Abrikosov–Josephson vortices in a layered tunnel structure

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Equations of nonlocal electrodynamics which describe Abrikosov–Josephson vortex structures in an SNSNS system of tunnel junctions are derived. Both exact and approximate solutions characterizing both the coherent individual vortices and the coherent multivortex structures appearing in the two Josephson junctions as a result of the interaction of their magnetic fields, which penetrate the superconductor separating the junctions, are obtained. Such a picture is obtained for both static and moving vortices. The current–voltage characteristics of the Josephson junctions corresponding to multivortex dynamics is considered in the strong-dissipation limit. The influence of the finite thickness of the outer superconducting electrodes on the size of the interacting coherent vortices and on their velocity in the nondissipative limit is revealed. The influence of the thickness of the superconducting layers on the spectra of low-amplitude short-wavelength electromagnetic waves propagating in Josephson superlattices is studied. © *1995 American Institute of Physics.*

1. INTRODUCTION

The vortices in layered Josephson structures have attracted the attention of researchers for a long time.[1–3] In recent years technological advances have made it possible to fabricate tunnel structures with a large number of layers.[4] This led to broadening of the research front in this area. For example, two- and three-layer structures were used in the experimental study described in Ref. 5. It was established there, in particular, that the velocity of a Josephson vortex can be determined by the number of tunnel junctions. Splitting of the spectrum of Swihart waves was also discovered. This effect was treated theoretically in Ref. 6. Subsequent experiments[7] displayed agreement with the theory in Ref. 6. An interesting experimental investigation of the phenomenon of vortex synchronization in two-layer Josephson structures was conducted in Refs. 8 and 9. The number of experimental investigations of magnetic vortices in layered Josephson tunnel structures will certainly increase (compare Ref. 10). At the same time, it must be stressed that all the papers just cited were devoted to the investigation of ordinary Josephson vortices, for which the characteristic scale of their spatial variation is much greater than λ, i.e., the London penetration depth of a magnetic field into the superconductor.

In the case of high-temperature superconductors, a situation in which the characteristic spatial scale of a vortex is smaller than the London length is possible owing to the large value of the parameter of the Ginzburg–Landau theory $\kappa=\lambda/\xi$ (where ξ is the correlation length).[11] It is also possible when there is a moderate magnetic field in a tunnel junction.[12] The electrodynamics of a Josephson tunnel junction under such conditions was developed in Refs. 11 and 13–15. A detailed discussion of the conditions under which nonlocal electrodynamics must be used instead of the ordinary Josephson electrodynamics based on the sine-Gordon equation was given in Refs. 11, 12, and 16. The vortex states appearing in nonlocal electrodynamics, unlike ordinary Josephson vortices, correspond to an array of magnetic-field force lines which are not pressed against the tunnel junction. Their structure is similar to the structure of the force lines of Abrikosov vortices, although, unlike ordinary Abrikosov vortices, they do not have a singular core. Therefore, the term Abrikosov–Josephson vortices can be used for the new vortex structures.

Some basic aspects of nonlocal Josephson electrodynamics as applied to layered tunnel structures were given in Ref. 17, where general relations were derived, the spectrum of generalized Swihart waves was obtained, an approach for constructing weakly nonlinear vortex structures was formulated, and the nonlinear one-dimensional picture of vortices in a structure with an infinite number of superconducting layers whose thicknesses are small compared with the London length was considered.

In this communication a structure containing two tunnel junctions (compare Ref. 5) is considered in the main text of the article. Equations for the phase differences of the Cooper pairs φ_1 and φ_2 in each of the tunnel layers are obtained for such a structure. A set of exact solutions is presented. In the case of strong coupling of the vortices, it is shown how the magnetic interaction causes changes in their spatial scale and their velocity. Coherent excitation of a perturbation in one tunnel junction by a vortex in the other junction is demonstrated in the case of weak coupling of the vortex structures. It is shown that solitary vortices excite a bisoliton perturbation. A description of the excitation of vortices undergoing cophasal motion in different tunnel junctions is given in the nondissipative limit.

A discussion of the current–voltage characteristic of a moving dissipative vortex structure in closely positioned tunnel junctions, including annular Josephson junctions, is given.

The influence of the thickness of the superconducting electrodes forming the outer layers of a system of two tunnel junctions separated by a superconductor layer is considered. The variation of the dimensions of a static vortex as a function of the electrode thickness is established. The laws gov-

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erning the influence of their thickness on the size of a traveling 4π kink and its velocity are ascertained. The dispersion equation for electromagnetic waves (generalized Swihart waves) in a layered superlattice consisting of an arbitrary number of tunnel junctions is derived using nonlocal Josephson electrodynamics. The spectra of generalized Swihart waves propagating in two coupled flat Josephson junctions and in a mirror-symmetric structure consisting of three Josephson junctions are found. It is shown that the group and phase velocities of short-wavelength electromagnetic waves depend strongly on the thickness of the superconducting layers. Unlike the case of the local theory,[7] it is found that the coupling of the waves due to the mutual influence of the neighboring tunnel junctions weakens significantly as the wave vector increases. It is shown in the appendix that the properties of Abrikosov–Josephson vortices in a superlattice consisting of N identical, closely arranged tunnel junctions are determined by the scale Nl, which is a multiple of l, i.e., the scale of a vortex in one junction.

2. BASIC EQUATIONS

Let us consider a tunnel structure consisting of three superconductors separated by two flat nonsuperconducting layers. We assume that the nonsuperconducting layers occupy regions of space in the ranges $-L-2d_1<x<-L$ and $L<x<L+2d_2$ and that Cooper pairs tunnel between them. We base the theory of vortices in such tunnel junctions on the equations of the nonlocal Josephson electrodynamics of layered superconducting structures presented in Ref. 17.

According to Ref. 17, for the phase differences of the wave functions φ_n and the magnetic fields on the boundaries of the Josephson junctions in the structure under study we have the following system of equations:

$$(\mathbf{e}_x \mathrm{curl}\mathbf{H}(x_n,\rho,t))=\frac{4\pi}{c}j_{nc}\left\{\sin\varphi_n(\rho,t)+\beta_n\omega_{jn}^{-2}\frac{\partial}{\partial t}\varphi_n(\rho,t)+\omega_{jn}^{-2}\frac{\partial^2}{\partial t^2}\varphi_n(\rho,t)\right\},\quad n=1,2, \tag{2.1}$$

$$\int d\rho'\int\frac{d\mathbf{k}}{(2\pi)^2}\exp[i\mathbf{k}(\rho-\rho')]\{-a_2(k)\mathbf{H}(x_2-d_2,\rho',t)+b_2(k)\mathbf{H}(x_1+d_1,\rho',t)+\lambda_1^2\sqrt{k^2+\lambda_1^{-2}}\mathbf{H}(x_1-d_1,\rho',t)+2d_1\mathbf{H}(x_1,\rho',t)\}=\frac{\hbar c}{2|e|}\left[\mathbf{e}_x\frac{\partial}{\partial\rho}\varphi_1(\rho,t)\right], \tag{2.2}$$

$$\int d\rho'\int\frac{d\mathbf{k}}{(2\pi)^2}\exp[i\mathbf{k}(\rho-\rho')]\{\lambda_3^2\sqrt{k^2+\lambda_3^{-2}}\times\mathbf{H}(x_2+d_2,\rho',t)+2d_2\mathbf{H}(x_2,\rho',t)+b_2(k)\times\mathbf{H}(x_2-d_2,\rho',t)-a_2(k)\mathbf{H}(x_1+d_1,\rho',t)\}=\frac{\hbar c}{2|e|}\left[\mathbf{e}_x\frac{\partial}{\partial\rho}\varphi_2(\rho,t)\right]. \tag{2.3}$$

Here the following notations are used: $x_1=-L-d_1$; $x_2=L+d_2$, ρ is a two-dimensional vector in the plane of the junction; j_{nc} is the critical Josephson current density in the nth tunnel junction; $\beta_n=4\pi\sigma_n/\varepsilon_n$; ε_n and σ_n are the dielectric constant of the nonsuperconducting nth layer; $\omega_{jn}=(16\pi|e|d_nj_{nc}/\hbar\varepsilon_n)^{1/2}$ is the Josephson frequency; λ_n is the London length in the nth superconductor; $\mathbf{e}_x=(1,0,0)$ is a unit vector along the x axis; $L_2=L$;

$$a_n(k)=\lambda_n^2\sqrt{k^2+\lambda_n^{-2}}\,\mathrm{cosech}(2L_n\sqrt{k^2+\lambda_n^{-2}}), \tag{2.4}$$

$$b_n(k)=\lambda_n^2\sqrt{k^2+\lambda_n^{-2}}\coth(2L_n\sqrt{k^2+\lambda_n^{-2}}). \tag{2.5}$$

