

Solitons of the untruncated Maxwell–Bloch equations

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The problem of pulse propagation in a medium consisting of two-level atoms with arbitrary detuning of the field frequency the frequency of the atomic subsystem and arbitrary ratio of the pulse duration and the period of the optical oscillations is investigated using the complete (untruncated) sytem of Maxwell–Bloch equations. It is shown that self-consistent solutions of this problem exist and the pulse profile and its dependence on different parameters of the medium are determined. © 1995 American Institute of Physics.

1. INTRODUCTION

The phenomenon of self-induced transparency has been studied intensively, both experimentally and theoretically, for the last thirty years. Over this period a powerful theoretical apparatus has been developed for determining the soliton solutions of the system of truncated Maxwell–Bloch equations for a field interacting with a system of two- or three-level atoms.^{1–6} The small parameter in the truncated equations or the equations for the slowly varying amplitudes is the ratio of the period of the optical oscillations to the pulse duration, i.e. $(\omega\tau_p)^{-1}$. The soliton solutions of the truncated equations corresponding to the case $\omega\tau_p \gg 1$ do not exhibit phase modulation. The question of the possibility of the existence of a phase-modulated soliton^{7–12} and the effect of phase modulation on the shape of a pulse propagating in a resonant medium have long been discussed. A perturbation theory in the parameter $(\omega\tau_p)^{-1}$ has been developed, numerical experiments have been performed, and the integrals of motion of the untruncated equations have been determined. However, the analytical form of the intensity profile of a pulse has been obtained only in the particular case of a pulse with no carrier.⁷ The untruncated system of Maxwell–Bloch equations is conventionally written for the vector E of the electric field intensity and the polarization P . In the slowly varying amplitude approximation the same equations are obtained with different choices of the independent variables (the vector potential A of the electromagnetic field and the current density j of the resonant transition or the vector E of the electric field intensity of the wave and the polarization P of the medium, and others). However, when we are dealing with effects that depend on the frequency of the optical field, it is desirable to employ the relativistically invariant form of the Hamiltonian of the field. For the case of circular polarization this Hamiltonian has the form

$$H_f = \frac{1}{4\pi} \int \left(\frac{1}{c^2} \frac{\partial A^+}{\partial t} \frac{\partial A^-}{\partial t} + \text{curl } A^+ \text{ curl } A^- \right) dV - \frac{1}{c} \int (j^+ A^- + j^- A^+) dV. \quad (1)$$

The system of equations for the variables E and P in the one-wave case admits a reduction^{13,14} which lowers the

second-order differential equations for the field are reduced to first-order equations. The system of equations for A and J does not admit such a reduction.

In the present paper the profiles of the self-consistent solutions of the untruncated system of Maxwell–Bloch equations for the variables A and j with arbitrary values of the parameter $\omega\tau_p$ are determined.

2. CONSERVATION LAWS FOR THE UNTRUNCATED SYSTEM OF MAXWELL–BLOCH EQUATIONS

The Hamiltonian (1) together with the Hamiltonian of a two-level atom leads to the following system of untruncated Maxwell–Bloch equations in the one-dimensional case:

$$\begin{aligned} \frac{\partial^2 A^\pm}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A^\pm}{\partial t^2} &= -\frac{4\pi}{c} \int j^\pm(\omega_0) f(\omega_0) d\omega_0, \\ \frac{\partial j^\pm}{\partial t} \mp i\omega_0 j^\pm &= \pm \frac{i|m|^2}{\hbar c} A^\pm \rho, \\ \frac{\partial \rho}{\partial t} &= \frac{2i}{\hbar c} (j^+ A^- - j^- A^+), \end{aligned} \quad (2)$$

where $f(\omega_0)$ is the contour of an inhomogeneously broadened line, $|m|$ is the matrix element of the current of the resonance transition of a separate atom, and ρ is the density of the population inversion.

