Modulational instability of gravity waves in deep water with allowance for nonlinear dispersion

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To describe the evolution of spectrally narrow wave packets of finite amplitude for a specific, nonlinear-dispersive system, a perturbation-theory method has been developed which allows one to reduce the initial system of nonlinear equations to a model equation for the envelope in the form of an asymptotic expansion of the time derivative in two independent small parameters which characterize the smallnesses of the amplitudes (ϵ) and the slowness of their spatial variation (μ). The influence of nonlinear dispersive terms of the "higher" nonlinear Schrödinger equation so obtained on the conditions of modulational instability of gravity waves in deep water is investigated. The dependence of the instability growth rate on the direction of propagation of the modulations relative to the direction of the main wave is determined.

1. INTRODUCTION

At present it is well known[1,2] that the evolution of the amplitude of the envelope of a weakly nonlinear quasimonochromatic wave as it propagates in a nonlinear dispersive medium is described to lowest order by the nonlinear Schrödinger (NLS) equation with cubic nonlinearity. The simplest exact solution of this equation, in the form of a plane wave of constant amplitude, is modulationally unstable, and the criterion for the existence of this instability depends on the sign of the product of the coefficients of the nonlinear and dispersive terms.[3,4] The NLS equation for gravity waves in deep water was first obtained by Zakharov by the spectral method,[5] later independently by Hasimoto and Ono[6] and Davey[7] by the multiscale expansion method, and also by Yuen and Lake[8] by the method of the averaged Lagrangian.

These and other[9,10] methods based on classical perturbation theory assume the existence of a single small parameter ϵ, which, for small amplitudes of the perturbations, characterizes the weak nonlinearity of the system. In the case of weak dispersion, slow space-time modulations of the envelope are characterized by one more small parameter μ, which is artifically related to ϵ. The result of using the expansion in one small parameter (instead of two) in the perturbation theory is that even at the outset one must assign a relation between the two parameters, and this in fact determines the form of the nonlinear and dispersive terms that will be contained in the leading order of the perturbation theory. All remaining terms appear in the higher orders an can be taken into account only as corrections after finding the solutions of the equation of the leading order (in our case, the NLS equation). A study of the evolution of the envelope with simultaneous treatment of the other terms with higher derivatives (both linear and nonlinear) using this perturbation theory is possible only under the assumption that the coefficients of the main terms also have a corresponding order of smallness in ϵ. In particular, for a liquid layer of finite depth, Johnson[11] and Kakutani *et al.*,[12] using the multiscale method, obtained a higher NLS equation valid near the critical value $kh = 1.363$ and analyzed the modulational instability of flow when the coefficient of the cubic nonlinearity in the NLS equation vanishes.

In the present paper, a perturbation theory method is developed to describe spectrally narrow wave packets of finite amplitude in the specific example of gravity waves on the surface of a deep liquid, which allows one to reduce the initial system of nonlinear equations in arbitrary order to a model evolution equation for the complex amplitude of the envelope. This equation is an asymptotic expansion of the first time derivative of the amplitude of the principal harmonic in the two independent parameters ϵ and μ. These parameters characterize the smallness of the amplitudes and their rates of variation in space, respectively.

When only the cubic nonlinearity, is retained the envelope in the problem evolves so that the second harmonic of the quasimonocharomatic signal gives no contribution in any order of the perturbation theory, while the contribution of the zeroth harmonic is associated with the appearance of nonlinear dispersion.

The instability of the exact homogeneous solution of the equation is investigated. Taking nonlinear dispersion into account, the criterion of modulational instability, the region of unstable wave numbers of the modulation for prescribed amplitude, the maximum growth rate of the amplitude in the initial stage of the development of the instability, the frequency shift, and the dispersion are all determined.

The nonlinear interaction of the zeroth and first harmonic leads to a difference of the growth rates for modulations propagating in the forward and backward directions relative to the direction of propagation of the main wave.

