Statistical description of plasma parameters in a high-pressure discharge in a stochastic microwave field

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An exact solution in quadratures is obtained for the distribution function of a plasma variable in a random microwave field whose evolution is described by a nonlinear homogeneous first-order differential equation with random force. Asymptotic estimates of the distribution function are obtained by the method of steepest descent. [S1063-7761(96)01704-0]

1. INTRODUCTION

It follows from the experiment of Ref. 1 that for long pulse durations a freely localized high-pressure microwave discharge takes the form of a tangle of intertwined bright luminous plasma filaments (highly ionized phase of the discharge). Estimates show that the plasma filaments can strongly scatter the heating microwaves, causing the field to have an inhomogeneous space–time structure in the discharge region. In such systems (for $D_{pl} \geqslant \lambda$, where D_{pl} is the characteristic diameter of the discharge region, and λ is the wavelength of the microwave field), the space–time dynamics of the plasma parameters in the highly ionized phase of the discharge can become stochastic due to the random nature of the scattered field.

In Ref. 2, the condition of equality of the amplitudes of the scattered and unperturbed waves in a model of spatially uncorrelated spherically symmetric inhomogeneities of small radius R ($R/\lambda \ll 1$), which act as "nucleating centers" of the highly ionized phase, was used to obtain a qualitative criterion for the transition of a microwave discharge into a stochastic combustion regime in the form

$$\frac{1}{8}|gq|\Gamma\left(\frac{2\pi D_{pl}}{\lambda}\right)^2 \geqslant 1, \quad g=\frac{4\pi}{\omega}\sigma, \quad q=\frac{3}{2+\varepsilon}, \qquad (1)$$

where Γ is the volume fraction of the discharge region occupied by the highly ionized phase, ω is the frequency of the microwave field, and σ and ε are the high-frequency electrical conductivity and the permittivity of the plasma in this phase.

2. STOCHASTIC MODEL OF MICROWAVE DISCHARGE

In Ref. 3, high-pressure microwave discharges with disordered structure were described, assuming complete randomization of the heating field in the discharge region. Very simple stochastic models of high-pressure microwave discharges making it possible to determine the probability characteristics of such discharges were proposed. To construct the statistics of the plasma formations in the microwave field in the approximation in which the plasma variables evolve locally, one introduces the distribution function $P_t(\mathbf{X})$ of the plasma variables, which determines the probability dW of finding a plasma variable $\mathbf{X}$ in the interval $d\mathbf{X}$: $dW=P_t(\mathbf{X})d\mathbf{X}$, $P_t(\mathbf{X})=\langle\delta(\mathbf{X}-\mathbf{X}(t))\rangle$ [where the symbol $\langle ... \rangle$ denotes averaging over all realizations of the random process $\mathbf{X}(t)$ or over the ensemble of inhomogeneities $\mathbf{X}_k(t)$, and k is the index of the inhomogeneity]. The hypothesis is also made that the statistical properties of the plasma and microwave field are homogeneous in the discharge region. In the simplest case, when the local state of the system is characterized by a single parameter X (we assume that $X \geqslant 0$), the investigation of the evolution of the original spatially inhomogeneous problem reduces to the construction of the distribution function for a nonlinear dynamical system under the influence of a random force $f(t)$, the role of which is played by some function of the amplitude of the microwave field:

$$\frac{dX}{dt}=F(X)+\sigma(X)f(t), \quad f=f(|\mathbf{E}|^2), \quad f_0=\langle f\rangle,$$

$$\delta f=f-f_0. \qquad (2)$$

Here $F(X)$ and $\sigma(X)$ are certain known functions. As parameter X of the state of the system one can use, depending on the particular problem, the electron temperature, the plasma concentration, the temperature or density of the neutrals, etc. The form of Eq. (2) is particularized for various special cases in Refs. 2 and 3.

