Quasioptics of smoothly inhomogeneous isotropic media

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A generalization of the quasioptical description of vector wave fields to arbitrary smoothly inhomogeneous media is proposed. A study is made of the effect of curvature and torsion of the propagation tracks on the focusing and defocusing properties of the equivalent quasioptical line and also on the polarization of the waves (pseudogyrotropy). The treatment is restricted to the so-called aberrationless approximation of quasioptics, taking into account field phase corrections of at most second order. © *1996 American Institute of Physics.* [S1063-7761(96)00203-7]

1. INTRODUCTION

The term quasioptics was adopted by physicists at the beginning of the sixties in connection with the development of microwave and laser technology.[1] Initially it had a predominantly ''instrumental'' direction and referred to devices similar to optical devices[1)] but requiring allowance for diffraction effects in the description of the wave processes in them (open laser resonators, mirror and lens transmission lines for microwave radiation, fiber optics, etc.). However, the generality of the methods of calculating fields in quasioptical systems based on the use of a truncated parabolic wave equation gradually led to an extended interpretation of the term—quasioptics came to describe the branch of physics concerned with waves of arbitrary nature (both linear and nonlinear) in processes with relatively narrow frequency and angular spectra.[2)]

There exists a deep and constructive analogy between the propagation of wave beams and packets in smoothly inhomogeneous media and in quasioptical transmission lines (of lens or mirror type). By virtue of the transverse localization of the fields around ''propagation paths,'' the inhomogeneous medium can be regarded as a collection of distributed phase correctors: linear (of prism type and responsible for bending of the track), quadratic (of lens type and determining focusing or defocusing), cubic, quartic, etc. (leading to aberrations, i.e., distortions of the ''ideal'' image transported by the equivalent transmission line). This similarity between a distributed system and a discrete transmission line makes it possible to generalize the quasioptical methods of description of wave fields with narrow angular and relatively narrow frequency instantaneous spectra to smoothly inhomogeneous media.

Probably the first quasioptical description of wide wave beams in distributed systems was given in Ref. 3 for the special case of lens-type media (distributed quadratic correctors with axial symmetry). A generalization of quasioptics to arbitrary (two-dimensional) smoothly inhomogeneous media was undertaken in Refs. 4 and 5; in particular, in Ref. 5 one of the present authors obtained a transformation of the coordinates and fields that mapped the field of a wave beam in vacuum to the field of a beam in an inhomogeneous medium. The parameters of this transformation (which is analogous to field mapping by an ideal thin lens[6)]) do not depend on the actual structure of the beam field and can be calculated using a rather trivial extension of the ray equations.

With regard to arbitrary inhomogeneous media not specially intended for the transmission and transformation of images, the terminology adopted in optics has become to a certain degree conventional. For example, in the general case it is not possible to regard aberrations as distortions of an ideal image; sometimes they completely determine the structure of the wave field. The studies of Refs. 7–9 were devoted to extension of the quasioptics of two-dimensional smoothly inhomogeneous media to the case of strong aberrations; these studies also obtained criteria for the applicability of the so-called aberrationless approximation, which takes into account only the quadratic distributed phase correctors. It was also shown in Refs. 7 and 8 that even when the aberrationless approximation cannot be regarded as an approximation to the true wave field it still remains informative and makes it possible to recover the correct solution by means of an asymptotic procedure analogous to the one developed in the diffraction theory of aberration.[2]

In this paper, we propose a generalization of quasioptics to three-dimensional distributed systems and vector fields. We are solely concerned with the aberrationless description of fields, but we allow for astigmatism of the phase correctors and torsion of the transmission paths.[3)]

2. THE EQUATION OF QUASIOPTICS IN A SMOOTHLY INHOMOGENEOUS MEDIUM

For definiteness, we shall consider the electrodynamic problem[4)] of the propagation of a beam of monochromatic electromagnetic waves in a smoothly inhomogeneous stationary medium with permittivity $\varepsilon(\omega,\mathbf{r})$. The complex amplitudes of the electric, $\mathbf{E}(\mathbf{r})$, and magnetic, $\mathbf{B}(\mathbf{r})$, fields are described by Maxwell's equations:

$$\operatorname{curl}\mathbf{E}=ik_0\mathbf{B},\quad \operatorname{curl}\mathbf{B}=-ik_0\varepsilon(\mathbf{r})\mathbf{E},\tag{2.1}$$

where $k_0=\omega/c$. Equations (2.1) can be significantly simplified if the transverse dimension Λ of the beam everywhere along the propagation path is, on the one hand, small on the scale $L_\varepsilon\sim\varepsilon/|\nabla\varepsilon|$ of the inhomogeneities of the medium but, on the other hand, large on the scale of the wavelength λ. In

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this case, the problem contains two small parameters: the width $\nu=\lambda/\Lambda$ of the angular spectrum of the beam and the ratio $\mu=\Lambda/L_\varepsilon$. The aim of the paper is to construct asymptotic solutions of the system (2.1) with respect to the parameters ν and μ.

It is intuitively clear (and this will be shown below) that for ν, $\mu\ll 1$ the wave beam is localized in a certain neighborhood of the geometric-optics ray (we shall call it the reference ray[5)]), the canonical equations of which have the form

$$\frac{d\mathbf{r}_0}{d\tau}=\mathbf{p},\quad \frac{d\mathbf{p}}{d\tau}=\frac{1}{2}\nabla\varepsilon. \tag{2.2}$$

Here $\mathbf{r}_0(\tau)$ is the radius vector of the points on the reference ray; $\mathbf{p}(\tau)$, which is normalized by k_0, is the instantaneous wave vector ($\mathbf{p}=\mathbf{l}\sqrt{\varepsilon(\mathbf{r}_0)}$, in which $\mathbf{l}$ is the unit vector tangent to the ray), and the variable τ is related to the path length s of the ray by $d\tau=ds/\sqrt{\varepsilon(\mathbf{r}_0)}$.

