both these features of the behavior are characterized by certain fractal exponents, which occur in the result.[1,11] Naturally, in the general case no rigorous analytic methods are known for deriving D_{eff} from the macroscopic equation (1), and use is made of the following nonrigorous argument based, as already said above, on the details of the microscopic motion of the individual particles of the admixture.

A contribution to the effective diffusion is made by particles carried in narrow flows for the time τ of their diffusion migration from the flow in the perpendicular direction (in the strip model analyzed here, this is a^2/D) over a distance $\lambda = v\tau$ (here $v_0 a^2/D$), which plays the role of an "effective mean free path." It is possible to calculate D_{eff} in accordance with the classical formula with allowance for the circumstance that this transport actually takes place in a small fraction α of the area of the two-dimensional medium occupied by the corresponding flows (here a/b), i.e., the diffusion coefficient "averaged" over the area is (see Refs. 1 and 11)

$$D_{\text{eff}} \sim v^2 \tau \alpha \equiv \frac{\lambda^2}{\tau}\,\alpha \sim \frac{(v_0 a)^2}{D}\,\frac{a}{b}$$

(in this method, we are not concerned with the numerical coefficient), which in the simple example at hand is in flagrant contradiction with the rigorous solution, which, as emphasized earlier, does not depend on b.

The reason for such a gross discrepancy is very simple. The effective mean free path in the diffusion coefficient is the correlation length over which the particle "forgets" its original motion (one often speaks of "loss of phase"). However, in the given example, even if a particle has escaped from a narrow flow it has a very high chance of returning to it and being carried again in the same direction. Such a displacement will be correlated with the previous displacement, i.e., even if the particle of the admixture repeatedly and for a long time escapes from the flow, making a random walk between the flows, on each of its returns the mean free paths, and not their squares, are added. This means that during prolonged nonparticipation in the transport the particles "remember" their phase. True "loss of phase" occurs only when they get into the other flow at a large distance from the first one.

The correct form of the "microscopic" arguments is as follows. The time required by the particles to diffuse over the distance b is $\tau_{\text{cor}} \sim b^2/D$, and during this entire time they have a chance of returning to the original flow in a correlated manner. Since the Brownian (diffusion) trajectory fills the plane uniformly, the total time τ' the particles are in the original flow is τ_{cor} reduced in proportion to the fraction of the area occupied by the flow: $\tau' = \tau_{\text{cor}}\, a/b$. Hence, the effective diffusion coefficient is equal to

$$D_{\text{eff}} \sim v^2 \tau' \alpha \sim \frac{(v_0 a)^2}{D},$$

i.e., to the correct value.

The effect of correlated drift analyzed here in a simple example is unavoidably present in many more complicated flows. Unfortunately, this subtlety remained unnoticed and undiscussed both in the original study of Ref. 11 and in the review of Ref. 1. From this one must not conclude that the nonrigorous method is incorrect (this is why we use the word "subtlety"). In topologically complicated flows, we are concerned with the correct choice among fairly numerous and diverse fractal exponents of the one that takes into account this effect. As we have already noted, the effective width of the flows in the two-dimensional problem, in contrast to the considered quasi-one-dimensional problem, is variable, and in the corresponding law of decrease one must take into account the parts of the fractal cluster of the flow (see Ref. 1) into which the particle can pass after the original exit from the flow. It is evident that in Ref. 11 the correct choice was made though without explicit recognition of the problem; nevertheless, this question requires additional investigation.

In any case in which nonrigorous calculations of D_{eff} are made for different flows testing for the effect of correlated drift makes it possible to reject in a simple and reliable manner unacceptable methods of averaging over the plane. For example, if in the problem with a "single-scale" $\Psi(x,y)$ we were to use, to represent the fraction of the area occupied by the flows, the so-called "internal dimension" exponent of the cluster (d_c on p. 993 in Ref. 1), this would certainly be incorrect, for this quantity characterizes the presence in the cluster of "internal voids," from which the particle, having entered them, necessarily returns to the same flow—it simply has nowhere else to go.