The equation that describes the evolution of the distribution function for the dynamical system (2) can be obtained from the theory of stochastic equations,[4] for which, however, it is necessary to known the statistical characteristics of the random force. The simplest model of the random force $f(t)$ under the condition that the characteristic time of evolution of X is appreciably greater than the time of variation of the field amplitude is a δ-correlated random Poisson process, i.e., a process constructed on a Poisson flow of points[4] with non-Gaussian distribution function $F(t)$ of the fluctuations. Then the differential equation for $P_t(X)$ has the form[3]

$$\frac{\partial P_t(X)}{\partial t}=-\frac{\partial}{\partial X}[F(X)+\sigma(X)f_0]P_t(X)$$
$$+\tau_c^{-1}\theta_t'\left[f_0\tau_c\frac{\partial}{\partial X}\sigma(X)\right]P_t(X), \qquad (3)$$

where τ_c is the characteristic correlation time of the fluctuations of the field amplitude. The operator expression $\theta_t'[v]$ must be understood in the sense of a cumulant expansion:

$$\theta_t'[v]=\sum_{n=0}^{\infty}\frac{i^n}{n!}B_n(t)v^n,\quad B_n=\frac{1}{i^n}\frac{d\theta_t'}{dv^n}\bigg|_{v=0}.$$

For a completely randomized microwave field, i.e., when the condition (1) is satisfied, and also when $f(t)=|\mathbf{E}|^2$, the operator expression takes the form[3]

$$\theta_t'[v]=\exp(v)\left(1+\frac{2}{3}v\right)^{-3/2}-1. \tag{4}$$

We introduce the dimensionless time $\tau=t/\tau_c$, the new dimensionless independent variable

$$Z=(\tau_c f_0)^{-1}\int_0^X\frac{dX'}{\sigma(X')},$$

the dimensionless distribution functions $Q(Z)=\tau_c f_0 P_t(Z)\sigma(Z)$, and the source term $\bar{F}(Z)/\sigma(Z)f_0$ (for $\bar{F}$, we shall use below the previous notation F). Then (3) reduces to the equation

$$\frac{\partial Q(Z)}{\partial\tau}=-\frac{\partial}{\partial Z}[Q(Z)+F(Z)Q(Z)]+\theta_t'\left[\frac{\partial}{\partial Z}\right]Q(Z), \tag{5}$$

which in the general case must be solved with the initial condition

$$Q(Z,0)=Q_0(Z) \tag{6}$$

and boundary conditions at $Z=0$

$$\frac{\partial Q^n(Z,\tau)}{\partial Z^n}=0,\quad n=0,\ldots,\infty. \tag{7}$$

(if $\theta_t'[v]$ is a polynomial of degree N, then $n=0,\ldots,N-1$). The choice of the boundary conditions at $Z=0$ in the form (7) for a positive-definite parameter Z is associated with the requirement of satisfying at any instant of time the normalization condition

$$\int_0^\infty P_t(X)dX=\int_0^\infty Q(Z)dZ=1.$$

Allowing for the fact that in the limit $Z\to\infty$ we also have fulfillment of (7), the choice of the boundary conditions in the form (7) automatically guarantees in the case

$$\int_0^\infty Q_0(Z)dZ=1$$

that the normalization condition is satisfied.

3. GREEN'S FUNCTION IN THE CASE OF A DETERMINISTIC SYSTEM (2) ($f=0$)

To construct a solution of Eq. (5) with the conditions (6) and (7), we find the Green's function $G(Z,Z_0,\tau,\tau_0)$ of the differential Liouville equation

$$\hat{L}G(Z,Z_0,\tau,\tau_0)=\delta(Z-Z_0)\delta(\tau-\tau_0),\quad \hat{L}=\frac{\partial}{\partial\tau}+\frac{\partial}{\partial Z}F(Z),\quad Z_0>0, \tag{8}$$

which gives the distribution function in the case of a completely deterministic dynamical system ($f=0$),

$$\frac{d\xi}{d\tau}=F(\xi),\quad \xi(\tau_0)=Z_0. \tag{9}$$

The solution of Eq. (8) that satisfies the boundary condition (7) can be readily found by making a Laplace transformation with respect to τ and determining a particular solution of the obtained ordinary differential equation by the method of variation of constants. After inverting the Laplace transformation, we have

$$G(Z,Z_0,\tau,\tau_0)=\begin{cases}\dfrac{\theta(Z-Z_0)}{F(Z)}\delta\left(\tau-\tau_0-\displaystyle\int_{Z_0}^{Z}\frac{dZ}{F(Z)}\right) & \text{for } F(Z_0)>0,\\[2ex] \dfrac{1-\theta(Z-Z_0)}{F(Z)}\delta\left(\tau-\tau_0-\displaystyle\int_{Z_0}^{Z}\frac{dZ}{F(Z)}\right) & \text{for } F(Z_0)<0,\end{cases} \tag{10}$$

where $\theta(\psi)$ is the step function:

$$\theta(\psi)=\begin{cases}1, & \psi\geq 0,\\ 0, & \psi<0.\end{cases}$$

By virtue of the properties of the δ function, the solution (10) can be represented in several equivalent forms. Since $\delta(\zeta)$ is nonzero only at the point $\zeta=0$, we can expand the argument of the δ function in the expression (10) in a Taylor series (in Z for fixed τ or in τ for fixed Z) at the point $\zeta=0$ and restrict the expansion to the first linear term, assuming that $\zeta(Z,\tau)=0$ corresponds to a single-valued function $Z(\tau)$. As a result, using the δ-function property

$$\delta(aZ)=\frac{1}{|a|}\delta(Z),$$

where a is some constant, we obtain the solution of (10) in the form

$$G(Z,Z_0,\tau,\tau_0)=\delta(Z-\xi(\tau))=\delta(Z_0-r(\tau)), \tag{11}$$

where $\xi(\tau)$ satisfies the relations (9), and $r(\tau)$ is determined by the equation

$$\frac{dr(\tau)}{d\tau}=-F(r),\quad r(\tau_0)=Z. \tag{12}$$

It follows from (11) that the evolution of the distribution function with initial condition in the form of the δ function (8) is not accompanied by distortion of the shape. Generally speaking, this is not true in the case when a distribution function of arbitrary form is specified at the initial time. The distortion of the shape is obviously due to the separation of the dynamical trajectories (9) for different Z, and the local distortion of the shape of the distribution function in time near an arbitrary point Z_1 is determined by $\Delta Z(\tau)=Z_1(\tau)-Z_2(\tau)$, where $Z_2(\tau_0)$ is some point near $Z_1(\tau_0)$: $\Delta Z(\tau_0)/Z_1\ll 1$. The evolution of $\Delta Z(\tau)$ can be described by the differential equation

$$\frac{d\Delta Z(\tau)}{d\tau}=F_z'(Z_1(\tau))\Delta Z(\tau),\quad \Delta Z(\tau_0)=Z_2(\tau_0)-Z_1(\tau_0),$$

which gives an exponential variation in time of $\Delta Z(\tau)$:

$$\Delta Z(\tau)=\Delta Z(\tau_0)\exp\left\{\int_{\tau_0}^{\tau}F'(Z_1(\tau))d\tau\right\}.$$

Assuming that the interval $[Z_1,Z_2]$ determines the region of localization of the initial distribution function $Q_0(Z)$ [outside the interval we have $Q_0(Z)\equiv 0$], we find that for any $\Delta Z(\tau_0)\neq 0$ and $F'_z(Z_1(\tau))\neq 0$, $\tau\geqslant\tau_0$ deformation of the original distribution function will be observed. An exception is the case $\Delta Z(\tau_0)\equiv 0$; then irrespective of the form of $F(Z)$ there will be no deformation of the shape of the distribution function: $\Delta Z(\tau)=\Delta Z(\tau_0)\equiv 0$. It is in this sense that (11) must be understood.

4. GREEN'S FUNCTION OF THE ORIGINAL PROBLEM

Using the obtained above Green's function of the problem (11) obtained above, we can rewrite the differential equation (5) in the integral form

$$Q(Z,\tau)=\int_0^{\tau}d\tau_0\int_0^{\infty}dZ_0\delta(Z_0-r(\tau))\left(-\frac{\partial}{\partial Z_0}Q(Z_0,\tau_0)+\theta'_t\left[\frac{\partial}{\partial Z_0}\right]Q(Z_0,\tau_0)\right). \quad (13)$$

Using the property of the δ function and differentiating (9) with respect to τ, we obtain a differential equation for Q in the new variables r and τ with constant coefficients:

$$\frac{\partial Q(r,\tau)}{\partial\tau}=-\frac{\partial}{\partial r}Q(Z)+\theta'_t\left[\frac{\partial}{\partial r}\right]Q(Z,\tau). \quad (14)$$

By hypothesis, $Z\geqslant 0$, and, therefore, the function $F(0)$ can be either equal to zero or greater than zero. This means that there are no trajectories on the plane $Z-\tau$ with $Z_0\geqslant 0$ that intersect the τ axis ($Z=0$), i.e., r is also positive definite. In addition, it is obvious that the solutions $r_i(\tau)$ of (12) for $Z_{01}>Z_{02}$ have the property $r_1(\tau)>r_2(\tau)$. As a result, the initial condition (6) and the boundary conditions (7) for Z applied to Eq. (3) go over to an analogous initial condition and boundary conditions at $r=0$ for r:

$$Q(r,\tau)=Q_0(r), \quad (15)$$

$$\frac{\partial Q^n(r,\tau)}{\partial r^n}=0,\quad n=0,\dots,\infty. \quad (16)$$