Neglecting the weak variation of the magnetic field in tunnel junctions, from Eqs. (2.2) and (2.3) we find

$$\mathbf{H}_1(\rho,t)=\mathbf{H}(x_1,\rho,t)=\mathbf{H}(x_1\pm d_1,\rho,t),$$

$$\mathbf{H}_2(\rho,t)=\mathbf{H}(x_2,\rho,t)=\mathbf{H}(x_2\pm d_2,\rho,t).$$

The magnetic fields $\mathbf{H}_n(\rho,t)$ characterizing the interaction between the tunnel junctions are determined by the integrals of the phase differences of the wave functions $\varphi_n(\rho,t)$ in accordance with the relations

$$\mathbf{H}_n(\rho,t)=\frac{\hbar c}{2|e|}\sum_{n'=1,2}\int d\rho' Q_{nn'}(\rho-\rho')\times\left[\mathbf{e}_x\frac{\partial}{\partial\rho'}\varphi_{n'}(\rho',t)\right], \tag{2.6}$$

where the kernels $Q_{nn'}(\rho)$ have the form

$$Q_{11}(\rho)=\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{\exp(i\mathbf{k}\rho)}{\Delta(k)}\times[b_2(k)+\lambda_3^2\sqrt{k^2+\lambda_3^{-2}}+2d_2], \tag{2.7}$$

$$Q_{12}(\rho)=Q_{21}(\rho)\int\frac{d\mathbf{k}}{(2\pi)^2}\exp(i\mathbf{k}\rho)\frac{a_2(k)}{\Delta(k)}, \tag{2.8}$$

$$Q_{22}(\rho)=\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{\exp(i\mathbf{k}\rho)}{\Delta(k)}\times[\lambda_1^2\sqrt{k^2+\lambda_1^{-2}}+b_2(k)+2d_1], \tag{2.9}$$

$$\Delta(k)=[\lambda_1^2\sqrt{k^2+\lambda_1^{-2}}+b_2(k)+2d_1]\times[b_2(k)+\lambda_3^2\sqrt{k^2+\lambda_3^{-2}}+2d_2]-a_2^2(k). \tag{2.10}$$

Finally, substituting the magnetic field (2.6) into (2.1), we arrive at a system of two coupled integrodifferential equations for the phase differences

$$\sin\varphi_n(\rho,t)+\beta_n\omega_{jn}^{-2}\frac{\partial}{\partial t}\varphi_n(\rho,t)+\omega_{jn}^{-2}\frac{\partial^2}{\partial t^2}\varphi_n(\rho,t)=\sum_{n'=1,2}\lambda_{0n}^3\frac{\partial^2}{\partial\rho^2}\int d\rho' Q_{nn'}(\rho-\rho')\varphi_{n'}(\rho',t), \tag{2.11}$$

where $\lambda_{0n}^3=\hbar c^2/8\pi|e|j_{nc}$.

According to (2.11), the distribution of the phase differences is highly dependent on the form of the kernel of the integral operator $Q_{nn'}(\rho)$. These kernels have an especially simple form in cases in which the structure is prepared from superconductors with identical London lengths, i.e., $\lambda_1=\lambda_2=\lambda_3=\lambda$, and the thicknesses of the tunnel junctions can be neglected. Then

$$\Delta(k)=2\lambda^2(1+k^2\lambda^2)[+\coth(2L\sqrt{k^2+\lambda^{-2}})], \tag{2.12}$$

and the kernels (2.7)–(2.9) equal

$$Q_{11}(\rho)=Q_{22}(\rho)=\frac{1}{2\lambda}\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{\exp(i\mathbf{k}\rho)}{\sqrt{1+k^2\lambda^2}}$$

$$=\frac{1}{4\pi\rho\lambda^2}\exp\left(-\frac{\rho}{\lambda}\right), \tag{2.13}$$

$$Q_{12}(\rho)=Q_{21}(\rho)=\frac{1}{2\lambda}\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{\exp(i\mathbf{k}\rho)}{\sqrt{1+k^2\lambda^2}}$$

$$\times\exp[-2L\sqrt{k^2+\lambda^{-2}}]$$

$$=\frac{1}{4\pi\lambda^2\sqrt{\rho^2+4L^2}}$$

$$\times\exp\left[-\frac{1}{\lambda}\sqrt{\rho^2+4L^2}\right]. \tag{2.14}$$

The system of equations (2.11) together with the kernels (2.7)–(2.10) or (2.12)–(2.14) make it possible to investigate vortex formations which are two-dimensional in the planes of Josephson junctions. To study simpler Josephson vortices, which vary only in the z direction, only the derivatives with respect to z should be retained in Eqs. (2.11), and the kernels (2.7)–(2.10) should be replaced by their one-dimensional analogs, which are obtained from (2.7)–(2.10) by substituting kz for $\mathbf{k}\rho$ and $\int dk/2\pi$ for $\int d\mathbf{k}/(2\pi)^2$. In the case of superconductors with identical London lengths, in the one-dimensional theory instead of (2.13) and (2.14) we have

$$Q_{11}(z)=Q_{22}(z)=\frac{1}{2\pi\lambda^2}K_0\left(\frac{|z|}{\lambda}\right), \tag{2.15}$$

$$Q_{12}(z)=Q_{21}(z)=\frac{1}{2\pi\lambda^2}K_0\left(\frac{1}{\lambda}\sqrt{z^2+4L^2}\right), \tag{2.16}$$

where K_0 is McDonald's function.

3. STATIONARY ABRIKOSOV–JOSEPHSON VORTICES

To establish the new properties of Abrikosov–Josephson vortex structures caused by the mutual influence of tunnel junctions, let us focus on the one-dimensional case, in which φ_1 and φ_2 depend only on the coordinate z. Just such a case offers information on the properties of Abrikosov–Josephson vortices in an individual tunnel junction, which is needed for a comparison with the new results presented. For simplicity, we shall assume below that $\lambda_1=\lambda_2=\lambda_3=\lambda$ are identical in all the superconductor layers. We shall also assume that the critical currents in the tunnel junctions are identical ($j_{c1}=j_{c2}=j_c$, and, therefore, $\lambda_{01}=\lambda_{02}=\lambda_0$). We neglect the thickness of the tunnel junctions. Then in the stationary case, according to (2.11), (2.15), and (2.16), we have

$$\frac{l}{\pi}\frac{d^2}{dz^2}\int_{-\infty}^{\infty}dz'\left\{K_0\left(\frac{1}{\lambda}\sqrt{(z-z')^2+4L^2}\right)\varphi_2(z')\right.$$

$$\left.+K_0\left(\frac{|z-z'|}{\lambda}\right)\varphi_1(z')\right\}$$

$$=\sin\varphi_1(z), \tag{3.1}$$

$$\frac{l}{\pi}\frac{d^2}{dz^2}\int_{-\infty}^{\infty}dz'\left\{K_0\left(\frac{1}{\lambda}\sqrt{(z-z')^2+4L^2}\right)\varphi_1(z')\right.$$

$$\left.+K_0\left(\frac{|z-z'|}{\lambda}\right)\varphi_2(z')\right\}=\sin\varphi_2(z), \tag{3.2}$$

where $l=\lambda_0^3/2\lambda^2=\lambda_j^2/\lambda$ and λ_j is the Josephson length.

We next assume that the thickness of a superconducting layer L is small compared with the London length, and we consider the consequences of the system of equations (3.1) and (3.2) in the case of vortices with a characteristic scale of spatial variation which is much smaller than the London length. We call such a case the strongly nonlocal limit and use the following asymptotic expansions of the kernels:

$$K_0=(|z|/\lambda)\simeq\ln(2\lambda/|z|), \tag{3.3}$$

$$K_0\left(\frac{1}{\lambda}\sqrt{z^2+4L^2}\right)\simeq\frac{1}{2}\ln\frac{4\lambda^2}{z^2+4L^2}. \tag{3.4}$$

Then the system of equations (3.1) and (3.2) takes the form

$$\frac{l}{\pi}\int_{-\infty}^{\infty}dz'\left\{\frac{1}{z'-z}\frac{d}{dz'}\varphi_1(z')\right.$$

$$\left.+\frac{z'-z}{(z'-z)^2+4L^2}\frac{d}{dz'}\varphi_2(z')\right\}=\sin\varphi_1(z), \tag{3.5}$$

$$\frac{l}{\pi}\int_{-\infty}^{\infty}dz'\left\{\frac{z'-z}{(z'-z)^2+4L^2}\frac{d}{dz'}\varphi_1(z')\right.$$

$$\left.+\frac{1}{z'-z}\frac{d}{dz'}\varphi_2(z')\right\}=\sin\varphi_2(z). \tag{3.6}$$

If L is sufficiently large, the coupling between these two equations can be neglected. We then obtain an equation which describes vortices in an individual tunnel junction

$$\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{d}{dz'}\varphi(z')=\sin\varphi(z). \tag{3.7}$$

Conversely, if L is small compared with the characteristic spatial scale of variation of the vortex, the system of equations (3.5) and (3.6) takes the form

$$\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{d}{dz'}[\varphi_1(z')+\varphi_2(z')]=\sin\varphi_1(z), \tag{3.8}$$

$$\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{d}{dz'}[\varphi_1(z')+\varphi_2(z')]=\sin\varphi_2(z). \tag{3.9}$$

An obvious solution of this system of two equations is $\varphi_1(z)=\varphi_2(z)=\psi(z)$, where $\psi(z)$ obeys the equation

$$\frac{2}{\pi}l\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{d}{dz'}\psi(z')=\sin\psi(z). \tag{3.10}$$

The difference between Eqs. (3.10) and (3.7) is confined to the doubling of the coefficient on the left-hand side of Eq.