The equations (2) have a well-known Bloch integral of motion

$$\begin{aligned} j^+(z,t)j^-(z,t) + \frac{|m|^2}{4} \rho^2(z,t) &= j^+(z,0)j^-(z,0) \\ &+ \frac{|m|^2}{4} \rho^2(z,0), \end{aligned} \quad (3)$$

which relates the absolute value of the current density of the resonance transition to the population inversion density. Using Eq. (3), we can introduce the Bloch angle $\theta(z,t)$ as

$$\begin{aligned} \rho(z,t) &= \rho_0(z) \cos \theta(z,t), \\ j^\pm(z,t) &= \frac{|m|}{2} \rho_0(z) \sin \theta(z,t) \exp[\pm i\Phi(z,t)], \end{aligned} \quad (4)$$

where $\rho_0(z)$ is the density distribution of the medium for $\theta=0$ and $\Phi(z,t)$ is the phase of the current density.

The law of conservation of energy for the system of equations (2) takes the form

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2} \frac{\partial A^+}{\partial t} \frac{\partial A^-}{\partial t} + \frac{1}{4\pi} \frac{\partial A^+}{\partial z} \frac{\partial A^-}{\partial z} - \frac{1}{c} \int (j^+ A^- \right. \\ \left. + j^- A^+) f(\omega_0) d\omega_0 + \int \frac{\hbar \omega_0}{2} \rho f(\omega_0) d\omega_0 \right] \\ - \frac{1}{4\pi} \frac{\partial}{\partial z} \left(\frac{\partial A^+}{\partial t} \frac{\partial A^-}{\partial z} + \frac{\partial A^+}{\partial z} \frac{\partial A^-}{\partial t} \right) = 0. \end{aligned} \quad (5)$$

For the equations (2) there is also a continuity equation

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{i}{2\pi\hbar c^2} \left(A^+ \frac{\partial A^-}{\partial z} - \frac{\partial A^+}{\partial z} A^- \right) + \int \rho f(\omega_0) d\omega_0 \right] \\ + \frac{\partial}{\partial z} \left[-\frac{1}{2\pi\hbar} \left(A^+ \frac{\partial A^-}{\partial z} - \frac{\partial A^+}{\partial z} A^- \right) \right] = 0. \end{aligned} \quad (6)$$

Using the indicated conservation laws, we shall seek the self-consistent solutions of Eq. (2) in the form

$$A^\pm(z, t) = A(t - z/v) \exp\{\pm i[\omega t - kz + \psi(t - z/v)]\}. \quad (7)$$

Here ω is the carrier frequency of the field and k is the wave vector represented in the form $k = \omega/v_1$, where v_1 can be chosen¹⁵ so that ω can be interpreted as the amplitude- and phase-modulation-independent carrying frequency of the pulse. To simplify the equations, however, in what follows we employ the following notation:

$$k = \omega/v_1 = \omega_1/c.$$

The phase velocity of the wave v_p can be different from both the velocity of light c and the propagation velocity v of the pulse, i.e., the group velocity. Indeed, v_p is found by differentiating with respect to time the condition expressing the constancy of the phase:

$$\Psi(z, t) = \omega t - kz + \psi(t - z/v) = C.$$

Therefore

$$\omega - k \frac{dz}{dt} + \dot{\psi} \left(1 - \frac{1}{v} \frac{dz}{dt} \right) = 0,$$

whence

$$v_p = \frac{dz}{dt} = v \frac{\omega + \dot{\psi}}{kv + \dot{\psi}}.$$

Therefore $v_p = v$ for $v_1 = v$.

It is convenient to introduce the dimensionless amplitude a of the field, defined as

$$A(z, t) = \left(\frac{2\pi\hbar c^2 N}{\omega V} \right)^{1/2} a(z, t), \quad (8)$$

where $|a|^2$ is the photon number density normalized to the density N/V of the number of resonant atoms. It is convenient to write the phase $\Phi(z, t)$ of the transition current density in the form

$$\Phi(z, t) = \omega t - kz + \varphi(t - z/v). \quad (9)$$

We introduce the dimensionless time $t' = t/\tau$ and the dimensionless coordinate $z' = z/L$, where $\tau = L/c$.