4. SUPERDIFFUSION IN THE LIMIT $\langle\Psi^2\rangle \to \infty$

As can be seen from the previous exposition, at the present time it is regarded as obvious that for any $\Psi(x,y)$ with finite $\langle\Psi^2\rangle$ and $\langle\mathbf{v}\rangle = 0$ the two-dimensional problem (1) reduces to an effective diffusion, albeit, possibly, with a different functional (power-law) dependence of D_{eff} on $\langle\Psi^2\rangle$. This can even be proved rigorously[12] (admittedly, it is very rare that one can calculate the value of the diffusion coefficient as rigorously—its numerical value is known only for three cases, including the one analyzed in the previous section, which is also interesting in that it gives the maximum possible value of D_{eff}, see Refs. 1 and 12). At the same time, finiteness of the mean square of the flow function is by no means an essential attribute of the real situation. It is again obvious that if this condition is violated, the process of spreading of the cloud of a passive scalar must be more rapid than in a diffusion process, i.e., we must be considering superdiffusion.[1,5]

Nevertheless, at the present time there are no examples of rigorous derivation of macroscopic equations corresponding to such a regime. On the other hand, as early as 1972 a simple example was known of superdiffusion in a flow with $\Psi(y)$ in which the estimates at the microscopic level of the motion of the individual particles of the admixture are trivial[4] (typically, this example also arose from investigation of the evolution of the magnetic field in the framework of electron magnetohydrodynamics). We are referring here to "strip" flow, which is a set of contiguous flows of equal width a and constant velocity v_0 whose sign is random, i.e., in each individual strip, the drift occurs with probability 1/2 in the positive or negative direction of the x axis independently of the sign of v_0 in the neighboring strips.

Such a choice of $\Psi(y)$, which ensures the absence of a regular averaged drift $\langle \mathbf{v} \rangle$, nevertheless differs very strongly from the example considered above with alternating sign of v_0 (and $b \equiv 0$), creating a stronger impression of "turbulence." Indeed, in the presence of diffusion motion at right angles to the system of flows, a particle of the admixture crosses $N \sim \sqrt{Dt}/a$ of them during the time t, as in the previous example, but because the direction of the velocity in each of the N flows is determined independently, the difference between the number of positive and negative signs of v_0 in this sequence is $\Delta N \sim \sqrt{N}$ (which is equivalent to divergence of the flow function at large scales $\Psi \propto \sqrt{y}$), whereas for the periodic case it is less than 1. The corresponding unbalanced (more precisely, insufficiently balanced) drift leads, as is readily seen, to the following law of displacement with respect to x (Refs. 1 and 4):

$$l_x \sim v_0 t \frac{\Delta N}{N} \propto t^{3/4}.$$

The existence of such a clear and transparent microscopic picture offers the possibility of using the analytic advantages of the "strip" geometry of $\Psi(y)$ mentioned in the Introduction and already justified in the previous sections and also deriving a macroscopic transport equation, i.e., to answer in a special case a question that exists in general form. This can indeed be done; moreover, it appears that the most complicated obstacle in the way of solving the problem is the choice of the correct language for describing the "randomness" of the function Ψ (or v) in the macroscopic equation (1).

It is first of all necessary, as is generally accepted in this field (Refs. 1, 6, 9, and 10), to separate in (1) the density of the passive scalar into smooth and strongly oscillating (with respect to y) components:

$$n = n(x) + \tilde{n}(x,y)$$

[it follows from the final equations that $\tilde{n}/n \sim (a^2/Dt)^{1/4}$], the evolution of which is described by the equations

$$\frac{\partial \tilde{n}}{\partial t} - D\frac{\partial^2 \tilde{n}}{\partial y^2} = -\frac{d\Psi}{dy}\frac{\partial n}{\partial x}, \tag{9}$$

$$\frac{\partial n}{\partial t} + \left\langle \frac{d\Psi}{dy}\frac{\partial \tilde{n}}{\partial x} \right\rangle = 0. \tag{10}$$

We have here omitted the second derivatives with respect to x—in (9) compared with the analogous operator with respect to y and in (10) compared with the retained superdiffusion operator—but, in contrast to the case with periodic Ψ (see the cited literature), we have not omitted the time derivative in (9). Further, it is convenient to make a Laplace transformation with respect to the time:

$$p\tilde{n}_p - D\frac{\partial^2 \tilde{n}}{\partial y^2} = -\frac{d\Psi}{dy}\frac{dn_p}{dx}, \tag{11}$$

$$pn_p + \left\langle \frac{d\Psi}{dy}\frac{\partial \tilde{n}_p}{\partial x} \right\rangle = n_0 \tag{12}$$