The reference ray is a space curve with its principal normal $\mathbf{n}=(d\mathbf{l}/ds)/K$ in the plane $(\mathbf{l},\nabla\varepsilon)$, while the curvature K and torsion T are given by

$$K=\left[\frac{\partial\varepsilon}{\partial n}\Big/2\varepsilon\right]_{\mathbf{r}=\mathbf{r}_0},\quad T=\left[\frac{\partial^2\varepsilon}{\partial l\,\partial m}\Big/\frac{\partial\varepsilon}{\partial n}\right]_{\mathbf{r}=\mathbf{r}_0}, \tag{2.3}$$

where $\mathbf{m}=\mathbf{l}\times\mathbf{n}$ is the unit vector of the binormal. The expression for the curvature follows directly from the ray equations, and to obtain the expression for the torsion it is necessary to differentiate the second equation of (2.2) with respect to τ.

If the reference ray has nonvanishing torsion, $T\neq 0$, the curvilinear coordinate system associated with the natural trihedral $(\mathbf{l},\mathbf{n},\mathbf{m})$ is not orthogonal (this follows directly from the Frenet–Serret formulas). As we move along the ray, the orthogonal basis $(\mathbf{e}_\tau,\mathbf{e}_1,\mathbf{e}_2)$ is rotated through angle $\theta(\tau)$ relative to the natural trihedral:

$$\mathbf{e}_1=\mathbf{n}\cos\theta+\mathbf{m}\sin\theta,$$
$$\mathbf{e}_2=\mathbf{m}\cos\theta-\mathbf{n}\sin\theta, \tag{2.4}$$
$$\frac{d\theta}{d\tau}=-\sqrt{\varepsilon}T.$$

The unit vectors $\mathbf{e}_1(\tau)$ and $\mathbf{e}_2(\tau)$ are parallel-transported ($\dot{\mathbf{e}}_1\cdot\mathbf{e}_2=\mathbf{e}_1\cdot\dot{\mathbf{e}}_2\equiv 0$) in the effective curvilinear space with metric $ds^2=\varepsilon(\mathbf{r})(dx^2+dy^2+dz^2)$, the geodesics of which coincide with the rays. The same applies to the polarization vector $\mathbf{e}(\mathbf{r})$ of the electric field in the geometric optics approximation;[2] therefore, the relation (2.4) is identical to Rytov's law[10] for the rotation of the plane of polarization in the case of electromagnetic waves propagating in an inhomogeneous medium.

For the quasioptical description of the field of the wave beam, it is convenient to go over to the curvilinear coordinate system (τ,ξ_1,ξ_2) associated with the reference ray:

$$\mathbf{r}=\mathbf{r}_0(\tau)+\xi_1\mathbf{e}_1(\tau)+\xi_2\mathbf{e}_2(\tau). \tag{2.5}$$

The Lamé coefficients of this coordinate system are

$$h_1=h_2=1,\quad h_\tau=h=\sqrt{\varepsilon_0(\tau)}[1-K(\xi_1\cos\theta-\xi_2\sin\theta)],$$

where $\varepsilon_0(\tau)$ are the values of the permittivity at the points on the reference ray. For some remarks concerning the properties of the comoving orthogonal coordinate system, see Appendix 1.

The local structure of the electromagnetic field in a wide (on the scale of λ) wave beam is always close to that of a plane wave, i.e., the fields $\mathbf{E}$ and $\mathbf{B}$ are almost perpendicular to the direction of propagation and to each other. Representing the field in the form

$$\mathbf{E}(\mathbf{r})=\mathbf{E}_\perp(\mathbf{r})+\mathbf{E}_\parallel(\mathbf{r})+\mathbf{O}(\nu^2,\mu^2),$$

and substituting in Eqs. (2.1) as expressed in the above curvilinear coordinate system (in which the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are to be assumed to be independent of the variable τ), we obtain

$$\frac{1}{h}\frac{\partial}{\partial\tau}\left(\frac{1}{h}\frac{\partial\mathbf{E}_\perp}{\partial\tau}\right)+\frac{1}{h}\frac{\partial}{\partial\xi_m}\left(h\frac{\partial\mathbf{E}_\perp}{\partial\xi_m}\right)+k_0^2\varepsilon(\mathbf{r})\mathbf{E}_\perp=0,$$
$$E_\parallel=\frac{1}{k_0^2\varepsilon_0}\frac{\partial}{\partial\tau}\operatorname{div}\mathbf{E}_\perp. \tag{2.6}$$

Here and in what follows, summation over repeated dummy indices is understood.

A wave beam with arbitrary polarization (including the case of inhomogeneity over the cross sections) can be represented in the form of a superposition of two beams with mutually orthogonal homogeneous polarizations (linear, circular, or elliptic).[6)] For each of the "partial" beams, it is possible to introduce the scalar field amplitude $E_\perp$ ($\mathbf{E}_\perp=E_\perp\mathbf{e}_\perp$, where $\mathbf{e}_\perp$ is a unit complex polarization vector).