The equations for the new atomic variables $\theta(z, t)$ and $\varphi(z, t)$ follow directly from the system of equations (2) and have the form

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= 2\sqrt{\beta} a \sin(\varphi - \psi), \\ \frac{\partial \varphi}{\partial t} &= \Omega + 2\sqrt{\beta} a \cot \theta \cos(\varphi - \psi), \end{aligned} \quad (10)$$

where

$$\beta = \frac{2\pi|m|^2 N}{\hbar \omega V} \tau^2, \quad \Omega = (\omega_0 - \omega)\tau. \quad (11)$$

It is convenient to use the conservation laws (5) and (6) to express the equations for the field variables $a(z, t)$ and $\phi(z, t)$. Assuming that the amplitude of the field is equal to zero at infinity and the medium is in either the ground state ($R_0 = -1$) or the excited state ($R_0 = 1$), we obtain from Eqs. (5) and (6)

$$\begin{aligned} (1 + 2\delta\dot{\psi})a^2 &= \Gamma \int \sin^2(\theta/2) f(\omega_0) d\omega_0, \\ a^2 + \dot{\psi}^2 a^2 + (\Delta/\delta)a^2 &= (\Gamma/\delta) \int [\Omega \sin^2(\theta/2) \\ &+ \sqrt{\beta} a \sin \theta \cos(\varphi - \psi)] \\ &\times f(\omega_0) d\omega_0, \end{aligned} \quad (12)$$

where

$$\begin{aligned} \delta &= \frac{1}{2v} \frac{c^2 - v^2}{(\omega_1 c - \omega v)\tau}, \quad \Gamma = \frac{\omega v R_0}{\omega v - \omega_1 c}, \\ \Delta &= \frac{(\omega_1^2 - \omega^2)v\tau}{2(\omega v - \omega_1 c)}. \end{aligned} \quad (13)$$

For $v_1 = v$, the parameters δ , Γ , and Δ determined by the expressions (13) assume the form

$$\delta = 1/2\omega\tau, \quad \Gamma = R_0 v^2 / (v^2 - c^2), \quad \Delta = -\omega\tau/2$$

and for $v_1 = c$ they have the form

$$\delta = (c+v)/2\omega\tau v, \quad \Gamma = R_0 v / (v-c), \quad \Delta = 0.$$

It is evident from the expressions presented above that δ is a small parameter, expanding with respect to which we obtain from the untruncated Maxwell-Bloch equations the equations for the slowly varying amplitudes. For $\delta=0$ we should obtain from Eqs. (10)–(13) the solitons of the truncated equations.

3. SOLITONS OF THE TRUNCATED MAXWELL-BLOCH EQUATIONS

For $f(\omega_0) = \delta(\omega_0 - \omega_a)$ Eqs. (10)–(13) have the form

$$\begin{aligned} \dot{\theta} &= 2\sqrt{\beta} a \sin(\varphi - \psi), \\ \dot{\varphi} &= \Omega + 2\sqrt{\beta} a \cot \theta \cos(\varphi - \psi), \\ (1 + 2\delta\dot{\psi})a^2 &= \Gamma \sin^2(\theta/2), \end{aligned} \quad (14)$$

$$\delta(\dot{a}^2 + \dot{\psi}^2 a^2) + \Delta a^2 = \Gamma[\Omega \sin^2(\theta/2) + \sqrt{\beta} a \sin \theta \cos(\varphi - \psi)].$$