[it is assumed here that $\tilde{n}|_{t=0} \equiv 0$, since asymptotically at $t \gg a^2/D$ the contribution of the initial condition to the solution (11) is nevertheless small] and operate with equations precisely in this representation. By averaging over the plane in (12) we must obviously mean

$$\lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{+L} \frac{d\Psi}{dy}\frac{\partial \tilde{n}_p}{\partial x}\, dy. \tag{13}$$

The subsequent sequence of operations is very simple: The solution of (11), expressed in terms of the Green's function of this diffusion equation in the p representation,

$$\tilde{n}_p = -\int_{-\infty}^{+\infty} \frac{\exp(-\sqrt{p/D}|y-y'|)}{2\sqrt{Dp}}\frac{d\Psi(y')}{dy'}\, dy'\, \frac{dn_p}{dx},$$

must be substituted in (13), and then the indicated limit in (12). It is readily seen that if Ψ is finite, then after some integrations by parts the corresponding term on the left-hand sides of (12) and (10) tends in the limit $p \to 0$ $(t \to \infty)$ in accordance with a readily established law to the known form $(\langle \Psi^2 \rangle / D)\partial^2 n_p/\partial x^2$ (see the previous section). In the case of a "random" Ψ, the product $(d\Psi(y')/dy')d\Psi(y)/dy \equiv v(y')v(y)$ occurring in the double integral (over y' and y) must be understood as the correlation function of the flow velocity (and this is the correct representation for the macroscopic problem):

$$v(y')v(y) = v_0^2 f(|y-y'|),$$

which possesses the property that $\int_{-\infty}^{+\infty} f(z)dz$ converges rapidly at distances of order a (this is a certain generalization of the example of Dreĭzin and Dykhne). Thus

$$\langle \ldots \rangle = \lim_{L\to\infty} \left[-\frac{1}{2L}\int_{-L}^{+L}\int_{-\infty}^{+\infty} \frac{\exp(-\sqrt{p/D}|y-y'|)}{2\sqrt{Dp}} \times v_0^2 f(|y-y'|)dy'dy \frac{d^2 n_p}{dx^2}\right]. \tag{14}$$

For $t \gg a^2/D$, we deduce from this (by going over from integration over y' and y to integration over $y-y'$ and $y+y'$)

$$\langle \ldots \rangle = \frac{v_0^2 a}{\sqrt{2Dp}}\frac{d^2 n_p}{dx^2},$$

and after multiplication of (12) by $\sqrt{p}$ and the inverse Laplace transformation this carries (10) into

$$\frac{\partial^2}{\partial t^2}\int_0^t \frac{n(t')}{\sqrt{\pi(t-t')}}-\frac{v_0^2 a}{\sqrt{2D}}\frac{\partial^2 n}{\partial x^2}=-\frac{n_0}{2\sqrt{\pi}t^{3/2}}, \qquad (15)$$

i.e., into a typical equation with fractional derivative $\partial^{3/2}/\partial t^{3/2}$ (Refs. 2 and 5). [In the nonlinear case of electron magnetohydrodynamics, the ordinary diffusion operator on the left-hand side of (15) goes over into a nonlinear operator: $(\partial/\partial x)n^2\partial n/\partial x)$].

It can be seen that the representation employed for Ψ (correlation function) differs from the representation in the microscopic problem of Ref. 4 (in practice, from the spectrum with respect to **k**: $\Psi_{\mathbf{k}}\propto 1/\sqrt{k}$, see Ref. 1), although, of course, it possesses similar properties. Very probably it was too strict adherence to the microscopic language that prevented the authors of Ref. 13 from solving this problem. They succeeded in deducing an effective equation, not for the process of spreading of the given cloud of admixture in a form averaged over the plane, but only for the characteristic spreading of different clouds averaged over different realizations of the flows (or experiments). Despite the apparent similarity of the problems, they are in reality very far from each other (see Ref. 14; the question is analyzed with particular care in Ref. 15). The second is usually "simpler," but, since it is not written in the usual physical space, it possesses quite different properties: In it, as a rule, we do not find the symmetry properties with respect to **r** and t that are in the original physical (1), and in Ref. 13 this is precisely the case.