In $E_\perp$, we isolate a phase factor with characteristic longitudinal scale λ:

$$E_\perp=\frac{1}{\varepsilon_0^{1/4}(\tau)}W(\mathbf{r})\exp\left[ik_0\int_0^\tau\varepsilon_0(\tau)d\tau\right]. \tag{2.7}$$

The field amplitude $W(\tau,\xi_1,\xi_2)$ of the beam, varying smoothly in space, has a characteristic transverse scale $\Lambda\gg\lambda$ ($\lambda/\Lambda\sim\nu\ll 1$) and characteristic longitudinal scale $L_\parallel\gg\Lambda$. In inhomogeneous media possessing focusing or defocusing properties, the longitudinal scale of variation of W can be determined both by diffraction effects (and then $\lambda/L_\parallel\sim\nu 2$) as well as by refraction effects (in this case $\lambda/L_\parallel\sim\nu\mu$). Substituting (2.7) in (2.6), we obtain up to terms of second order (in ν and μ) the equation

$$\frac{2i}{k_0}\frac{\partial W}{\partial\tau}+\frac{1}{k_0^2}\left(\frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}\right)W-\alpha_{mn}(\tau)\xi_m\xi_n W=0,$$
$$\alpha_{mn}(\tau)=\left[\frac{3}{4\varepsilon}\frac{\partial\varepsilon}{\partial\xi_m}\frac{\partial\varepsilon}{\partial\xi_n}-\frac{1}{2}\frac{\partial^2\varepsilon}{\partial\xi_m\partial\xi_n}\right]_{\mathbf{r}=\mathbf{r}_0}. \tag{2.8}$$

The sign in front of the quadratic form in (2.8) is chosen in such a way that positive values of the coefficients α_{mn} correspond to focusing properties of the medium. More precisely, if the signature Σ of the quadratic form is 2, the medium possesses focusing properties with respect to all directions; if $\Sigma=0$, then in one direction the medium focuses and in the other it defocuses; if $\Sigma=-2$, the medium defocuses in all directions.

In actual calculations, it is more convenient to represent the coefficients α_{mn} in (2.8) in terms of derivatives in the direction of the principal normal **n** and binormal **m** "attached" to the gradient of the permittivity in the medium:

$$\alpha_{lk}=R_{lp}(\tau)R_{kq}(\tau)\beta_{pq},$$

$$\beta_{11}=3\varepsilon_0K^2-\frac{1}{2}\frac{\partial^2\varepsilon}{\partial n^2},\quad \beta_{22}=-\frac{1}{2}\frac{\partial^2\varepsilon}{\partial m^2},$$

$$\beta_{12}=-\frac{1}{2}\frac{\partial^2\varepsilon}{\partial n\partial m},\quad \hat{R}(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \tag{2.9}$$

The angle θ (τ) in the rotation operator $\hat{R}(\theta)$ is determined by Rytov's law (2.4) (the derivatives in the above expressions are taken at points on the reference beam).

It can be seen from (2.9) that curvature of the reference beam is always a focusing factor (in the rectifying plane). In addition, the focusing and defocusing properties of the medium are also determined by the second derivatives of the permittivity along the transverse directions.

Equation (2.8) is written down in the orthogonal curvilinear coordinate system (τ,ξ_1,ξ_2), but the form of the differential operators retained in it after truncation makes it possible to establish a direct analogy with the problem of wave propagation in a lens-like medium elongated along the τ axis. This analogy can also be taken further to discrete optical lines—chains of quadratic phase correctors (thin lenses).[7)] The absence of axial symmetry ($\alpha_{11}\neq\alpha_{22}$, $\alpha_{12}\neq 0$) leads to astigmatism—each individual element of the line maps a point into two crossed segments. As few as two lenses for which the planes of the principal normal sections do not coincide (torsion effect) blur the image. In geometrical optics, astigmatism is usually included among the aberrations, but in quasioptics it can be taken into account in an approximation that (with a certain degree of license) we shall call aberrationless.

Pseudogyrotropy

Sometimes it is advantageous, for one reason or another, to represent the field of a wave beam in a skew coordinate system (τ,η_1,η_2) associated with the natural trihedral. For this, it is necessary to apply a rotation transformation to the coordinates $[\xi_m=R_{mn}(\theta)\eta_n]$ and field $[E_\perp^{(m)}=R_{mn}(\theta)\tilde{E}_\perp^{(n)}]$. Bearing in mind that the derivative of the rotation operator is $\dot{R}_{mn}(\theta)=\dot{\theta}R_{mn}(\theta+\pi/2)$, we obtain after truncating (2.6) the system of equations

$$\frac{2i}{k_0}\hat{D}_\tau\tilde{W}_1+\frac{1}{k_0^2}\Delta_\perp\tilde{W}_1-\beta_{mn}(\tau)\eta_m\eta_n\tilde{W}_1=\frac{2i}{k_0}\sqrt{\varepsilon_0}T\tilde{W}_2,$$

$$\frac{2i}{k_0}\hat{D}_\tau\tilde{W}_2+\frac{1}{k_0^2}\Delta_\perp\tilde{W}_2-\beta_{mn}(\tau)\eta_m\eta_n\tilde{W}_2=-\frac{2i}{k_0}\sqrt{\varepsilon_0}T\tilde{W}_1, \tag{2.10}$$

$$\hat{D}_\tau=\left(\frac{\partial}{\partial\tau}+T(\tau)e_{mn}\sqrt{\varepsilon_0}\eta_m\frac{\partial}{\partial\eta_n}\right),\quad \Delta_\perp=\frac{\partial^2}{\partial\eta_1^2}+\frac{\partial^2}{\partial\eta_2^2},$$

where e_{mn} is the unit antisymmetric tensor.