2. BASIC EQUATIONS OF THE WEAKLY NONLINEAR THEORY

We will investigate potential flow of an infinitely deep, inviscid and incompressible liquid with a free surface in a uniform gravity field as a planar problem of hydrodynamics. We choose the Cartesian coordinate sytem so that the Y axis points vertically upward and the unperturbed free surface coincides with the $Y=0$ plane. The profile of the surface is defined by the equation $y=\eta(x,t)$.

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The velocity potential $\mathbf{v}=\nabla\phi$, in the region D^- occupied by the liquid, is found by solving the boundary problem (in dimensionless variables)

$$\phi_{xx}+\phi_{yy}=0, \quad x,y\in D^-, \tag{1}$$

$$\phi_{tt}+\phi_y+(v^2)_t+\frac{1}{2}\mathbf{v}\nabla(v^2)=0, \quad y=\eta(x,t), \tag{2}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=-\infty}=0. \tag{3}$$

The quantities ϕ, $\mathbf{v}$, and also x and t, are written in dimensionless form. Assuming that the problem has some characteristic length l, the remaining normalization values, namely those of the velocity, the potential, and time are

$$v_l=\sqrt{gl}, \quad \phi_l=lv_l, \quad t_l=\frac{l}{v_l}, \tag{4}$$

where g is the acceleration due to gravity. The time dependence of the potential is parametric. Differentiation with respect to x, y, and t will be denoted by subscripts.

In the weakly nonlinear approximation $\epsilon=|\eta/l|\ll 1$ the boundary condition (2) reduces to the plane $y=0$ and the resulting boundary problem for the half-plane can be reformulated in terms of the complex potential[13,14]

$$W(z,t)=\phi(x,y,t)+i\psi(x,y,t), \quad z\in D^-, \tag{5}$$

where $\phi(x,y,t)$ is the hydrodynamic potential, $\psi(x,y,t)$ is the flow function, and the variable t is a parameter.

For the boundary value of the complex potential

$$w(x,t)=\lim_{y\to -0} W(z,t) \tag{6}$$

a nonlinear integrodifferential equation with a Caudry kernel was obtained in Refs. 13 and 14:

$$iw_{tt}-w_x-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{w^{(2)}(\xi,t)+w^{(3)}(\xi,t)+\dots}{\xi-x+i0}\,d\xi=0. \tag{7}$$

The quadratic and cubic terms in the integrand in Eq. (7) are equal to

$$w^{(2)}=(|w_x|^2)_t-\mathrm{Re}\, w_t\, \mathrm{Re}\, s_x, \tag{8}$$

$$\begin{aligned} w^{(3)}=&\mathrm{Re}(w_x^2\bar{w}_{xx})-2\,\mathrm{Re}\, w_t\, \mathrm{Im}(\bar{w}_{xx}w_x)_t\\ &-\frac{1}{2}|w_x|^2\,\mathrm{Re}\, s_x-\mathrm{Re}\, w_t\, \mathrm{Im}\, \bar{w}_{xt}\, \mathrm{Re}\, s_x\\ &-\frac{1}{2}(\mathrm{Re}\, w_t)^2\, \mathrm{Im}\, s_{xx}. \end{aligned} \tag{9}$$

Here Re and Im denote the real and imaginary parts of the complex functions, the bar over a variable indicates its complex conjugate, and $s(x,t)$ is the lower limit of the auxiliary complex Keldysh functions $S(z,t)$ (Ref. 15):

$$\begin{aligned} S(z,t)=iW_{tt}(z,t)-W'(z,t), \quad s(x,t)=&\lim_{y\to -0} S(z,t)\\ =iw_{tt}-w_x, \quad z\in D^-. \end{aligned} \tag{10}$$