(3.10). Therefore, for example, in the case of a solitary Abrikosov-Josephson vortex Eq. (3.10) gives

$$\psi(z)=\pi+2\arctan\frac{z}{2l}, \tag{3.11}$$

which is distinguished from the corresponding solution of Eq. (3.7) (Ref. 11)

$$\psi(z)=\pi+2\arctan\frac{z}{l} \tag{3.12}$$

by a twofold increase in the scale of spatial variation of the vortex.

In the case of a stationary train of Abrikosov-Josephson vortices with a nonzero mean magnetic field

$$\bar{H}=\frac{\phi_0}{4\pi\lambda L_H}, \tag{13}$$

where $\phi_0=\hbar c/|e|=2.05\times10^{-7}$ Oe/cm^2 is the magnetic flux quantum and L_H characterizes the periodicity of the vortex structure, Eq. (3.10) gives

$$\psi(z)=\pi+2\arctan\left[\left(\sqrt{\frac{L_H^2}{4l^2}+1}+\frac{L_H}{2l}\right)\tan\frac{z}{2L_H}\right]. \tag{3.14}$$

The difference between this expression and the corresponding solution of Eq. (3.7) (Ref. 16)

$$\varphi(z)=\pi+2\arctan\left[\left(\sqrt{\frac{L_H^2}{l^2}+1}+\frac{L_H}{l}\right)\tan\frac{z}{2L_H}\right] \tag{3.15}$$

is similar to the case of (3.11).

We now turn to the limit of large $L\gg l$, in which the terms containing L on the left-hand sides of Eqs. (3.5) and (3.6) can be assumed to be small. Then we can consider the coherent perturbation which appears under the action of the vortex in the first tunnel junction

$$\varphi_1\simeq\pi+2\arctan\frac{z}{l} \tag{3.16}$$

in the other junction. For a small perturbation $\varphi_2(z)$ we can write the following equation:

$$\varphi_2(z)-\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{d}{dz'}\varphi_2(z')=\frac{2lz}{z^2+(2L+l)^2}. \tag{3.17}$$

The solution of this equation can be written in quadratures

$$\varphi_2(z)=\exp[(l+2L+iz)/l]\mathrm{Ei}[-(l+2L+iz)/l]+\text{h.c.}, \tag{3.18}$$

where

$$\mathrm{Ei}(-x)=-\int_x^{\infty}e^{-t}dt/t$$

is an exponential integral. However, it is clear that the second integral term on the left-hand side of Eq. (3.17) is relatively small when $L\gg l$. Therefore, it can be asserted with high accuracy that the vortex (3.16) in the first junction is accompanied by the bisoliton perturbation

$$\varphi_2(z)\simeq-\frac{2lz}{z^2+4L^2} \tag{3.19}$$

in the other junction.

In analogy to the treatment just presented for the case of one solitary vortex [the 2π kink (3.16)] in the first junction, we can consider the case of the train of vortices (3.15) in the first junction. According to Eq. (3.6), at sufficiently large $L\gg l$ the following cophased perturbation then appears in the other junction:

$$\begin{aligned}\varphi_2(z)\simeq&\frac{l^2}{\pi L_H^2}\int_{-\infty}^{\infty}\\ &\times\frac{dz'(z'-z)}{(z'-z)^2+4L^2}\frac{1}{\sqrt{1+(l/L_H)^2}-\cos(z'/L_H}\\ =&-\frac{l}{L_H}\sin\left(\frac{z}{L_H}\right)\left[\sqrt{1+\left(\frac{l}{L_H}\right)^2}\cosh\left(\frac{2L}{L_H}\right)\right.\\ &\left.+\frac{l}{L_H}\sinh\left(\frac{2L}{L_H}\right)-\cos\left(\frac{z}{L_H}\right)\right]^{-1}.\end{aligned} \tag{3.20}$$

Thus, comparatively simple limiting equations describing static coherent vortex structures which exhibit cophasal behavior in two tunnel junctions separated by a superconducting layer have been derived in this section.

4. NONDISSIPATIVE TRAVELING ABRIKOSOV–JOSEPHSON VORTICES

In this section we shall discuss consequences of the theory which describe the interaction of vortices in different tunnel junctions in cases in which such an interaction is realized for nonstationary traveling vortices. In the process we shall bear in mind that in the nondissipative case,[1)] for an individual isolated Josephson junction in the strongly nonlocal limit, in which the phase difference between the wave functions of the Cooper pairs φ is described by the equation

$$\omega_j^{-2}\frac{\partial^2}{\partial t^2}\varphi(z,t)+\sin\varphi(z,t)=\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{\partial}{\partial z'}\varphi(z',t), \tag{4.1}$$

there is a known solution corresponding to a 4π kink[14,15]

$$\varphi(z,t)=4\arctan\frac{z-vt}{l}, \tag{4.2}$$

moving with a velocity $v=l\omega_j$, and there is a known solution corresponding to a periodic structure of Abrikosov–Josephson vortices[16]

$$\varphi(z,t)=4\arctan\left[\sqrt{\frac{l\omega_j+v}{l\omega_j-v}}\tan\frac{z-vt}{2L_H}\right], \tag{4.3}$$

moving with a velocity given by the expression

$$v=\omega_jL_H\left[\frac{1}{2}(\sqrt{1+4l^2/L_H^2}-1)\right]^{1/2}. \tag{4.4}$$

According to Eqs. (3.5) and (3.6), the system of equations describing the dynamics of vortices in two interacting tunnel junctions with the neglect of dissipation in the strongly nonlocal limit has the following form

$$\omega_j^{-2}\frac{\partial^2}{\partial t^2}\varphi_1(z,t)+\sin\varphi_1(z,t)=\frac{l}{\pi}\int_{-\infty}^{\infty}dz'\left[\frac{1}{z'-z}\frac{\partial}{\partial z'}\varphi_1(z',t)+\frac{z'-z}{(z'-z)^2+4L^2}\frac{\partial}{\partial z'}\varphi_2(z',t)\right], \quad (4.5)$$

$$\omega_j^{-2}\frac{\partial^2}{\partial t^2}\varphi_2(z,t)+\sin\varphi_2(z,t)=\frac{l}{\pi}\int_{-\infty}^{\infty}dz'\left[\frac{z'-z}{(z'-z)^2+4L^2}\frac{\partial}{\partial z'}\varphi_1(z',t)+\frac{1}{z'-z}\frac{\partial}{\partial z'}\varphi_2(z',t)\right]. \quad (4.6)$$

Here it is assumed that the Josephson frequencies in the two tunnel junctions are identical.

For vortices traveling with a constant velocity v, where

$$\varphi_{1,2}(z,t)=\varphi_{1,2}(\xi), \quad \xi=z-vt, \quad (4.7)$$

the system of equations (4.6) and (4.5) has the form

$$\frac{v^2}{\omega_j^2}\varphi_1''(\xi)+\sin\varphi_1(\xi)=\frac{l}{\pi}\int_{-\infty}^{\infty}d\xi'\left[\frac{1}{\xi'-\xi}\varphi_1'(\xi')+\frac{\xi'-\xi}{(\xi'-\xi)^2+4L^2}\varphi_2'(\xi')\right], \quad (4.8)$$

$$\frac{v^2}{\omega_j^2}\varphi_2''(\xi)+\sin\varphi_2(\xi)=\frac{l}{\pi}\int_{-\infty}^{\infty}d\xi'\left[\frac{1}{\xi'-\xi}\varphi_2'(\xi')+\frac{\xi'-\xi}{(\xi'-\xi)^2+4L^2}\varphi_1'(\xi')\right], \quad (4.9)$$

In the limit $L\ll l$ these equations take the form

$$\frac{v^2}{\omega_j^2}\varphi_{1,2}''(\xi)+\sin\varphi_{1,2}(\xi)=\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\times[\varphi_1'(\xi')+\varphi_2'(\xi')]. \quad (4.10)$$

Accordingly, we have the following solution, which corresponds to cophasal 4π kinks

$$\varphi_1(z-vt)=\varphi_2(z-vt)=4\arctan\frac{z-vt}{2l} \quad (4.11)$$

traveling in two tunnel junctions with the velocity

$$v=2l\omega_j. \quad (4.12)$$

The interaction of the vortices in two closely positioned identical tunnel junctions led to doubling of the characteristic scale of variation of each 4π kink and a twofold increase in its velocity (4.12).

If the distance between the tunnel junctions is comparatively large ($\lambda>L>l$), in an approximation we have

$$\varphi_1(z,t)\simeq 4\arctan(z/l-\omega_j t), \quad (4.13)$$

and for the weak vortex coherently induced in the second tunnel junction by the 4π kink (4.13), we have the following equation:

$$l^2\varphi_2''(\xi)+\varphi_2(\xi)-\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\varphi_2'(\xi')=-\frac{4l\xi}{\xi^2+4L^2}. \quad (4.14)$$

Bearing in mind the smallness of l in comparison to L, from (4.14) we obtain the simple approximate solution

$$\varphi_2(\xi)\simeq-\frac{4l\xi}{\xi^2+4L^2}, \quad (4.15)$$

which describes a bisoliton excitation traveling in the second tunnel junction, which is coherently induced by the 4π kink traveling in the first junction.