Equations (2.10) can be decoupled for waves with circular polarizations: $\tilde{W}_\pm=\tilde{W}_1\pm i\tilde{W}_2$.[8)] Multiplying the second of Eqs. (2.10) by the imaginary unit, and adding it to the first (and then subtracting from it), we obtain

$$\frac{2i}{k_0}\hat{D}_\tau\tilde{W}_\pm+\frac{1}{k_0^2}\Delta_\perp\tilde{W}_\pm-\left[\pm\frac{2\sqrt{\varepsilon_0}}{k_0}T+\beta_{mn}(\tau)\eta_m\eta_n\right]\tilde{W}_\pm=0. \tag{2.11}$$

The corrections of opposite signs to the effective refractive indices of waves with right and left circular polarizations establish a definite similarity between (2.11) and the equations for waves in optically active media (sugar solution, turpentine). One can say that the torsion of the propagation path leads to "quasi-" or "pseudogyrotropy" effects in media without spatial dispersion. We may mention in passing that in other situations, when $\Lambda\gg L_\varepsilon$, it is more convenient to interpret the gyrotropy due to multiple scattering by macroscopic inhomogeneities as a manifestation of structural spatial dispersion of long-range order.[11]

The skew nature of the coordinate system (τ,η_1,η_2) also leads to the replacement in (2.11) of the derivative with respect to τ by the operator $\hat{D}_\tau$, which in the comoving cylindrical coordinates $(r_\perp,\phi)$ reduces to the form

$$\hat{D}_\tau=\frac{2i}{k_0}\left(\frac{\partial}{\partial\tau}+\sqrt{\varepsilon_0}T\frac{\partial}{\partial\phi}\right).$$

A similar operator occurs in the Schrödinger equation for an electron in a constant magnetic field;[12] for its influence on the structure of the field, see Appendix 3.

3. BEAMS IN SYSTEMS ADMITTING SEPARATION OF THE VARIABLES

If it is admissible to go over to coordinates (x_1,x_2) in which the quasioptical equation is invariant with respect to the substitutions $x_1\to -x_1$ and $x_2\to -x_2$,[9)] it is possible to use the method of separation of variables, representing the complex field amplitude in the form

$$W(x_1,x_2)=X_1(x_1)X_2(x_2). \tag{3.1}$$

Substituting (3.1) in (2.8), where $\alpha_{12}\equiv 0$, we obtain two equations of the same kind. We write down only one of them, omitting the subscripts:

$$2ik_0\frac{\partial X}{\partial\tau}+\frac{\partial^2X}{\partial x^2}-k_0^2\alpha(\tau)x^2X=0. \tag{3.2}$$

We go over in (3.2) to the dimensionless variables (z,y) by means of the substitution

$$y(\tau)=\frac{x}{\sigma(\tau)},\quad z(\tau)=\int_0^\tau\frac{d\tau}{k_0\sigma^2},$$

$$X=\sqrt{\frac{1}{\sigma}}V(z,y)\exp\left(\frac{ik_0}{2}\sigma\dot{\sigma}y^2\right), \tag{3.3}$$

where $\sigma(\tau)$ is an as yet arbitrary function. As a result, Eq. (3.2) is reduced to the form

$$2i\frac{\partial V}{\partial z}+\frac{\partial^2V}{\partial y^2}-k_0^2\sigma^3[\ddot{\sigma}+\alpha(\tau)\sigma]y^2V=0. \tag{3.4}$$

It is obvious that (3.4) can be reduced in two ways to canonical forms by equating the coefficient of y^2V to unity or zero. We consider each of these possibilities separately.

3.1. Expansion with respect to the modes of the discrete spectrum

Setting in (3.4)

$$\ddot{\sigma}+\alpha(\tau)\sigma=\frac{1}{k_0^2\sigma^3}, \tag{3.5}$$

we obtain the equation of a "quantum-mechanical oscillator":

$$\left(2i\frac{\partial}{\partial z}+\frac{\partial^2}{\partial y^2}-y^2\right)V=0. \tag{3.6}$$

The general solution of this equation can be represented in the form of a series in Hermite functions:[4]

$$V=\frac{C_n}{\sqrt{\pi^{1/2}2^n n!}}\exp\left[-i\left(n+\frac{1}{2}\right)z-\frac{y^2}{2}\right]H_n(y), \tag{3.7}$$

where

$$H_n(y)=(-1)^n\exp(y^2)\frac{d^n}{dy^n}\exp(-y^2)$$

is a Hermite polynomial.

The general solution of Eq. (3.5) for the characteristic width of the eigenfunctions contains two arbitrary constants, on the values of which the form of the expansion (3.7) in the dimensionless variables (z,y) does not depend. However, in the "real" space (τ,x) the structure of the modes depends critically on the choice of the constants of integration (3.5). The possibility arises of making an optimum choice of the modes of the discrete spectrum in different applied problems. We shall return to this question and mention here only one detail—expansion of the wave field with respect to the modes of the discrete spectrum is possible not only in the case of focusing layers $[\alpha(\tau)>0]$ or, putting it differently, waveguide channels, but also in the case of defocusing inhomogeneities $(\alpha<0)$ (then, of course, the characteristic widths of the modes will increase exponentially along the reference ray).

3.2. Expansion in modes of the continuous spectrum

If we choose the normalization parameter $\sigma(\tau)$ in such a way that it satisfies the linear equation

$$\ddot{\sigma}+\alpha(\tau)\sigma=0, \tag{3.8}$$

then (3.4) can be reduced to the "vacuum" form

$$\left(2i\frac{\partial}{\partial z}+\frac{\partial^2}{\partial y^2}\right)V=0. \tag{3.9}$$

In the quasioptics of homogeneous media, the solution of (3.9) is, as a rule, represented in the following two forms:
1) expansions in Green's functions,

$$V(z,y)=\frac{1}{\sqrt{2\pi iz}}\int_{-\infty}^{\infty}V_0(y')\exp\left[\frac{i}{2z}(y-y')^2\right]dy'; \tag{3.10}$$

2) expansions in plane waves,

$$V(z,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{V}_0(q)\exp\left(-\frac{iz}{2}q^2+iyq\right)dq. \tag{3.11}$$

It can be seen from the integral representations themselves that $V_0(y)$ and $\widetilde{V}_0(q)$ are the "initial" (at $z=0$) distributions of the field and of its Fourier spectrum. However, upon transition to the dimensional variables (τ,x), this remains true only for the quite definite choice of the solution of the characteristic equation (3.8) corresponding to the following initial conditions:

$$\sigma|_{\tau=0}=\sigma_0,\quad \dot{\sigma}|_{\tau=0}=0$$

(for σ_0 chosen equal to the unit of length, rescaling of the initial distribution does not occur: $V_0(x/\sigma_0)=\sqrt{\sigma_0}X(0,x)$).