3. EVOLUTION EQUATIONS OF HIGHER ORDER

Let us consider the evolution of a spectrally narrow wave packet of finite amplitude according to the nonlinear equation (7). We assume that the packet is characterized by two spatial scales $l=k^{-1}$ and $L=\kappa^{-1}$ such that $\mu=l/L=\kappa/k\ll 1$, where k and κ are the characteristic wave numbers of the carrier and modulation waves. In the weakly nonlinear approximation we represent the solution in the form of a Fourier series in the multiple harmonics of the carrier wave. It is assumed that, due to nonlinearity effects, the main wave excites multiple harmonics and the zeroth harmonic, which in turn leads to its modulation. For simplicity, we will take account of only the zeroth and second harmonics in the expansion, in addition to the principal harmonic (of order ϵ). These two harmonics are generatede by the quadratic terms and therefore are of order ϵ^2

$$\begin{aligned} w(x,t)=&\epsilon A(\mu x,t)e^{i(t-x)}+\epsilon^2(B(\mu x,t)\\ &+C(\mu x,t)e^{2i(t-x)})+O(\epsilon^3). \end{aligned} \tag{11}$$

In writing dowm the expansion (11), we have used dimensionless variables, with k^{-1} as the normalization length, and the frequency and wave vector of the principal harmonic satisfy the dispersion relation for linear gravity waves, $\omega^2=gk$. The slowness with which the amplitudes of the envelopes vary in space is characterized by the independent small parameter μ and is taken into account explicitly by introducing scale stretching: $\xi=\mu x$.

The space and time derivatives of the potential (11) of order l and m, respectively ($w_{lx,mt}\equiv w_{l,m}$), have the same structure

$$\begin{aligned} w_{l,m}(x,t)=&\epsilon\hat{A}_{l,m}(\mu x,t)e^{i(t-x)}+\epsilon^2[B_{l,m}(\mu x,t)\\ &+\hat{C}_{l,m}(\mu x,t)e^{2i(t-x)}]+O(\epsilon^3), \end{aligned} \tag{12}$$

$$\hat{A}_{l,m}(x,t)=\sum_{n,p=0}^{l,m}\mu^n a_{np}^{lm}A_{n\xi,pt}(\xi,t),$$

$$a_{np}^{lm}=i^{l+m-p-1}\binom{l}{n}\binom{m}{p}, \tag{13}$$

$$\hat{C}_{l,m}(x,t)=\sum_{n,p=0}^{l,m}\mu^n 2^{l+m-n-p}a_{np}^{lm}C_{n\xi,pt}(\xi,t),$$

$$B_{l,m}(x,t)=\mu^l B_{l\xi,mt}(\xi,t), \tag{14}$$

where the subscripts on A, B, and C, separated by a comma, denote n- and p-tuple differentiation with respect to ξ and t.

Correspondingly, for the boundary value of the Keldysh function (10) we have

$$\begin{aligned} s(x,t)=&\epsilon s^{(A)}(\mu x,t)e^{i(t-x)}+\epsilon^2(s^{(B)}(\mu x,t)+s^{(C)}\\ &\times(\mu x,t)e^{2i(t-x)})+O(\epsilon^3), \end{aligned} \tag{15}$$

where

$$\begin{aligned} &s^{(A)}(x,t)=i\hat{A}_{tt}-\hat{A}_x=iA_{tt}-2A_t-\mu A_\xi,\\ &s^{(B)}(x,t)=iB_{tt}-\mu B_\xi,\\ &s^{(C)}(x,t)=iC_{tt}-4C_t-\mu C_\xi, \end{aligned} \tag{16}$$

$$s_{l,m}(x,t)=\epsilon\hat{s}^{(A)}_{l,m}(\mu x,t)e^{i(t-x)}+\epsilon^2[s^{(B)}_{l,m}(\mu x,t)$$
$$+\hat{s}^{(C)}_{l,m}\mu x,t)e^{2i(t-x)}]+O(\epsilon^3)\,.$$

The expressions for $\hat{s}_{l,m}$ are analogous to expressions (13) and (14). If we substitute relations (11)–(16) into the master equation (7) and make use of the asymptotic limit[16]

$$\int_{-\infty}^{\infty}\frac{f(\xi)e^{ik\xi}}{\xi-x-i0}\,d\xi=2\pi i f(x)e^{ikx}+O(k^{-\infty}),\quad k\to\infty \tag{17}$$

we obtain a system of differential equations for the amplitudes A, B, and C:

$$s^{(A)}(x,t)+2i\epsilon^2(\omega^{(2)}_{AB}+\omega^{(2)}_{AC}+\omega^{(3)}_{A})=0, \tag{18}$$