The differential equation (14) with initial condition (15) and boundary conditions (16) can be readily solved by the operator method. In the important special case in which $Q_0(r)=\delta(r-r_0)$, the solution is the Green's function of the original problem (14), which for $r-r_0<0$ vanishes identically and for $r-r_0\geqslant 0$ [for an operator expression of the form (4), i.e., in the case when $\theta'_t(p)\to 0$ for $\mathrm{Re}\,p=0$ and $\mathrm{Im}\,p\to\pm\infty$] is

$$Q(r,\tau)=\exp(-\tau)[\delta(r-r_0-\tau)+Q'(r,\tau)],$$

$$Q'(r,\tau)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}dp\,\exp[p(r-r_0-\tau)]\times\{\exp[\theta'_t(p)\tau]-1\}. \quad (17)$$

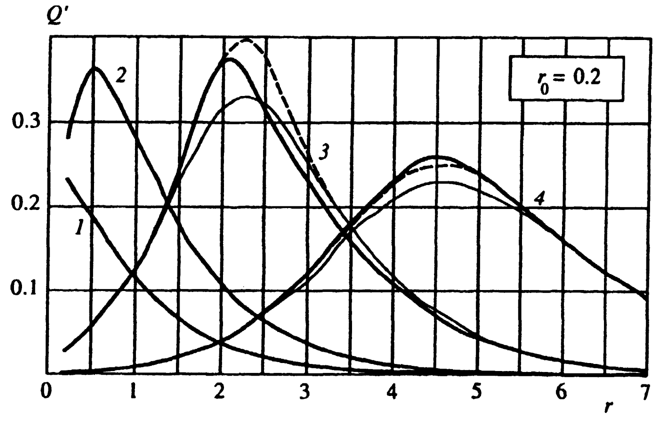

FIG. 1. Evolution in time of the function $Q'(r)$.

5. ASYMPTOTIC BEHAVIOR OF THE SOLUTION (17) FOR $\tau>0$, $|r-r_0-\tau|\to\infty$ (METHOD OF STEEPEST DESCENT)

The asymptotic behavior of the solution $Q(r,\tau)$ for $\tau>0$, $|r-r_0-\tau|\to\infty$ is readily obtained by the method of steepest descent.[5] Investigations showed that the main contribution to the integral (17) is made by a saddle point y^* that lies on the real axis of p and is determined by the condition $dS(y)/dy=0$ for $y=y^*$, $S(y)=\theta'_t(y^*)\tau+y^*(r-r_0-\tau)$. In the case of practical interest $\tau\to\infty$, bearing in mind that the range of variation of r in which the function $Q(r)$ is essentially nonzero is localized near $|r-r_0-\tau|\to 0$, we can obtain an explicit expression for the saddle point y^* in the form of an expansion with respect to the small parameter $\Omega=(r-r_0-\tau)/\tau$.

The leading term of the asymptotic expansion has the form

$$Q'(r,\tau)=\sqrt{-\frac{2\pi}{S''_{yy}}}\exp[\theta'_t(y)\tau+y(r-r_0-\tau)],$$

$$S''_{yy}=\frac{d^2S}{dy^2}. \quad (18)$$

Figure 1 shows the graph of the function $Q'(r)$ at four instants of time [*1*) $\tau=0.5$; *2*) 1.0; *3*) 2.5; *4*) 5.0]. The heavy curves correspond to the numerical calculation in accordance with (17), the thin continuous curves to the leading term in the asymptotic behavior (18), and the dashed curves to the asymptotic behavior with allowance for the first two terms of the expansion. It can be seen from Fig. 1 that there is good agreement between the numerical and asymptotic solutions in the complete range of variation of r and not only as $|r-r_0-\tau|\to\infty$.

The expression for the Green's function of Eq. (14) obtained directly in the integral form (17) or in the form of the asymptotic estimate (18) can be used to calculate the statistical characteristics of the discharge plasma in stochastic microwave fields and also in analogous problems described by Eq. (2).

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