If, however, we choose as normalization factor the solution (3.8) with the initial conditions

$$\sigma|_{\tau=0}=\sigma_0,\quad \dot{\sigma}|_{\tau=0}=-\sigma_0/F,$$

then (3.10) upon transition to the dimensional variables becomes an expansion in initially focused waves.[10] As is shown in Ref. 8, it is conveniently used in the description of aberration distortions of the beam field, the parameter F being varied in such a way that in the region of observation each partial wave either "collapses" to a point or has a plane phase front.

It should be noted that the characteristic equation (3.8) is identical to the equations of rays that are differentially close to the reference ray. As a consequence, all the results obtained above admit a transparent geometrical interpretation.

Fundamental system of rays

The quasioptical equation (3.2) can be brought into correspondence with the equations of geometrical optics (in a somewhat unusual form, since the advance of phase along the reference ray is eliminated by the change of variables (2.7)). Representing the complex field amplitude in the form $X=A\exp(ik_0\Psi)$ and substituting in (3.2), we obtain the following equation for the eikonal Ψ:

$$\frac{\partial\Psi}{\partial\tau}+\frac{1}{2}\left[\left(\frac{\partial\Psi}{\partial x}\right)^2+\alpha x^2\right]=0. \tag{3.12}$$

The equations of rays corresponding to (3.12) have the form

$$\frac{dx}{d\tau}=p,\quad \frac{dp}{d\tau}=-\alpha(\tau)x, \tag{3.13}$$

where $p=\partial\Psi/\partial x$ is the projection of the instantaneous wave vector (normalized by k_0) onto the plane normal to the reference ray. It is readily seen that Eq. (3.13) after elimination of p is identical to Eq. (3.8) for the characteristic parameter σ.

The linearity of the ray equations (3.13) makes it possible to introduce a fundamental system of solutions $P(\tau)$ and $S(\tau)$ that satisfy the initial conditions

$$P(0)=1,\quad \dot{P}(0)=0,\quad S(0)=0,\quad \dot{S}(0)=1,$$

with Wronskian $\mathscr{W}[P,S]=\dot{P}S-P\dot{S}=1$. The solution of Eqs. (3.13) with initial conditions $x(0)=x'$, $p(0)=\dot{x}(0)=p'$ can be written in the form

$$x=P(\tau)x'+S(\tau)p',\quad p=\dot{P}(\tau)x'+\dot{S}(\tau)p'. \tag{3.14}$$

Here $P(\tau)$ determines the rays of an initially plane wave, and $S(\tau)$ determines rays that diverge as a fan from the origin (these circumstances determine the choice of notation in accordance with the initial letters of the words *plane* and *source*). However, P and S also have a different "geometrical" meaning—they are also a fundamental system of the equation of the normal sections of the ray tubes (this equation is obtained by differentiating the ray equations with respect to the parameter and, by virtue of the linearity of these equations, is identical to them). The solution of any problem in geometrical optics can be expressed in terms of the fundamental system of rays P and S (in the small-angle approximation).

For example, fixing x and x' in (3.14), we obtain (in the approximation of geometrical optics) an expression for the eikonal $\Psi(x,x')$ of the two-point Green's function:

$$\frac{\partial\Psi}{\partial x'}=-p'=S^{-1}(Px'-x),$$

$$\frac{\partial\Psi}{\partial x}=p=S^{-1}(\dot{S}x-x'), \tag{3.15}$$

$$\Psi(x,x')=\frac{1}{2}S^{-1}(\dot{S}x^2-2xx'+Px'^2).$$

The amplitude of the Green's function in the approximation of geometrical optics is determined from the law of conservation of the energy flux within a ray tube surrounding the ray and joining the points x' and x. Consequently, the amplitude $A\sim 1/\sqrt{S}$ and does not depend explicitly on the transverse coordinates. Therefore, in the approximation of geometrical optics we obtain the exact Green's function of the quasioptical equation of the aberrationless approximation, which, naturally, corresponds to the expansion (3.10) given in the previous section. The same also applies to the expansions with respect to originally plane and originally focused waves.

We rewrite the expansions in modes of the continuous spectrum in terms of the fundamental ray system:

1) The expansion in Green's functions is

$$X(\tau,x)=\sqrt{\frac{k_0}{2\pi iS}}\int_{-\infty}^{\infty}X_0(x')\exp[ik_0\Psi(x,x')]dx', \tag{3.16}$$

where Ψ is given by the expression in (3.15).

2) The expansion in originally plane waves in

$$X(\tau,x)=\sqrt{\frac{1}{2\pi P}}\int_{-\infty}^{\infty}\tilde{X}_0(q)\times\exp\left[\frac{ik_0}{2P}(\dot{P}x^2+2xq-Sq^2)\right]dq, \tag{3.17}$$

where

$$\tilde{X}_0(q)=\frac{k_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty}X(x)\exp(-ik_0qx)dx.$$

The expansion of the field in modes of the discrete spectrum (3.7) can also be expressed in terms of the fundamental system P, S. It can be seen from (3.16) that in the aberrationless approximation a Gaussian beam always remains Gaussian. At the same time, its characteristic width σ can be expressed in terms of the initial width σ_0 as follows:

$$\sigma(\tau)=\sqrt{\sigma_0^2\left(P-\frac{S}{F}\right)^2+\left(\frac{S}{k_0\sigma_0}\right)^2}. \tag{3.18}$$

It can be shown by direct substitution (although it is also obvious) that $\sigma(\tau)$ in the form (3.18) satisfies Eq. (3.5).