The solution of (15) for any initial $n_0(x)$ can be expressed in terms of the self-similar Green's function

$$n(x,t)=\int_{-\infty}^{+\infty} n_0(x')G(x-x',t)dx',$$

$$G(x,t)=\frac{2}{3\sqrt{A}t^{3/4}}\frac{1}{2\pi i}\int_C \exp(\zeta^{4/3}-\zeta|\xi|)d\zeta,$$

$$A=\frac{v_0^2 a}{\sqrt{2D}},\quad \xi=\frac{x}{\sqrt{A}t^{3/4}},$$

where the integration contour C in the complex plane of ζ consists of two rays with polar angles $\varphi=\pm 3\pi/4$. Asymptotically for $|\xi|\gg 1$

$$G\simeq\frac{1}{\sqrt{A}t^{3/4}}\frac{3}{4}\sqrt{2\pi}|\xi|\exp\left(-\frac{27}{256}\xi^4\right)$$

(cf. Ref. 5). As in any stochastic transport, (15) describes the tendency of an initial profile of n_0 to tend to the universal finite-parameter (one can change the amplitude and position of the maximum) function G (Ref. 5).

To conclude the section, we must consider two important circumstances. First, the present method can be readily generalized to other classes of random functions $v(y)$ different from the example of Dreĭzin and Dykhne (an attempt in this direction was also already attempted in Ref. 13): If the correlation function of the velocity f has a power-law "tail," ensuring divergence of $\int_{-\infty}^{+\infty}f(z)dz$ at large $|z|$, then instead of (15) there arises a superdiffusion equation with a different (of higher degree) fractional derivative with respect to t. Second, here it is indeed (see the Introduction) necessary to work with a completely different type of superdiffusion equations compared with the one introduced in Ref. 5, in which exclusively fractional derivatives with respect to x (or, rather, fractional powers of the Laplacian Δ) were obtained. It appears that also for general problems of the transport of a passive scalar the new type of equation is more characteristic,[1,13,16] although it should by no means be concluded from this that fundamentally macroscopic physical systems (such as electron magnetohydrodynamics) cannot be described by other equations. For example, the rather standard skin problem of the diffusion of the magnetic field in a thin film is characterized by an equation with $\Delta^{1/2}$ (Ref. 17). Thus, equations with an ordinary diffusion operator (the Laplacian) and a fractional derivative with respect to the time can describe physical processes with both sub- and superdiffusion, whereas the alternative form (fractional power of Δ and ordinary $\partial/\partial t$) is suitable only for superdiffusion. The obstacle here is purely mathematical—a superdiffusion equation of such type cannot ensure that its Green's function is positive definite.[5]

5. CONCLUSIONS

Thus, considering the example of a topologically simple class of flows $\Psi(y)$, we have analyzed in this paper aspects of the effective stochastic transport of an admixture that are also present in the general problem. We have presented a regular derivation of the corresponding macroscopic equations, which we can also expect to encounter in other forms of convection (of course, with different fractional exponents). At the least, the simple topology is often encountered in practice.

Equations of this type are characterized by the following properties.

1. Despite the fractional nature of the time derivative (which, in general, can have any degree), to solve the initial-value problem for these equations it is necessary and sufficient to know only $n(x,y)|_{t=0}$, as in the original equation (1). We may mention in passing that for some reason the problem of solving the initial-value problem has not been investigated at all in the mathematical literature on fractional derivatives—see the monograph of Ref. 2.

2. They possess an "ineradicable" difference from equations with derivatives of integer degree. Investigators are very often tempted to eliminate the unusual nonlocal operators by double (or multiple) application of fractional derivatives to the derived equations (see, for example, Ref. 3). But for equations of the class in question this device does not work (or rather, it does not give the desired result)—the presence of n_0 on the right-hand side [see (6) and (15)] is an obstacle. For example, it is only for $n_0\equiv 0$ that Eq. (6) can be transformed into the classical diffusion equation (cf. Ref. 3 and the remark in this connection in Sec. 2); in other cases, the "correction" of the left-hand side is done by transforming the right-hand side [cf. (6) and (8) in the limit $\eta\to\infty$], so that the solutions of the initial-value problem for equations with fractional derivative with respect to t will never be the same as the solutions of ordinary equations with the familiar [vanishing in the ordinary (x,t) space] right-hand side.

These two features can be combined by the remark that the right-hand sides of the equations of the class determine their type, and also the properties of the solutions, to no less an extent than the "standard" left-hand sides. Once again, we must regret the absence of a discussion of this problem in Ref. 2.

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