Let us now turn to the case of a traveling periodic structure of Abrikosov–Josephson vortices. In this case Eqs. (4.10) have the following solution instead of the structure (4.3) for a single tunnel junction:

$$\varphi_1(z-vt)=\varphi_2(z-vt)=4\arctan\left[\sqrt{\frac{2l\omega_j+v}{2l\omega_j-v}}\tan\frac{z-vt}{2L_H}\right], \quad (4.16)$$

where, in contrast to (4.4), the velocity of the traveling periodic train of Abrikosov–Josephson vortices is given by the relation

$$v^2=\frac{1}{2}\omega_j^2L_H^2\left(\sqrt{1+16\frac{l^2}{L_H^2}}-1\right). \quad (4.17)$$

Thus, when the distance between the tunnel junctions is small, the interaction of vortices in different junctions results in significant modification of both the velocity of the vortices and the scale of their spatial variation.

If the distance between the tunnel junctions is great and if we assume in an approximation that $\varphi_1(z,t)\simeq\varphi(z,t)$ in one of the junctions, where φ is given by (4.3) and (4.4), we have the following equation for $\varphi_2(z,t)\simeq\varphi_2(\xi)$:

$$\frac{v^2}{\omega_j^2}\varphi_2''(\xi)+\varphi_2(\xi)-\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\varphi_2'(\xi')=-\frac{2lv}{L_H}\sin\frac{\xi}{L_H}\times\left[l\omega_j\cosh\frac{2L}{L_H}+\sqrt{l^2\omega_j^2-v^2}\times\sinh\frac{2L}{L_H}-v\cos\frac{\xi}{L_H}\right]^{-1}. \quad (4.18)$$

If $l\ll 2L\ll L_H$, from this expression we obtain

$$\varphi_2(\xi)=-\frac{l}{L_H}\sin\frac{\xi}{L_H}\left[\sin^2\left(\frac{\xi}{2L_H}\right)+\frac{L^2}{L_H^2}\right]^{-1}. \quad (4.19)$$

The smallness of the vortex perturbation (4.19) is stipulated by the large value of $2L_H/l$.

If both l and L_H are small compared with the distance between the tunnel junctions, the perturbation in the second tunnel junction is given, according to Eq. (4.18), by the expression

$$\varphi_2(\xi)=-\frac{4lv}{L_H}\frac{\sin(\xi/L_H)}{l\omega_j+\sqrt{l^2\omega_j^2-v^2}}\exp\left(-\frac{2L}{2L_H}\right). \quad (4.20)$$

The relations obtained in this section characterize simple coherent couplings of traveling vortex structures in two Josephson junctions caused by a magnetic field penetrating through the superconductor separating the junctions.

5. DISSIPATIVE ABRIKOSOV–JOSEPHSON VORTEX STRUCTURES

Numerous exact solutions of the equation of the nonlocal Josephson electrodynamics of an isolated tunnel junction that describe vortices under conditions under which dissipation is quite great are presently known.[2)] The corresponding equation for the phase difference between the wave functions of Cooper pairs which describes small-scale structures of Abrikosov–Josephson vortices has the form

$$\beta\omega_j^{-2}\frac{\partial}{\partial t}\varphi(z,t)+\sin\varphi(z,t)+i=\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{\partial}{\partial z'}\varphi(z',t). \quad (5.1)$$

Here $\omega_j^2/\beta=j_cR_s(2|e|/\hbar)$, and $i=j/j_c$, where j is the current density and R_s is the resistance per unit area of the tunnel junction. We compare the expressions obtained below which describe Abrikosov–Josephson vortices modified by their interaction in two Josephson junctions with some known solutions of Eq. (5.1) for Abrikosov–Josephson vortices in an isolated junction. We, first of all, write down the solution of Eq. (5.1) describing a traveling solitary vortex (a 2π kink):[18]

$$\varphi(z,t)=-\theta+\pi+2\arctan\left(\frac{z-vt}{l}\sqrt{1-i^2}\right), \quad (5.2)$$

where

$$\theta=\arcsin i,\quad v=l\omega_j^2\beta^{-1}i/\sqrt{1-i^2}. \quad (5.3)$$

In addition, a train of traveling dissipative Abrikosov–Josephson vortices in one tunnel junction is described by the expression

$$\varphi(z,t)=-\theta+\pi+2\arctan\left\{\frac{\tan[(z-vt)/2L_H]}{\tanh(\alpha/2)}\right\}, \quad (5.4)$$

where

$$\cos^2\theta=l^2/L_H^2\sinh^2\alpha, \quad (5.5)$$

$$\sinh^2\alpha=\left[\frac{1}{4}\left(1-i^2-\frac{l^2}{L_H^2}\right)^2+\frac{l^2}{L_H^2}\right]^{1/2}-\frac{1}{2}\left(1-i^2-\frac{l^2}{L_H^2}\right), \quad (5.6)$$

$$v=\frac{\omega_j^2}{\beta}L_H\left\{\left[\frac{1}{4}\left(1-i^2-\frac{l^2}{L_H^2}\right)^2+\frac{l^2}{L_H^2}\right]^{1/2}-\frac{1}{2}\left(1-i^2-\frac{l^2}{L_H^2}\right)\right\}^{1/2}. \quad (5.7)$$

In the case of two Josephson tunnel junctions discussed below, the dissipative small-scale Abrikosov–Josephson vortices are described by the system of equations

$$i+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi_1(z,t)+\sin\varphi_1(z,t)=\frac{l}{\pi}\int_{-\infty}^{\infty}dz'\left[\frac{1}{z'-z}\frac{\partial}{\partial z'}\varphi_1(z',t)+\frac{z'-z}{(z'-z)^2+4L^2}\frac{\partial}{\partial z'}\varphi_2(z',t)\right], \quad (5.8)$$

$$i+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi_2(z,t)+\sin\varphi_2(z,t)=\frac{l}{\pi}\int_{-\infty}^{\infty}dz'\left[\frac{1}{z'-z}\frac{\partial}{\partial z'}\varphi_2(z',t)+\frac{z'-z}{(z'-z)^2+4L^2}\frac{\partial}{\partial z'}\varphi_1(z',t)\right], \quad (5.9)$$

Here it is assumed that the resistances of the junctions are identical. For vortices traveling with a constant velocity v, the dependence on the coordinate z and the time t is described by the variable $\xi=z-vt$[compare (4.7)]. Accordingly, Eqs. (5.8) and (5.9) give

$$i+\frac{v\beta}{\omega_j^2}\varphi_1'(\xi)+\sin\varphi_1(\xi)=\frac{l}{\pi}\int_{-\infty}^{\infty}d\xi'\left[\frac{1}{\xi'-\xi}\varphi_1'(\xi')+\frac{\xi'-\xi}{(\xi'-\xi)^2+4L^2}\varphi_2'(\xi')\right], \quad (5.10)$$

$$i+\frac{v\beta}{\omega_j^2}\varphi_2'(\xi)+\sin\varphi_2(\xi)=\frac{l}{\pi}\int_{-\infty}^{\infty}d\xi'\left[\frac{1}{\xi'-\xi}\varphi_2'(\xi')+\frac{\xi'-\xi}{(\xi'-\xi)^2+4L^2}\varphi_1'(\xi')\right]. \quad (5.11)$$

In the limit of closely arranged tunnel junctions, where $L\ll l$, these equations have the form

$$i+\frac{v\beta}{\omega_j^2}\varphi_{1,2}'(\xi)+\sin\varphi_{1,2}(\xi)=\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}[\varphi_1'(\xi')+\varphi_2'(\xi')]. \quad (5.12)$$

The solution of these equations describing 2π kinks moving synchronously in two layers has the form

$$\varphi_1(\xi)=\varphi_2(\xi)=-\theta+\pi+2\arctan\left(\frac{z-vt}{2l}\sqrt{1-i^2}\right), \tag{5.13}$$

where

$$\theta=\arcsin i,\quad v=2l\omega_j^2\beta^{-1}i/\sqrt{1-i^2}. \tag{5.14}$$

The synchronized vortices in two tunnel junctions described by (5.13) have a doubled size in comparison to a vortex in an isolated tunnel junction at the same value of the current i, and their velocity is also doubled.