4. BEAMS IN SYSTEMS WITH "TORSION"

When the reference ray is an arbitrary curve with variable torsion, it is not possible to separate variables in Eq. (2.8). However, the important property of the aberrationless approximation noted above still holds—geometrical optics for the Green's function of originally plane and focused waves gives the exact result. The ray equations remain (as in one-dimensional systems) linear:

$$\frac{dx_m}{d\tau}=p_m,\quad \frac{dp_m}{d\tau}=-\alpha_{mn}(\tau)x_n. \tag{4.1}$$

Their solution with the initial conditions

$$x_m(0)=x'_m,\quad p_m(0)=p'_m,$$

can be represented in the form

$$x_m=P_{mn}(\tau)x'_n+S_{mn}(\tau)p'_n,$$

$$p_m=\dot{P}_{mn}(\tau)x'_n+\dot{S}_{mn}(\tau)p'_n. \tag{4.2}$$

The matrices $\hat{P}$ and $\hat{S}$ are themselves solutions of the equations for the cross sections of the ray tubes:

$$\ddot{P}_{mn}+\alpha_{mk}(\tau)P_{kn}=0,$$

$$\ddot{S}_{mn}+\alpha_{mk}(\tau)S_{kn}=0,$$

with initial conditions

$$P_{mn}(0)=\delta_{mn},\quad \dot{P}_{mn}=0,$$

$$S_{mn}(0)=0,\quad \dot{S}_{mn}=\delta_{mn}.$$

It is readily shown that by virtue of the symmetry $\alpha_{mn}=\alpha_{nm}$ the fundamental matrices are related to each other by

$$P_{km}\dot{S}_{kn}-\dot{P}_{km}S_{kn}=\delta_{mn}.$$

The formalism of matrix algebra makes it possible to generalize the two-dimensional results of the previous section to three-dimensional systems. For this, it is necessary to replace the operations of multiplication and division of a scalar by a scalar by the operations of multiplication of a vector by a matrix and by the inverse matrix. Thus, the integral representations of the field analogous to (3.16) and (3.17) are transformed as follows:

1) Expansion in Green's functions:

$$W(\tau,\mathbf{x})=\frac{k_0}{2\pi i\sqrt{D_S}}\int\int \times W_0(\mathbf{x}')\exp[ik_0\Psi(\mathbf{x},\mathbf{x}')]dx_1'dx_2', \tag{4.3}$$

$$\Psi=\frac{1}{2}S_{mk}^{-1}(P_{mn}x_k'x_n'-2x_mx_k'+\dot{S}_{nk}x_nx_m), \quad D_S=\det\|S\|.$$

2) Expansion in originally plane waves:

$$W(\tau,\mathbf{x})=\frac{1}{2\pi\sqrt{D_P}}\int\int \times \tilde{W}_0(\mathbf{q})\exp[ik_0\tilde{\Psi}(\mathbf{x},\mathbf{q})]dq_1dq_2, \tag{4.4}$$

$$\tilde{\Psi}=\frac{1}{2}P_{mk}^{-1}(-S_{mn}q_kq_n+2x_mq_k+\dot{P}_{nk}x_nx_m),$$

$$D_P=\det\|P\|.$$

Note that the expansion in Green's functions is completely invertible, i.e., in (4.3) it is possible to interchange the distribution of the complex field amplitude in the initial section and in the plane of observation:

$$W_0(\mathbf{x}')=\frac{ik_0}{2\pi\sqrt{D_S}}\int\int W(\tau,\mathbf{x}) \times\exp[-ik_0\Psi(\mathbf{x},\mathbf{x}')]dx_1dx_2.$$

In the aberrationless approximation, a wave beam with Gaussian distribution of the complex field amplitude in the initial section ($\tau=0$) always remains Gaussian. Substituting into (4.3)

$$W_0(\mathbf{x}')=\sqrt{\frac{1}{D_\sigma}}\exp\left[-\frac{1}{2}(\sigma_{mn}^{-1}-ik_0R_{mn}^{-1})x_m'x_n'\right],$$

we obtain

$$W(\mathbf{x},\tau)=\sqrt{\frac{1}{D_\gamma}}\exp\left[\frac{ik_0}{2}\gamma_{mk}^{-1}\dot{\gamma}_{nk}x_mx_n\right], \tag{4.5}$$

$$\gamma_{mn}=\sigma_{mk}(P_{nk}+S_{nl}R_{lk}^{-1})+\frac{i}{k_0}S_{mn}$$

(D_σ and D_γ are the determinants of the corresponding matrices). The symmetric matrices $\hat{\sigma}$ and $\hat{R}^{-1}$ give the characteristic dimensions of the beam and the curvature of the phase front in the initial section $\tau=0$. In any other normal section, the corresponding quantities are determined by

$$\tilde{\sigma}_{mn}^{-1}=\mathrm{Im}(k_0\gamma_{mk}^{-1}\dot{\gamma}_{nk}), \quad \tilde{R}_{mn}^{-1}=\mathrm{Re}(\gamma_{mk}^{-1}\dot{\gamma}_{nk}).$$

The matrices $\|\tilde{\sigma}_{mn}^{-1}\|$ and $\|\tilde{R}_{mn}^{-1}\|$ are symmetric, but in the general case they cannot be simultaneously reduced to diagonal form by a rotation since the ellipses of equal intensities and of equal phases are rotated relative to each other, and the angle of rotation changes along the propagation path.