We shall show that the soliton solutions for the slowly varying amplitudes follow from Eqs. (14) with $\delta = \Delta = 0$. The last two equations of the system (14) with $\delta = \Delta = 0$ assume the form

$$a = \sqrt{\Gamma} \sin(\theta/2), \quad (15)$$

$$\Omega \sin(\theta/2) = -2\sqrt{\beta} a \cos(\theta/2) \cos(\varphi - \psi).$$

First, let $\Omega = 0$. Then we have $\varphi - \psi = \pi/2$, and substituting the first equation from Eqs. (15) into the first equation in Eqs. (14) we obtain

$$\dot{\theta} = 2\sqrt{\Gamma\beta} \sin(\theta/2). \quad (16)$$

This equation has the well-known solution

$$a = \frac{a_0}{\cosh\left[\frac{1}{\tau_p} \left(t - \frac{z}{v}\right)\right]}, \quad (17)$$

where $a_0 = \sqrt{\Gamma}$ and $1/\tau_p = \sqrt{\Gamma\beta}$. The area of the pulse (17) is equal to 2π .

Now let $\Omega = \omega_a - \omega \neq 0$. Then we obtain from Eq. (15)

$$\cos \frac{\theta}{2} \cos(\varphi - \psi) = -\frac{\Omega}{2\sqrt{\Gamma\beta}}. \quad (18)$$

Substituting the expression obtained into Eq. (14), we obtain

$$\dot{\theta} = 2\sqrt{\Gamma\beta} \sin \frac{\theta}{2} \left[1 - \frac{\Omega^2}{4\Gamma\beta} \frac{1}{\cos^2(\theta/2)}\right]^{1/2}. \quad (19)$$

The solution of this equation has the form

$$a = \frac{a_0}{\cosh\left[\frac{1}{\tau_p} \left(t - \frac{z}{v}\right)\right]},$$

where

$$a_0 = \left[\Gamma \left(1 - \frac{\Omega^2}{4\Gamma\beta}\right)\right]^{1/2}, \quad \frac{1}{\tau_p} = \left[\Gamma\beta \left(1 - \frac{\Omega^2}{4\Gamma\beta}\right)\right]^{1/2}. \quad (20)$$

To within constant factors, this expression is identical to the well-known expression for a soliton with detuning.³ The population inversion varies in time according to the law

$$R = R_0 \cos \theta = R_0 \left\{ 1 - \frac{2}{1 + \Omega^2 \tau_p^2 / 4} \times \frac{1}{\cosh^2\left[\frac{1}{\tau_p} \left(t - \frac{z}{v}\right)\right]} \right\}.$$

4. SOLITONS OF THE UNTRUNCATED MAXWELL-BLOCH EQUATIONS

The last two equations of the system (14) can be rewritten in the following form

$$\dot{\psi} = \frac{1}{2\delta} \left(\Gamma \frac{\sin^2(\theta/2)}{a^2} - 1 \right), \quad (21)$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \dot{\psi}^2 + \frac{\Delta}{\delta} = \frac{\Gamma \sin^2(\theta/2)}{\delta a^2} \left(\Omega + 2\sqrt{\beta} \times \frac{\sin(\theta/2)}{a} \cos \frac{\theta}{2} \cos(\varphi - \psi) \right).$$

It is evident from Eq. (21) that the field variables depend only on the combination $a/\sin(\theta/2)$ and the variable $\cos(\theta/2)\cos \eta$, where $\eta = \varphi - \psi$. For this reason, we introduce the new variables x , y , and F defined as follows:

$$x = \cos(\theta/2) \sin \eta, \quad (22)$$

$$y = \cos(\theta/2) \cos \eta,$$

$$a = \frac{D}{\sqrt{\beta}} F \sin \frac{\theta}{2},$$

where D is a constant. The equations for the variables x and y follow from Eq. (14) and have the form

$$\dot{x} = \left(2DFy + \Omega_1 - \frac{B}{F^2} \right) y - DF(1 - x^2), \quad (23)$$

$$\dot{y} = - \left(DFy + \Omega_1 - \frac{B}{F^2} \right) x,$$

where we have written $\dot{\psi}$ in the form

$$\dot{\psi} = \frac{B}{F^2} - \frac{1}{2\delta}, \quad (24)$$

and Ω_1 and B are determined by the following expressions:

$$\Omega_1 = \Omega + \frac{1}{2\delta}, \quad B = \frac{\Gamma\beta}{2\delta D^2}. \quad (25)$$