For trains of traveling Abrikosov–Josephson vortices moving synchronously in two Josephson junctions, Eqs. (5.12) have the solutions $\varphi_1=\varphi_2=\varphi_2(\xi)$, where $\varphi(\xi)$ is given by (5.4). Here the parameters θ, α, and v are described by the expressions

$$\cos^2\theta=4l^2/L_H^2\sinh^2\alpha, \tag{5.15}$$

$$\sinh^2\alpha=\left[\frac{1}{4}\left(1-i^2-4\frac{l^2}{L_H^2}\right)^2+4\frac{l^2}{L_H^2}\right]^{1/2}-\frac{1}{2}\left(1-i^2-4\frac{l^2}{L_H^2}\right), \tag{5.16}$$

$$v=\frac{\omega_j^2}{\beta}L_H\left\{\left[\frac{1}{4}\left(1-i^2-4\frac{l^2}{L_H^2}\right)^2+4\frac{l^2}{L_H^2}\right]^{1/2}-\frac{1}{2}\left(1-i^2-4\frac{l^2}{L_H^2}\right)\right\}^{1/2}, \tag{5.17}$$

which are distinguished from (5.5)–(5.7) by the replacement of l by $2l$.

Let us now dwell on the current-voltage characteristic of two tunnel junctions in which periodic trains of dissipative traveling Abrikosov–Josephson vortices move synchronously. For the value of the electric potential averaged over the period of a train we have (compare Ref. 18)

$$V=\hbar v/2|e|L_H, \tag{5.18}$$

where v is given by (5.17). Therefore, for the dependence of the voltage on the dimensionless current $i=j/j_c$ we obtain

$$\frac{V}{j_cR_s}=\left\{\left[\frac{1}{4}\left(1-\frac{j^2}{j_c^2}-4\frac{l^2}{L_H^2}\right)^2+4\frac{l^2}{L_H^2}\right]^{1/2}-\frac{1}{2}\left(1-\frac{j^2}{j_c^2}-4\frac{l^2}{L_H^2}\right)\right\}^{1/2}. \tag{5.19}$$

This equation can be written in the form of the dependence of the current density on the voltage

$$\frac{j^2}{j_c^2}=\frac{V^2}{R_s^2j_c^2}+\left[1+\left(2\frac{l}{L_H}\frac{j_cR_s}{V}\right)^2\right]^{-1}. \tag{5.20}$$

If

$$V\gg 2lj_cR_s/L_H, \tag{5.21}$$

Eq. (5.20) gives[3)]

$$j^2=j_c^2+V^2/R_s^2, \tag{5.22}$$

which corresponds to the ordinary Ohm's law when

$$V>R_sj_c. \tag{5.23}$$

In the limit

$$V\ll 2lj_cR_s/L_H \tag{5.24}$$

it follows from (5.20) that

$$j=\frac{V}{R_s}\sqrt{1+L_H^2/4l^2}. \tag{5.25}$$

In the case of $4l^2\ll L_H^2$, Eq. (5.25) gives

$$j=\frac{V}{R_s}\frac{L_H}{2l}. \tag{5.26}$$

At small voltages (see 5.24) this corresponds to a decrease in the resistance of the two tunnel junctions with moving trains of Abrikosov–Josephson vortices by a factor of $2l/L_H$ in comparison to the ohmic resistance.

The linear increase in the current according to the law (5.26) ceases upon attainment of a voltage $V\sim 2lj_cR_s/L_H$, at which the current density approaches the critical value. If $2l\ll L_H$, the current density subsequently varies slowly in the range of voltages up to $V\sim j_cR_s$. At larger voltages the variation of the current approximates an ohmic law.

We use (5.19) to describe the current-voltage characteristic of a structure having the form of a ring. We assume that the radius of such a ring is much greater than the London length and the distance L between two distributed Josephson junctions. Accordingly, we use Eqs. (5.10)–(5.11) for an approximate description of the Abrikosov–Josephson vortices in such a structure (compare Refs. 24 and 25). If it is assumed that $R=mL_H$ in the solution of (5.4) and (5.15)–(5.17), it will describe the picture of a set of cophasal vortices traveling along two annular tunnel junctions, which, for example, approximate m single Abrikosov–Josephson vortices when the value of L is large. According to Eq. (5.19), in this case the current-voltage characteristic can be written in the form

$$\frac{V}{R_sj_c}=\left\{\left[\frac{1}{4}\left(\frac{j^2}{j_c^2}+4\frac{m^2l^2}{R^2}-1\right)^2+4\frac{m^2l^2}{R^2}\right]^{1/2}+\frac{1}{2}\left(\frac{j^2}{j_c^2}-4\frac{m^2l^2}{R^2}-1\right)\right\}^{1/2}, \tag{5.27}$$

or

$$\frac{j^2}{j_c^2}=\left(\frac{V}{R_sj_c}\right)^2+\left[1+\left(2\frac{ml}{R}\frac{j_cR_s}{V}\right)^2\right]^{-1}. \tag{5.28}$$

Figure 1 presents the dependence of the current on the voltage averaged around the ring when $l/R=0.1$. The dimensionless current $i=j/j_c$ is plotted along the vertical axis, and V/R_sj_c is plotted along the horizontal axis. The dashed straight line corresponds to Ohm's law $V=jR_s$. Curves 1, 2, and 3 correspond to $m=1$, 2, and 3, respectively, i.e., to the cases of one, two, and three vortices in the annular structure. It is seen from Fig. 1 that the current increases sharply with increasing voltage when $V/R_sj_c<ml/R$. The increase subsequently slows, and when $V/R_sj_c\gg 2ml/R$ the dependence of the current on the voltage is described by the approximate law[19–22]

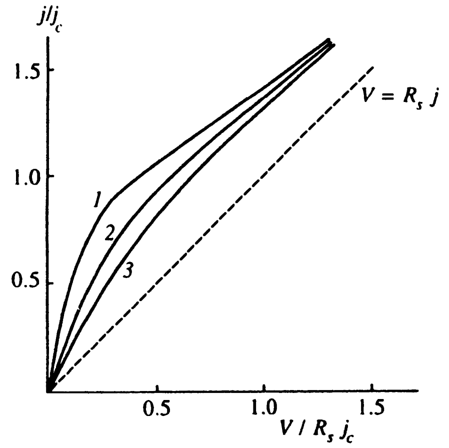

FIG. 1. Dependence of the current on the voltage averaged around the ring in the case of two closely arranged annular Josephson junctions for $l/R=0.1$ and $m=1, 2, 3$, where $m=R/L_H$.

$$j/j_c=\sqrt{1+(V/R_s j_c)^2}, \tag{5.29}$$

which transforms into Ohm's law when $V \gg R_s j_c$. Plots of the dependence of the current on the voltage for $m=1$ and several values of l/R are presented in Fig. 2. According to Fig. 2, an increase in the ratio l/R results in the formation of a smoother current–voltage characteristic.

6. VORTICES IN STRUCTURES OF FINITE THICKNESS

In an experiment a layered structure of Josephson junctions has a finite thickness. In this section we shall follow the work of Alfimov and Popkov[27] and turn to a study of the properties of vortices in distributed Josephson junctions, in which the outer superconducting layers (the electrodes) have a finite thickness. In contrast to Ref. 27, we shall focus on the case of two tunnel junctions. In addition, we shall establish the relationship to the treatment in Ref. 26. Let us, first of all, examine how the finite thickness of the outer superconductors influences the equations for the phase differences of the wave functions under the condition that there is a magnetic field outside the structure. In this case, neglecting the thicknesses of the nonsuperconducting layers, in accordance with the general rules in Ref. 17 for determining the magnetic fields in tunnel junctions we have the system of equations [compare (2.2) and (2.3)]:

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2} \exp[i\mathbf{k}(\rho-\rho')]\{-a_2(k)\mathbf{H}_2(\rho',t) \\ -a_1(k)\mathbf{H}_\infty(\rho',t)+[b_1(k)+b_2(k)]\mathbf{H}_1(\rho',t)\} \\ =\frac{\hbar c}{2|e|}\left[e_x \frac{\partial}{\partial \rho}\varphi_1(\rho,t)\right], \tag{6.1}$$

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2} \exp[i\mathbf{k}(\rho-\rho')]\{-a_2(k)\mathbf{H}_1(\rho',t) \\ -a_3(k)\mathbf{H}_\infty(\rho',t)+[b_2(k)+b_3(k)]\mathbf{H}_2(\rho',t)\} \\ =\frac{\hbar c}{2|e|}\left[e_x \frac{\partial}{\partial \rho}\varphi_2(\rho,t)\right], \tag{6.2}$$

where $\mathbf{H}_\infty$ is the magnetic field at the boundaries of the layered structure. It was assumed in Eqs. (6.1) and (6.2) that the thicknesses of the superconducting electrodes are equal to L_1 and L_3, respectively, and that the thickness of the superconducting layer between the tunnel junctions is equal to L_2. Accordingly, $\mathbf{H}_1$ and $\mathbf{H}_2$ are the magnetic fields in the first and second tunnel junctions. Treating (6.1) and (6.2) as a system, we note that when the thickness of the inner superconducting layer is equal to zero, in which case $\mathbf{H}_1(\rho,t)=\mathbf{H}_2(\rho,t)=\mathbf{H}(\rho,t)$ and $b_2(k)-a_2(k)=0$, the summation of Eqs. (6.1) and (6.2) gives an equation which describes a single Josephson junction between superconductors of finite thickness

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2} \exp[i\mathbf{k}(\rho-\rho')]\{[b_1(k) \\ +b_3(k)]\mathbf{H}(\rho',t)-+[a_1(k)+a_3(k)]\mathbf{H}_\infty(\rho',t)\} \\ =\frac{\hbar c}{2|e|}\left[e_x \frac{\partial}{\partial \rho}\varphi(\rho,t)\right], \tag{6.3}$$

where $\varphi(\rho,t)=\varphi_1(\rho,t)+\varphi_2(\rho,t)$ is the phase difference of the single Josephson junction formed in the limiting transition. In the special case of a uniform external magnetic field $\mathbf{H}_\infty(\rho,t)=\mathbf{H}_\infty(t)$, Eq. (6.3) is a natural generalization of the one-dimensional equation for the magnetic field obtained in Ref. 26 under conditions under which the magnetic field varies in two directions in the plane of the tunnel junction.