5. CONCLUSIONS

The procedure described above for integrating wave equations in the quasioptical (aberrationless) approximation can be conveniently implemented on a computer. It reduces to the solution of the system of ordinary differential equations (2.2), (2.4), and (4.1) and subsequent calculation of convolutions of the type of (4.3) or (4.4). We emphasize that the first block of calculations is completely independent of the actual structure of the required wave field and reflects only universal properties of the propagation path. This is an important difference between our approach and the traditional generalizations of geometric optics (Maslov's method, the Kravtsov–Ludwig method of reference functions, etc.), in which the wave field is found in an appropriate ray approximation. In the second stage, one can simultaneously calculate the fields of wave beams with different initial distributions of the amplitude, phase, and polarization in all cross sections that are of interest (without calculating of the field in the intermediate regions); moreover, these distributions need not be smooth (their integrability is sufficient).

APPENDIX 1: "HELICAL" COORDINATE SYSTEM

The geometrical properties of the comoving coordinate system introduced in Sec. 2 does not depend on whether or not the reference curve is a ray. In this Appendix, we illustrate some of these properties for the example of a helical reference curve having in a cylindrical coordinate system (r,ϕ,z) an equation of the form

$$r=a, \quad \phi=\phi, \quad z=\tilde{h}\phi,$$

where a is the radius of the surface around which the reference curve is "wound"; $\tilde{h}=h/2\pi$, where h is the pitch of the spiral. The curvature and torsion of the helical curve are

$$K=\frac{a}{a^2+\tilde{h}^2}, \quad T=\frac{\tilde{h}}{a^2+\tilde{h}^2}. \tag{A.1.1}$$

The principal normal $\mathbf{n}$ is directed along the radius; the vector $\mathbf{l}$ tangent to the curve makes an angle $\gamma=\tan^{-1}(a/\tilde{h})$ with the z axis. The element of arc length is $ds = d\phi\sqrt{a^2+\tilde{h}^2}$. The natural trihedral revolves about the z axis and makes a complete revolution when the variable ϕ changes by 2π. The orthogonal basis $(\mathbf{l},\mathbf{e}_1,\mathbf{e}_2)$ rotates to the left (for a right-handed spiral $h>0$) with respect to the natural trihedral:

$$\frac{d\theta}{d\phi}=-\frac{\tilde{h}}{\sqrt{a^2+\tilde{h}^2}},$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{e}_1$. In one period of the helical curve, the lag of the orthogonal basis behind the natural dihedral is measured by the angle $\theta_p = h/\sqrt{a^2+\tilde{h}^2}$.

It can be seen that for a compact spiral ($\tilde{h}\ll a$) the angle $\theta_p\to 0$, i.e., the orthogonal basis hardly lags behind the natural trihedral. In the other limiting case of a significantly stretched spiral ($\tilde{h}\gg a$), we have $\theta_p\to -2\pi$, i.e., the lag is greatest, and the orthogonal basis hardly rotates relative to

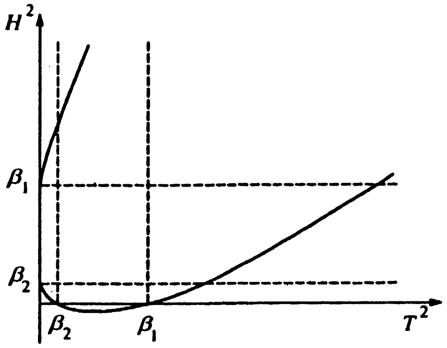

FIG. 1. Characteristic curves.

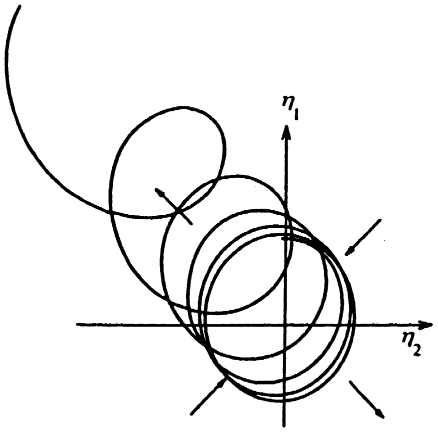

FIG. 2. Projection of a ray close to the reference ray onto the plane (η_1,η_2) for $\beta_2<T^2<\beta_1$. The arrows indicate the directions of the additional focusing and defocusing due to the torsion.

the z axis. However, over extended paths (over many turns of the spiral) it is necessary to take into account the integrated effects, which lead to a rotation of the orthogonal basis about the z axis through an angle $\Delta\phi\simeq za^2/2\tilde{h}^3$.

APPENDIX 2: ON THE GENERALIZATION OF RYTOV'S LAW IN QUASIOPTICS

Rytov's law for the rotation of the polarization vector (2.4) is obtained in the geometrical optics approximation.[10] In wave optics, the field at the point of observation is formed by the interference of signals that arrive along different rays from the region of the sources. The torsion of these interference rays (indices i) differs in principle from the torsion of the reference ray, and one can speak of their relative torsion $\tilde{T}=T_i-T_0$. Therefore, the polarizations of the partial signals differ from each other. However, in quasioptics (in the small-angle approximation) the interference rays consist of segments of strongly elongated spirals that wind around the reference ray. As was shown in Appendix 1, in this case the orthogonal basis (and, therefore, the polarization vector) for the partial ray will not twist (up to terms of order $\tau\Lambda^2/L_\varepsilon^3$) relative to the orthogonal basis of the reference ray. Therefore, for paths with moderate extension τ, Rytov's law is also fairly well satisfied in the quasioptical approximation.