The last equation in Eqs. (21) can be written in the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4BDy}{F} + \frac{2B\Omega_1}{F^2} - \frac{B^2}{F^4} - K, \quad (26)$$

where

$$K = \frac{\Delta}{\delta} + \frac{1}{4\delta^2} = \frac{1}{4\delta^2} \left(\frac{\omega c - \omega_1 v}{\omega_1 c - \omega v} \right)^2. \quad (27)$$

On the other hand, using the last equation in Eqs. (22), we obtain for \dot{a}/a

$$\frac{\dot{a}}{a} = DFx + \frac{\dot{F}}{F}. \quad (28)$$

Equating the expressions (26) and (28), we obtain

$$\frac{\dot{F}}{F} = \sqrt{G} - DFx, \quad (29)$$

where

$$G = \frac{4BDy}{F} + \frac{2B\Omega_1}{F^2} - \frac{B^2}{F^4} - K. \quad (30)$$

Premultiplying both parts of Eqs. (29) by $(DFy + \Omega_1 - B/y^2)$ and using Eq. (23), we can easily rewrite Eq. (29) in the form

$$\frac{d}{dt} \sqrt{G} = -2B \left(\frac{Dy}{F} + \frac{\Omega_1}{F^2} - \frac{B}{F^4} \right). \quad (31)$$

We note that according to Eq. (26)

$$\frac{\dot{a}}{a} = \pm \sqrt{G}.$$

The sign on the right-hand side is chosen so as to satisfy the conditions, discussed above, at infinity.

5. CONDITIONS FOR SELF-CONSISTENT PULSE PROPAGATION AND THE FORM OF THE SOLITON

The equations (31) are, in principle, sufficient to determine the conditions for the appearance of soliton solutions, since by comparing Eqs. (30) and (31) we can see that the right- and left-hand sides of Eq. (31) depend on the same combinations of variables. However, we shall make a more incisive and detailed analysis of the combined system of equations (23) and Eq. (29) or its equivalent Eq. (31). We expand the variables x and y in power series in F :

$$y = \sum_{n=-\infty}^{\infty} f_n F^n, \quad x = \sum_{n=-\infty}^{\infty} g_n F^n. \quad (32)$$

Substituting these expansions into Eq. (23) and equating the derivatives \dot{F} obtained from the first and second equations in Eq. (23), we obtain the following equation:

$$\sum_{n=-\infty}^{\infty} F^n \left\{ n \sum_{m=-\infty}^{\infty} F^m \left[\sum_{k=-\infty}^{\infty} F^k (2f_n f_m f_k + g_n g_m f_k + f_n g_m g_k) + \frac{\Omega_1}{D} (f_n f_{m+1} + g_n g_{m+1}) - \frac{B}{D} (f_n f_{m+3} + g_n g_{m+3}) \right] - f_n \right\} = 0. \quad (33)$$

The coefficients f_n and g_n do not depend on F under the following conditions:

$$\begin{aligned} 2f_n f_m f_k + g_n g_m f_k + f_n g_m g_k &= 0 \quad \text{for } m+k \neq 0, \\ f_n f_{m+1} + g_n g_{m+1} &= 0 \quad \text{for } m \neq 1, 3. \end{aligned} \quad (34)$$