The solution of the system of integral equations (6.1) and (6.2) has the form [compare (2.6)]

$$\mathbf{H}_n(\rho,t)=\frac{\hbar c}{2|e|}\sum_{n'=1,2}\int d\rho' Q_{nn'}(\rho-\rho') \\ \times\left[\mathbf{e}_x \frac{\partial}{\partial \rho'}\varphi_{n'}(\rho',t)\right] \\ +\int d\rho' Q_{nH}(\rho-\rho')\mathbf{H}_\infty(\rho',t). \tag{6.4}$$

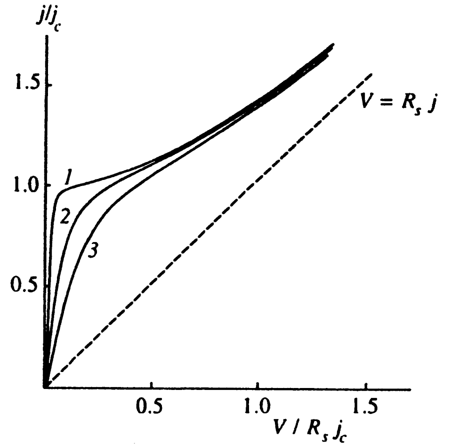

FIG. 2. Current–voltage characteristic of two closely arranged annular Josephson junctions for $R=L_H$ and three values of l/R: *1*) 0.01; *2*) 0.05; *3*) 0.1.

Here the kernels of the integral operators are described by the expressions

$$Q_{11}(\rho)=\int \frac{d\mathbf{k}}{(2\pi)^2}\exp(i\mathbf{k}\rho)[b_2(k)+b_3(k)]/\Delta_L(k), \quad (6.5)$$

$$Q_{12}(\rho)=Q_{21}(\rho)=\int \frac{d\mathbf{k}}{(2\pi)^2}\exp(i\mathbf{k}\rho)a_2(k)/\Delta_L(k), \quad (6.6)$$

$$Q_{22}(\rho)=\int \frac{d\mathbf{k}}{(2\pi)^2}\exp(i\mathbf{k}\rho)[b_1(k)+b_2(k)]/\Delta_L(k), \quad (6.7)$$

$$Q_{1H}(\rho)=\int \frac{d\mathbf{k}}{(2\pi)^2}\exp(i\mathbf{k}\rho)\{a_1(k)[b_2(k)+b_3(k)] +a_2(k)a_3(k)\}/\Delta_L(k), \quad (6.8)$$

$$Q_{2H}(\rho)=\int \frac{d\mathbf{k}}{(2\pi)^2}\exp(i\mathbf{k}\rho)\{a_3(k)[b_1(k)+b_2(k)] +a_1(k)a_2(k)\}/\Delta_L(k), \quad (6.9)$$

where we used the notation

$$\Delta_L(k)=[b_1(k)+b_2(k)][b_2(k)+b_3(k)]-a_2^2(k). \quad (6.10)$$

Equations (6.4)–(6.10) are distinguished from the Eqs. (2.6)–(2.10), which describe an unbound system of two Josephson junctions, first, by the appearance of the additional term stipulated by the magnetic field outside the structure in (6.4) and, second, by the changes in the form of the kernels $Q_{nn'}(\rho)$. Now the $Q_{nn'}(\rho)$ contain the dependence on the thickness of the outer superconductors. Plugging the magnetic field (6.4) into the equation for the phase difference (2.1), we find

$$\sin\varphi_n(\rho,t)+\frac{\beta_n}{\omega_{jn}^2}\frac{\partial}{\partial t}\varphi_n(\rho,t)+\frac{1}{\omega_{jn}^2}\frac{\partial^2}{\partial t^2}\varphi_n(\rho,t) =\lambda_{0n}^3\sum_{n'=1,2}\frac{\partial^2}{\partial\rho^2}\int d\rho' Q_{nn'}(\rho-\rho')\varphi_{n'}(\rho',t) +\frac{c}{4\pi j_{nc}}\int d\rho' Q_{nH}(\rho-\rho')(\mathbf{e}_x \operatorname{curl}\mathbf{H}_\infty(\rho',t)). \quad (6.11)$$

Unlike (2.11), these equations contain the new kernels $Q_{nn'}(\rho)$ of the integral operators and the source of phase differences created by the nonuniformity of the magnetic field outside the structure. The system of equations (6.11) makes it possible to discuss the excitation of vortices by an external magnetic field, as well as to investigate Abrikosov–Josephson vortices with a magnetic field outside the layered structure itself. In the case of a uniform magnetic field outside the structure, in which $\mathbf{H}_\infty(\rho,t)=\mathbf{H}_\infty(t)$, the source of phase differences in (6.11) vanishes (see also Ref. 22).

If the phase differences and the magnetic field $\mathbf{H}_\infty=(0,H_\infty,0)$ are nonuniform only in the direction of the z axis, Eqs. (6.11) take the form

$$\sin\varphi_n(z,t)+\beta_n\omega_{jn}^{-2}\frac{\partial}{\partial t}\varphi_n(z,t)+\omega_{jn}^{-2}\frac{\partial^2}{\partial t^2}\varphi_n(z,t) =\lambda_{0n}^3\sum_{n'=1,2}\frac{\partial^2}{\partial z^2}\int_{-\infty}^{\infty}dz' Q_{nn'}(z-z')\varphi_{n'}(z',t) -\frac{c}{4\pi j_{nc}}\int_{-\infty}^{\infty}dz' Q_{nH}(z-z')\frac{\partial}{\partial z'}H_\infty(z',t), \quad (6.12)$$

where the kernels $Q_{nn'}$ and Q_{nH} are described by the relations (6.5)–(6.10), in which $\mathbf{k}\rho$ should be replaced by kz and $\int d\mathbf{k}/(2\pi)^2$ should be replaced by $\int dk/2\pi$. Let us examine Eqs. (6.12) in the case in which the superconductors have the identical London lengths $\lambda_n=\lambda$ and the critical current densities in the Josephson junctions are equal to j_c.

In the static limit and in the case of one tunnel junction, the system of equations (6.12) transforms into the equation obtained in Ref. 27. We note that the nonstationary treatment used here corresponds to a quasimagnetostatic approach. We are interested in vortices having dimensions smaller than the London length. In addition, we assume that the thickness of the inner superconductor $L_2=L$ is much smaller than both the dimensions of a vortex and the thicknesses of the outer superconductors, which we assume are equal to $L_1=L_3=L_B$. Then, focusing on the stationary distribution of the phase differences in a layered structure with a uniform external magnetic field, from (6.12) we have

$$\sin\varphi_1(z)=\frac{l}{4L_B}⨍_{-\infty}^{\infty}dz' \sinh^{-1}\left[\frac{\pi}{4L_B}(z'-z)\right][\varphi_1'(z') +\varphi_2'(z')], \quad (6.13)$$

$$\sin\varphi_2(z)=\frac{l}{4L_B}⨍_{-\infty}^{\infty}dz' \sinh^{-1}\left[\frac{\pi}{4L_B}(z'-z)\right] \times[\varphi_1'(z')+\varphi_2'(z')]. \quad (6.14)$$

Using the result in Ref. 28, which was obtained in the theory of dislocations (see also Ref. 27), we can write the cophasal solution of the system of equations (6.13) and (6.14)

$$\varphi_1(z)=\varphi_2(z)=\pi+2\arctan\left[\frac{\sinh(\pi z/D)}{\cos(2\pi L_B/D)}\right], \quad (6.15)$$

where the spatial scale $D>4L_B$ is found from the equation

$$\frac{2\pi l}{D}=\cot\left(\frac{2\pi}{D}L_B\right). \quad (6.16)$$

When $L_B\gg l$, the solution consisting of (6.15) and (6.16) transforms into (3.11), which was found in the third section for unrestricted outer superconductors. Conversely, when $l\gg L_B$, from (6.16) we have

$$D=2\pi\sqrt{lL_B}\ll 2\pi l, \quad (6.17)$$

which describes the decrease in the size of a vortex in a structure of small thickness. We also note that the solution consisting of (6.15) and (6.16) is distinguished from the solution found in Ref. 27 for a single Josephson junction by the replacement of l by the larger value $2l$.