APPENDIX 3: INFLUENCE OF TORSION OF THE REFERENCE RAY ON THE FOCUSING PROPERTIES OF THE EQUIVALENT OPTICAL LINE

Torsion of the reference ray not only leads to the Rytov polarization effect (pseudogyrotropy) but also changes the focusing properties of the equivalent optical line. In this Appendix, we consider the special case with constant torsion (T=const) of the reference ray (a helical curve) and τ-independent (in the frame of reference attached to the natural trihedral) parameters of the medium:

$$\beta_{11}=\beta_1=\text{const},\quad \beta_{22}=\beta_2=\text{const},\quad \beta_{12}=0 \tag{A.3.1}$$

[see (2.9)]. Without loss of generality, we can also set $\varepsilon_0\equiv 1$. Such properties are possessed, for example, by a gradient-index glass fiber that spirals round a cylindrical surface.[11)]

As was noted in Sec. 4, the properties of the equivalent optical line are completely determined by the behavior of the geometrical-optics rays, the equations of which in the co-moving skew coordinate system (τ,η_1,η_2) have the form

$$\frac{d\eta_1}{d\tau}=p_1-T\eta_2,\quad \frac{d\eta_2}{d\tau}=p_2+T\eta_1,$$
$$\frac{dp_1}{d\tau}=-\beta_1\eta_1-Tp_2,\quad \frac{dp_2}{d\tau}=-\beta_2\eta_2+Tp_1. \tag{A.3.2}$$

The solution of the system of equations (A.3.2) can be represented in the form

$$\eta_i=\text{Re}\left[\sum_{j=1}^{4} c_{ij}\exp(iH_j\tau)\right],$$

where c_{ij} are constant complex coefficients determined from the initial conditions. The "spatial frequencies" H_j of the ray oscillations satisfy the characteristic relation

$$H^2=T^2+\frac{\beta_1+\beta_2}{2}\pm\frac{1}{2}\sqrt{(\beta_1-\beta_2)^2+8(\beta_1+\beta_2)T^2}. \tag{A.3.3}$$

Figure 1 is the graph of the dependence $H^2(T^2)$.

It can be seen from Fig. 1 that torsion of the reference ray leads to an enhancement of the focusing properties of the effective optical line in certain directions and to a weakening in others. Moreover, there exists a range of values for the torsion, $\beta_2<T^2<\beta_1$, in which one of the ray modes is unstable ($H^2<0$), and, therefore, the optical line acquires defocusing properties,[12)] even though it is made of collecting lenses for all directions ($\beta_{1,2}>0$).

For this effect, there is a two-dimensional analog. Let us consider, for example, an axisymmetric parabolic wave channel whose focusing properties vary along the axis in accordance with an harmonic law:

$$\varepsilon=\varepsilon_0-\beta(1+a\sin\Omega\tau)r_\perp^2.$$

In such a channel, the rays are described by the Mathieu equation

$$\frac{d^2r_\perp}{d\tau^2}+\beta(1+a\sin\Omega\tau)r_\perp=0,$$

which has a discrete set of instability bands (the first band $\Omega_2\sim 4\beta$). In contrast to an axisymmetric system, in a twisted elliptical wave channel there exists only one continuous band of parametric instability. An example of such an instability is given in Fig. 2.

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1)Systems are generally called optical if they obey the laws of geometrical optics sufficiently well.

2)Quasioptics as a branch of science has a fairly ancient tradition. The investigations of Fresnel on diffraction by an opening in a screen and at the edge of a screen were made in the approximation of quasioptics. Schrödinger wrote down an equation that, essentially, is the quasioptical approximation of a more rigorous relativistic equation. Of course, in all these (and other) "prehistoric" cases a different terminology was used—the small-angle, for example, or paraxial approximation.[2]

3)It should be noted that in optics effects associated with astigmatism of the phase correctors (the difference of the principal radii of curvature of the refracting surfaces) are usually classed as aberration effects. In quasioptics there is a somewhat different definition—the aberrationless approximation is assumed to be one in which a Gaussian wave beam remains Gaussian (with parabolic phase front).

4)Apart from the terminology and different notation, all results will, of course, be valid for waves of arbitrary nature—acoustic, seismic, information, etc.

5)The choice of the reference ray is not unique and can vary depending on the type of problem to be solved. In problems in which the wave beam is formed by a system of emitters or collimators, it is sensible to choose as the reference ray the ray that emanates from the center of the beam-forming aperture (or other device) in the direction of the maximum of the beam pattern.

6)The polarization degeneracy inherent in (2.6) is lifted in the general case in the following order in ν and μ; this can lead to significant effects over very extended propagation tracks.

7)Note that wave beams in a discrete optical line can be described by the same equation (2.8) if we set $\alpha_{mn}(\tau)=\delta(\tau-\tau_k)\alpha_{mn}^k$, where $\delta(\tau)$ is the Dirac delta function, and k is the number of the corrector.

8)In optics and electrodynamics, the directions of the circular polarizations are defined differently. In the given case, the "plus" sign corresponds to right-circular polarization in the electrodynamic sense—the direction of rotation of the polarization vector is related to the wave vector by the right-hand screw rule.

9)This symmetry can be nominally called mirror symmetry, though it is only such for the equivalent (rectified) optical line. In the real space, mirror symmetry may not be present.

10)The completeness and orthogonality of such an expansion can be proved as follows. An ideal quadratic corrector with optical strength $-F^{-1}$ is positioned in the section $\tau=0$. At its "output," the field is expanded in a Fourier integral, and one then positions a compensating corrector with optical strength F^{-1}, which transforms each plane wave of the expansion into a focused wave.

11)If the profile of the permittivity of the fiber glass has the form $\varepsilon=\varepsilon_0-\gamma r_\perp^2$ and it is wound onto a cylinder of radius a with pitch $2\pi\tilde{h}$, then $\beta_1\approx\gamma+3\varepsilon_0[a/(a^2+\tilde{h}^2)]^2$, $\beta_2\approx\gamma$.

12)In discrete optical lines and in cavities, astigmatism can also lead to instability of quasioptical modes.[13]

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