The equation (33) then assumes the form

$$\begin{aligned} n \sum_{m=-\infty}^{\infty} \left[(f_n f_m + g_n g_m) f_{-m} + f_n (f_m f_{-m} + g_m g_{-m}) \right. \\ \left. + \frac{\Omega_1}{D} n (f_n f_1 + g_n g_1) - \frac{B}{D} n (f_n f_3 + g_n g_3) \right] \\ - f_n = 0. \end{aligned} \quad (35)$$

Comparing Eqs. (34) and (35), it is easy to see that only the coefficients f_n and g_n with $n = -3, -1, 1,$ and 3 can be different from zero, i.e. y and x can be represented in the form

$$y = \frac{f_1}{F^3} + \frac{f_2}{F} + f_3 F + f_4 F^3, \quad (36)$$

$$x = \frac{g_1}{F^3} + \frac{g_2}{F} + g_3 F + g_4 F^3.$$

On the other hand, the variables x and y are determined by the expressions (22), whose dependence on η allows us to conclude that the expansions of x and y in power series in F should not be symmetric, since y at $\eta=0$ has a maximum and so its derivative is zero, whereas x is equal to zero and its derivative is a maximum. Comparing likewise the expressions for a from Eqs. (22) and (15), we can see that in the zeroth-order approximation in δ the variable y has the form

$$y = -\frac{\Omega}{2D} \frac{1}{F}.$$

It can therefore be conjectured that the expansion of y in a power series in F should also contain negative powers of F even for $\delta \neq 0$, so that $y(F)$ for $\delta \neq 0$ will have the form

$$y = \frac{f_1}{F^3} + \frac{f_2}{F} + f_3 F. \quad (37)$$

After the expression (37) is substituted in Eqs. (30) and (31), Eq. (31) assumes the form

$$\frac{d}{dt} \left(a_1 + \frac{2b_1}{F^2} - \frac{c_1}{F^4} \right)^{1/2} = 2 \left(a_2 + \frac{b_2}{F^2} - \frac{c_2}{F^4} \right), \quad (38)$$

where

$$\begin{aligned} a_1 &= 4BDf_3 - K, \quad b_1 = B(\Omega_1 + 2Df_2), \\ c_1 &= B^2 - 4BDf_1, \\ a_2 &= BDf_3, \quad b_2 = B(\Omega_1 + Df_2), \quad c_2 = B^2 - BDf_1. \end{aligned}$$

The equation (38) can be simplified by introducing the new variable α :

$$G^2 = \frac{a_1 c_1 + b_1^2}{c_1} \sin^2 \alpha. \quad (39)$$

Then

$$\frac{1}{F^2} = \frac{b_1}{c_1} \pm \frac{\sqrt{a_1 c_1 + b_1^2}}{c_1} \cos \alpha. \quad (40)$$

Substituting the expressions (30) and (40) into Eq. (38) we obtain

$$\frac{da}{dt} = -2 \left(\frac{A_1}{\cos \alpha} - 2B_1 - C_1 \cos \alpha \right), \quad (41)$$

where

$$A_1 = c_2 \sqrt{\frac{c_1}{a_1 c_1 + b_1^2}} \left(\frac{a_2}{c_2} + \frac{b_1}{c_1} \left(\frac{b_2}{c_2} - \frac{b_1}{c_1} \right) \right),$$

$$B_1 = \pm \frac{c_2}{\sqrt{c_1}} \left(\frac{b_1}{c_1} - \frac{1}{2} \frac{b_2}{c_2} \right),$$

$$C_1 = \frac{c_2}{c_1} \sqrt{\frac{a_1 c_1 + b_1^2}{c_1}}.$$

The solution of Eq. (41) can be written in a general form, but it is quite complicated. The equation (41) leads to solutions in the form of a solitary pulse, whose envelope approaches zero at infinity for

$$A_1 > B_1 \pm \sqrt{B_1^2 + A_1 C_1}. \quad (42)$$

In the case when at least one of the inequalities (42) is not satisfied, the solution is oscillatory.