Now let us examine the one-dimensional nonstationary synchronized distribution of the phase differences in a system of two closely positioned nondissipative junctions formed by identical superconductors of finite thickness. Here, for the functions $\varphi_1(z-vt)=\varphi_2(z-vt)=\varphi(\xi)$, where $\xi=z-vt$, we have the equation

$$\frac{v^2}{\omega_j^2}\varphi''(\xi)+\sin\varphi(\xi)=\frac{l}{2L_B}\int_{-\infty}^{\infty} d\xi'\,\sinh^{-1}\left[\frac{\pi}{4L_B}(\xi'-\xi)\right]\varphi'(\xi'). \quad (6.18)$$

The integral equation (6.18) has a solution of the traveling 4π kink type

$$\varphi(z-vt)=4\arctan\left\{\frac{\sinh[\pi(z-vt)/D]}{\cos(2\pi L_B/D)}\right\} \quad (6.19)$$

with a spatial scale D and a velocity v, which depends on the thickness L_B of the outer superconductors (electrodes) in accordance with the relations

$$\cot\left(\frac{2\pi}{D}L_B\right)=\left[\pi\frac{1}{D}+\sqrt{\left(\pi\frac{1}{D}\right)^2+\frac{1}{27}}\right]^{1/3}-\left[\sqrt{\left(\pi\frac{1}{D}\right)^2+\frac{1}{27}}-\pi\frac{l}{D}\right]^{1/3}, \quad (6.20)$$

$$v=\omega_j\frac{D}{\pi}\cot\left(2\pi\frac{L_B}{D}\right),\quad D>4L_B. \quad (6.21)$$

The dependences described by (6.20) and (6.21) of the spatial scale D and the vortex velocity v on the thickness of the superconductors $2L_B$ are shown in Fig. 3. If $L_B\gg l$, Eqs. (6.19)–(6.21) describe a 4π kink (4.11) traveling with the velocity $v=2l\omega_j$ (4.12). Conversely, when $l\gg L_B$, from (6.20) and (6.21) we find

$$D=2\pi l^{1/4}L_B^{3/4}, \quad (6.22)$$

$$v=2\omega_j\sqrt{lL_B}. \quad (6.23)$$

Thus, as the thickness of the superconductors serving as electrodes decreases, the size of the vortex and its velocity also decrease. A solution like (6.19)–(6.21) is also valid for a single Josephson junction consisting of superconductors of finite thickness, if the scale l is reduced by a factor of 2 in Eq. (6.20).

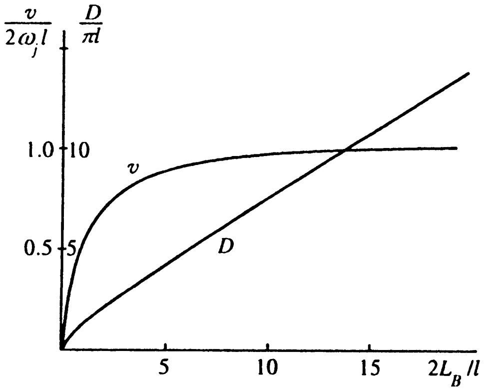

FIG. 3. Dependence of the spatial scale D and the velocity v of a traveling 4π kink on the thickness of the outer superconductors (electrodes).

7. GENERALIZED SWIHART WAVES

The electromagnetic waves in Nb–(Al/AlO$_x$–Nb)$_n$ structures consisting of two ($n=2$) or three ($n=3$) Josephson junctions were investigated experimentally in a recent study,[7] and a description of the experimental data was given in the context of the local Josephson electrodynamics of layered structures. In view of the interest in the investigation of linear waves in multilayer Josephson structures and the new possibilities for preparing the latter,[4,29] in this section we shall discuss small-amplitude electromagnetic waves (generalized Swihart waves) under conditions under which nonlocality effects are significant. Applying the basic equations of the nonlocal electrodynamics of Josephson superlattices[17] to linear waves of the form $\varphi_n\exp(-i\omega t+i\mathbf{k}\rho)$ in a structure of N tunnel junctions, we obtain the following dispersion equations:

$$\begin{vmatrix} A_1(k), & -a_2(k), & 0, & & & \\ -a_2(k), & A_2(k), & -a_3(k), & & & \\ 0, & -a_3(k), & A_3(k), & & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & & & A_{N-2}(k), & -a_{N-1}(k), & 0 \\ & & & -a_{N-1}(k), & A_{N-1}(k), & -a_N(k), \\ & & & 0, & -a_N(k), & A_N(k), \end{vmatrix}=0, \quad (7.1)$$

where we used the notations

$$A_n(k)=B_n(k)-k^2\lambda_{0n}^3\omega_{jn}^2(\omega^2+i\beta_n\omega-\omega_{jn}^2)^{-1}, \quad (7.2)$$

$$B_n(k)=b_n(k)+b_{n+1}(k)+2d_n. \quad (7.3)$$

Equation (7.1) transforms into the equation obtained in the local theory,[7] if the $a_n(k)$ and $b_n(k)$ are replaced by the

$a_n(0)$ and $b_n(0)$, i.e., a transition to local electrodynamics occurs at wave numbers much smaller than the inverse London lengths $k\lambda_n \ll 1$.

Let us first consider the dispersion equation (7.1) in the case of two arbitrary Josephson junctions ($N=2$). Then, neglecting the weak dissipation of the waves ($\beta_n=0$), we find

$$\omega_{\pm}^2(k)=\frac{1}{2}[\omega_1^2(k)+\omega_2^2(k)]\pm\frac{1}{2}\{[\omega_1^2(k)-\omega_2^2(k)]^2 + 4k^4\lambda_{01}^3\lambda_{02}^3\omega_{j1}^2\omega_{j2}^2 Q_i^2(k)\}^{1/2}. \quad (7.4)$$

Here the characteristic frequencies of the electromagnetic waves $\omega_n(k)$ have the form

$$\omega_n^2(k)=\omega_{jn}^2[1+k^2\lambda_{0n}^3 Q_n(k)], \quad (7.5)$$

where $Q_i(k)$ and $Q_n(k)$ are the Fourier components of the kernels of the integral equations for the phase differences of the wave functions of the Cooper pairs,

$$Q_{1,2}(k)=B_{2,1}(k)[B_1(k)B_2(k)-a_2^2(k)]^{-1}, \quad (7.6)$$

$$Q_i(k)=a_2(k)[B_1(k)B_2(k)-a_2^2(k)]^{-1}. \quad (7.7)$$

The dispersion law for electromagnetic waves takes an especially simple form in the case of a mirror-symmetric structure of two flat junctions, in which the critical currents j_{nc}, the dielectric constants ϵ_n, and the thicknesses d_n of the nonsuperconducting layers, as well as the London lengths and the thicknesses of the outer superconductors are equal: $j_{c1}=j_{c2}$, $\epsilon_1=\epsilon_2$, $d_1=d_2$, $\lambda_1=\lambda_3=\lambda$, $L_1=L_3=L_B$. Then $\omega_{jn}=\omega_j$, $\lambda_{0n}=\lambda_0$, $B_1(k)=B_2(k)$, $Q_1(k)=Q_2(k)$, and from (7.4)–(7.7) we obtain

$$\omega_{\pm}^2(k)=\omega_j^2\left[1+\frac{k^2\lambda_0^3}{B_2(k)\mp a_2(k)}\right], \quad (7.8)$$

where $B_2(k)$ and $a_2(k)$ are described by (7.3), (2.4), and (2.5). Here the amplitudes of the waves obey the following relations: $\varphi_1=\varphi_2$ for ω_+ and $\varphi_1=-\varphi_2$ for ω_-. According to (7.4) and (7.8), the modification of the dispersion laws in the individual junctions is specified by the coupling parameter $a_2(k)$ (2.4), which characterizes the mutual influence of the magnetic fields in the neighboring junctions. We recall that in the local theory[7] the coupling of the waves was specified by the constant $a_2(0)$. Now the coupling parameter $a_2(k)$ is a monotonically decreasing function of the wave vector k. In particular, in the short-wavelength region, where $k\lambda_2 \gg 1$ and $2kL_2 \gg 1$, the coupling parameter is exponentially small: $a_2(k)\simeq 2k\lambda_2\exp(-2kL_2)$. The weakening of the interaction of the waves is especially significant when the thickness of the superconducting layer separating the junctions is greater than their wavelength. At short wavelengths the Josephson frequency, i.e., the first term in the square brackets in (7.5) and (7.8), can be neglected. In this limit the local theory[7] leads to a linear dispersion law for electromagnetic waves propagating with the constant Swihart velocities $c_\pm=\omega_\pm/k=\text{const}$. Totally different wave dispersion laws hold in the nonlocal theory. For example, neglecting the small thicknesses of the tunnel junctions in a mirror-symmetric structure, in the short-wavelength portion of the spectrum, where $k\lambda \gg 1$ and $k\lambda_2 \gg 1$, we find

$$\omega_+=\omega_j\sqrt{2kl}[\cot(2kL_B)+(\lambda_2/\lambda)^2\tanh(kL_2)]^{-1/2}, \quad (7.9)$$

$$\omega_-=\omega_j\sqrt{2kl}[\cot(2kL_B)+(\lambda_2/\lambda)^2\coth(kL_2)]^{-1/2}. \quad (7.10)$$

According to (7.9) and (7.10) the phase and group velocities of the generalized Swihart waves are functions of the wave vector and do not coincide. They are highly dependent on both the thickness of the superconducting layer separating the junctions L_2 and the thickness of the outer superconductors (the electrodes) L_B. For a thin inner superconducting layer ($kL_2 \ll 1$) and thin electrodes ($2kL_B \ll 1$), from (7.9) and (7.10) we have

$$\omega_+/k=c_+=2\omega_j\sqrt{lL_B}, \quad (7.11)$$

$$\omega_-/k=c_-=2\omega_j\sqrt{lL_B}[1+2(L_B/L_2)(\lambda_2/\lambda)^2]^{-1/2}. \quad (7.12)$$

In this limit dependences typical of the local theory,[7] in which the group and phase velocities of the waves are determined by the Swihart velocity c_+ or c_-, are realized. Conversely, for comparatively thick superconducting layers, for which $kL_2 \gg 1$ and $2kL_B \gg 1$, we obtain

$$\omega_+=\omega_-=\omega_j\sqrt{k\lambda_0^3/(\lambda^2+\lambda_2^2)}. \quad (7.13)$$

The latter expression is precisely equivalent to the dispersion law for a generalized Swihart wave in an isolated Josephson junction.[13,15]

To conclude this section, as in Ref. 7, let us also consider electromagnetic waves in a mirror-symmetric structure consisting of three Josephson junctions, in which $j_{c1}=j_{c3}$, $\epsilon_1=\epsilon_3$, $d_1=d_3$, $\lambda_1=\lambda_4$, $\lambda_2=\lambda_3$, $L_1=L_4$, and $L_2=L_3$. Then, in the case of identical capacitances in the tunnel junctions ($\omega_{jn}^2\lambda_{0n3}=\omega_j^2\lambda_0^3$), neglecting the weak dissipation of the waves and the small Josephson frequencies in the denominator in (7.2), from the dispersion equation (7.1) we find

$$\omega_m/k=\omega_j\lambda_0\sqrt{\lambda_0/B_1(k)}, \quad (7.14)$$

$$\frac{\omega_\pm}{k}=\omega_j\lambda_0\sqrt{2\lambda_0}[B_1(k)+B_2(k) \mp\sqrt{[B_1(k)-B_2(k)]^2+8a_2^2(k)}]^{-1/2}. \quad (7.15)$$

The main fact distinguishing (7.14) and (7.15) from the expressions obtained in the local theory[7] is that the phase and group velocities of the electromagnetic waves do not coincide and are functions of the wave vector. The new relations are manifested most explicitly in the wavelength region for short London lengths and in not excessively thin superconducting layers.

8. CONCLUSIONS

Summarizing the material presented, we can state that the nonlocal Josephson electrodynamics describing Abrikosov–Josephson vortex structures made it possible to obtain comparatively simple and physically visualizable equations, which give a picture of coherent vortices in two tunnel junctions. This became possible both as a result of the derivation of nonlinear integral equations describing vortices in two junctions and as a result of the attainment of exact and approximate solutions corresponding to an interaction between the vortices mediated by a magnetic field penetrating through the superconductor separating the tunnel junctions. Modification of the picture of interacting vortices in two separate junctions stipulated by the finite thickness of the superconducting electrodes forming the outer layers of the tunnel structure was revealed both in the case of static vortices and in the case of moving quasimagnetostatic vortices.

A systematic nonlocal description of the magnetic fields made it possible to reveal new spectral properties of electromagnetic waves (generalized Swihart waves) in the wavelength region for short London lengths which are not commensurate with the thicknesses of the superconducting layers.

APPENDIX A:

Let us discuss the mutual influence of N identical Josephson junctions separated by thin superconductors having a thickness $2L$, which is smaller than both the London length and the dimensions of a vortex, where the inequality $2L\sqrt{k^2+\lambda^{-2}}\ll 1$ holds. The thickness of the outermost superconductors with the numbers $n=1$ and $n=N+1$ is assumed to be much greater than λ or the dimensions of a vortex. Then, neglecting the thickness of the tunnel junctions themselves, for the magnetic field in them we have (compare Ref. 17)

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2}\exp[i\mathbf{k}(\rho-\rho')]\{-a_{n+1}\mathbf{H}_{n+1}(\rho',t) -a_n\mathbf{H}_{n-1}(\rho',t)+[b_n+b_{n+1}]\mathbf{H}_n(\rho',t)\} =\frac{L}{\lambda^2}\frac{\hbar c}{|e|}\left[e_x\frac{\partial}{\partial\rho}\varphi_n(\rho,t)\right], \tag{A1}$$

where $a_n=0$, $b_n=2L\sqrt{k^2+\lambda^{-2}}\ll 1$ for $n=1$, $N+1$ and where $a_n=b_n=1$ for $n\neq 1, N+1$. In a superlattice of closely arranged identical Josephson junctions having a thickness smaller than the dimensions of a vortex, it is natural to assume that the magnetic fields and the phase differences are equal: $\mathbf{H}_n=\mathbf{H}$, $\varphi_n=\varphi$. In this case, summing all N Eqs. (A1), we obtain

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2}\exp[i\mathbf{k}(\rho-\rho')]\lambda^2\sqrt{k^2+\lambda^{-2}}\mathbf{H}(\rho',t) =\frac{\hbar c}{4|e|}N\left[\mathbf{e}_x\frac{\partial}{\partial\rho}\varphi(\rho,t)\right]. \tag{A2}$$

We hence find the magnetic field [see (2.6)] and then the equation for the phase difference

$$\sin\varphi(\rho,t)+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi(\rho,t)+\frac{1}{\omega_j^2}\frac{\partial^2}{\partial t^2}\varphi(\rho,t) =\lambda_0^3\frac{\partial^2}{\partial\rho^2}\int d\rho' Q_N(\rho-\rho')\varphi(\rho',t), \tag{A3}$$

where the kernel $Q_N(\rho)$ contains the dependence on the number of tunnel junctions

$$Q_N(\rho)=\frac{N}{4\pi\rho\lambda^2}\exp\left(-\frac{\rho}{\lambda}\right). \tag{A4}$$

A comparison of (A4) with (2.13) reveals an N-fold increase in the kernel of the integral equation (A3) in comparison to the case of a single isolated tunnel junction. Such an increase in the kernel results in an N-fold increase in the dimensions of the Abrikosov–Josephson vortices, and in the case of moving vortices it results in a corresponding change in their velocity.

The equation for the phase difference in a superlattice consisting of N identical closely arranged Josephson junctions with identical outer superconductors (electrodes) having a finite thickness $2L_n=2L_B\gg 2L$ $(n=1,\ N+1)$ is derived in a totally similar manner. For the case of identical phase differences in all the junctions we find

$$\sin\varphi(\rho,t)+\beta\omega_j^{-2}\frac{\partial}{\partial t}\varphi(\rho,t)+\omega_j^{-2}\frac{\partial}{\partial t}\varphi(\rho,t) =\lambda_0^3\frac{\partial^2}{\partial\rho^2}\int d\rho' Q_{NL}(\rho-\rho')\varphi(\rho',t) +\frac{c}{4\pi j_c}\int d\rho' Q_H(\rho-\rho')(\mathbf{e}_x\mathrm{curl}\mathbf{H}_\infty(\rho',t)), \tag{A5}$$

where the kernels of the integral operators have the form

$$Q_{NL}(\rho)=\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{N\exp(i\mathbf{k}\rho)}{2\lambda^2\sqrt{k^2+\lambda^{-2}}}\tanh(2L_B\sqrt{k^2+\lambda^{-2}}), \tag{A6}$$

$$Q_H(\rho)=\int\frac{d\mathbf{k}}{(2\pi)^2}\exp(i\mathbf{k}\rho)\ \mathrm{sech}(2L_B\sqrt{k^2+\lambda^{-2}}). \tag{A7}$$

In the case of electrodes of finite thickness there is also an N-fold increase in the kernel of the integral equation for the phase difference Q_{NL} describing the mutual influence of the tunnel junctions. At the same time, the kernel Q_H, which describes the influence of a magnetic field on the boundaries of the structure, remains unchanged.

[1)] We note that the analog of Eq. (4.1) in local Josephson electrodynamics is the sine-Gordon equation and that the investigation of its consequences which are used in the theory of Josephson junctions has been the subject of numerous papers.

[2)] Neglect of the second derivative with respect to time corresponds to the so-called case of the resistive model in local Josephson electrodynamics.[19–22] In the case of the intrinsic Josephson effects[23] in high-temperature BiSrCaCuO superconductors, the resistive limit corresponds to a conductivity smaller than $10^{10}\ s^{-1}$.

[3)] Equation (5.22), which holds in the limit of large voltages (5.21), does not depend on the length parameter l, which characterizes the small-scale Abrikosov–Josephson vortices studied here. Therefore, the limiting equation (5.22) corresponds to the results of ordinary electrodynamics obtained in Refs. 19–21 (see also Ref. 22).

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