The simplest pulse profile obtains in the case when $B_1 = 0$. This can be achieved, for example, by the following choice of the coefficient f_2 :

$$f_2 = -\frac{\Omega_1(B + 2Df_1)}{3BD}. \quad (43)$$

In this case the envelope and the modulation of the frequency of the pulse are determined by the expressions

$$a = \frac{a_0}{\cosh^\gamma \left[\frac{2}{\tau_0} \left(\tau - \frac{z}{v} \right) \right]}, \quad (44)$$

$$\dot{\psi} = B \frac{b_1}{c_1} - \frac{1}{2\delta} \pm B \left[\frac{a_1 c_1 + b_1^2}{c_1^2} - \left(\frac{a_1}{c_1} - \frac{a_2}{c_2} \right) \tanh^2 \left[\frac{2}{\tau_0} \left(t - \frac{z}{v} \right) \right] \right]^{1/2},$$

where

$$\frac{1}{\tau_0} = \frac{c_2}{\sqrt{c_1}} \left[\frac{a_1}{c_1} - \frac{a_2}{c_2} \right]^{1/2}, \quad \gamma = \frac{1}{2} \frac{c_1}{c_2}.$$

As one can see from the formulas presented above, in the presence of frequency modulation the pulse shape is different from the case $\delta = 0$. It follows from the expressions for the coefficients $c_{1,2}$ that $\gamma \leq 1/2$.

6. KERR NONLINEARITY

In media with a Kerr nonlinearity the equation for the field has the form

$$\frac{\partial^2 A^\pm}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A^\pm}{\partial t^2} = -\frac{4\pi}{c} \int j^\pm(\omega_0) f(\omega_0) d\omega_0 - 4\pi\kappa^4 \chi_3 |A|^2 A^\pm.$$

The Kerr nonlinearity will be manifested in the final equations only as an additional term in the second equation of the system of field equations (21). This system now assumes the form

$$\left(\frac{\dot{a}}{a} \right)^2 + \dot{\psi}^2 + \frac{\Delta}{\delta} - \chi a^2 = \frac{\Gamma \sin^2(\theta/2)}{\delta a^2} \times \left(\Omega + 2\sqrt{\beta} \frac{\sin(\theta/2)}{a} \times \cos \frac{\theta}{2} \cos(\varphi - \psi) \right),$$

where

$$\chi = \frac{2\pi^2 \hbar c^2 N}{\delta \omega} \frac{N}{V} \kappa^4 \chi_3.$$

The form of the system of equations (23) and (29) will remain unchanged, only the expression for G will change:

$$G = \frac{4BDy}{F} + \frac{2b\Omega_1}{F^2} - \frac{B^2}{F^4} - K + \frac{D^2\chi}{\beta} F^2(1 - x^2 - y^2). \quad (45)$$

The expansions (36) of the variables x and y in powers of F will remain the same, since their derivation employed only Eq. (23) for the atomic variables.

In summary, Eqs. (23) and (29), in which G is determined by the expression (45), comprise a complete system of equations describing the self-consistent propagation of a pulse in media with Kerr nonlinearity. We can see that the Kerr nonlinearity will not lead to any qualitative changes in the structure of the equations describing the dynamics of self-consistent pulse propagation.

7. CONCLUSIONS

The present investigations show that the complete (un-truncated) system of Maxwell-Bloch equations contains solutions which correspond to the self-consistent propagation of pulses in a medium of two-level atoms. The solutions obtained do not contain any restrictions on the magnitude of the ratio $\delta^{-1} = \omega\tau_0$ of the pulse duration τ_0 and the period of the optical pulses or on the magnitude of the detuning $\Omega = \omega_0 - \omega$ of the frequency ω of the field from the frequency ω_0 of the atomic subsystem. The width τ_0 of the soliton and its amplitude D are determined by the corresponding characteristics of the incident pulse and depend on the magnitude of the detuning. The relations between these parameters are determined by the formulas (44), and to find them explicitly it is necessary to know the explicit form of the coefficients f_n and g_n in the expansions (36) and (37). The corresponding formulas are quite complicated, but the characteristic parameters of the pulse profile and the frequency modulation of the pulse can be obtained in different limiting case just from the general formulas (44).

The pulse profile is determined by the parameters γ and τ_0 . Substituting into them the expressions for the coefficients a_n , b_n and c_n , we obtain in the general case

$$\frac{1}{\tau_0} = \left[\frac{(B - Df_1)[3B^2Df_3 - K(B - Df_1)]}{(B - 4Df_1)^2} \right]^{1/2}, \quad (46)$$

$$\gamma = \frac{1}{2} \frac{B^2 - 4BDf_1}{B^2 - BDf_1}.$$

The modulation of the frequency of the pulse is determined by the expression

$$\dot{\psi} = \Omega + \left[\Omega_1^2 + \frac{B^2Df_3}{B - Df_1} + B \frac{3B^2Df_2 - K(B - Df_1)}{(B - Df_1)(B - 4Df_1)} \operatorname{sech}^2 \left[\frac{2}{\tau_0} \left(t - \frac{z}{v} \right) \right] \right]^{1/2}. \quad (47)$$

It follows from Eqs. (46) and (47) that the minimum pulse width and the maximum amplitude of the frequency modulation are achieved for $K=0$. The condition $K=0$ holds for $\omega_1 v = \omega c$. In this case, as we noted in Sec. 2, the phase velocity of the pulse is equal to the group velocity, i.e. the pulse is stationary not only with respect to the envelope profile but also with respect to the high-frequency distribution of the field.

It is also evident from Eqs. (46) and (47) that the formulas simplify considerably in the case $f_1 \ll 1$. For this reason, setting $f_1=0$ and $K=0$, we obtain from Eqs. (45) and (46)

$$\frac{1}{\tau_0} = \sqrt{3BDf_3}, \quad \gamma = \frac{1}{2},$$

$$\dot{\psi} = \Omega + \left[\Omega_1^2 + BDf_3 + 3BDf_3 \operatorname{sech}^2 \left[\frac{2}{\tau_0} \left(t - \frac{z}{v} \right) \right] \right]^{1/2}.$$

It is evident from the last formula that the amplitude of the frequency modulation is largest for $\Omega_1=0$. Therefore, for pulses containing several periods of the optical oscillations, the equality $\omega - \omega_0 \approx 1/2\delta$ will probably hold. In this case the amplitude of the frequency modulation is

$$\Delta\omega = \dot{\psi}(0) - \dot{\psi}(\infty) = \sqrt{BDf_3},$$

and the product $\Delta\omega\tau_0$ is

$$\Delta\omega\tau_0 = 1/\sqrt{3}.$$

The analysis performed above pertains to both absorbing ($R_0 = -1$) and amplifying ($R_0 = 1$) media. It follows from the form of Eqs. (12) that the parameter Γ must be positive. In the case $K=0$, we obtain $\Gamma = R_0 v^2 / (v^2 - c^2)$. For $R_0 = -1$ this quantity is positive for $v < c$ and for $R_0 = 1$ it is positive for $v > c$, i.e., just as for the case of slowly varying equations, the velocity of the pulse in an absorbing medium is less than the velocity of the light and in an amplifying medium it is greater than the velocity of light.

The solutions found above describe self-consistent propagation of pulses. To determine whether or not these solutions are solitons, numerical experiments can be performed on collisions of pulses. Such experiments will make it possible to determine the pulse stability. As follows from the results of Sec. 3, for $\delta=0$ the profile of the pulse intensity is the same both when the equations for the field intensity E and the vector potential A are used. For $\delta \neq 0$ these expressions are different, and for this reason real physical experiments would make it possible to resolve the question of the validity of different representations for the field.

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