$$s^{(B)}(x,t)-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\omega^{(2)}_{BA}}{\xi-x+i0}\,d\xi=0, \tag{19}$$

$$s^{(C)}(x,t)+2i\omega^{(2)}_{CA}=0, \tag{20}$$

where

$$\omega^{(2)}_{BA}=(|\hat{A}_x|^2)_t-\frac{1}{4}(\hat{A}_t\hat{s}^{*(A)}_x+\text{c.c.}), \tag{21}$$

$$\omega^{(2)}_{CA}=-\frac{1}{4}\hat{A}_t\hat{s}^{(A)}_x, \tag{22}$$

$$\omega^{(2)}_{AB}=(i+\partial/\partial t)\hat{A}_xB^*_x-\frac{1}{4}(B_t+B^*_t)\hat{s}^{(A)}_x$$
$$-\frac{1}{4}\hat{A}_t(s^{(B)}_x+s^{*(B)}_x), \tag{23}$$

$$\omega^{(2)}_{AC}=(i+\partial/\partial t)\hat{C}_x\hat{A}^*_x-\frac{1}{4}(\hat{C}_t\hat{s}^{*(A)}_x+\hat{A}^*_t\hat{s}^{(C)}_x, \tag{24}$$

$$\omega^{(3)}_{A}=\frac{1}{2}\hat{A}^2_x\hat{A}^*_{xx}-\hat{A}_t\,\mathrm{Im}(\hat{A}_{xt}\hat{A}^*_{xx}+\hat{A}_x\hat{A}^*_{xxt})-\frac{1}{4}|\hat{A}_x|^2\hat{s}^{(A)}_x$$
$$-\frac{1}{4}\hat{s}^{(A)}_x\,\mathrm{Im}(\hat{A}_t\hat{A}^*_{xt})-\frac{i}{8}\hat{A}_t\hat{A}_{xt}\hat{s}^{*(A)}_x$$
$$+\frac{i}{8}\left(\left|\hat{A}_t\right|^2\hat{s}^{(A)}_{xx}-\frac{1}{2}\hat{A}^2_t\hat{s}^{*(A)}_{xx}\right). \tag{25}$$

We have retained the nonlinear terms of order ϵ^3 in the equation for the amplitude of the principal harmonic [Eq. (18)], whereas in Eqs. (19) and (20) we have retained only the terms of order ϵ^2.

It is evident that stretching the coordinate ($\xi=\mu x$) induces an ordering of the spatial derivatives, as a result of which the system (18)–(20) is found by expanding in the parameters ϵ and μ and also contains time derivatives of the amplitudes A, B, and C of various orders. In order to establish a similar hierarchy of time derivatives, it is necessary to stretch the time coordinate.

The key idea in the construction of the systematic procedure enabling one to reduce systems of the form (18)–(20) to simpler forms is to represent the first time derivatives $A_t(\xi,t)$, $B_t(\xi,t)$, and $C_t(\xi,t)$ of the complex amplitudes by asymptotic power-series expansions in the parameters ϵ and μ of the most general form with undetermined constant coefficients. Since all higher time derivatives can be expressed in terms of the first time derivatives, the problem reduces to finding these coefficients. In this regard, the character of the expansion is determined by the specific structure of Eqs. (18)–(20).

Restricting ourselves, as before, to an account of the nonlinear terms with powers no higher than cubic, we represent the expansion for $A_t(\xi,t)$ in the form

$$A_t(\xi,t)=A^{(L)}_t+\epsilon^2A^{(NL)}_t+O(\epsilon^4), \tag{26}$$

$$A^{(L)}_t=\sum_{n=0}^{\infty}\mu^na_nA_{n\xi}(\xi,t). \tag{27}$$

The nonlinear terms of order ϵ^3 in various powers of μ can in general contain such terms as the following:

μ^0: A^2A^*; AB^*, AB; A^*C;

μ^1: AA^*A_ξ, $A^2A^*_\xi$; AB^*_ξ, AB_ξ,

$A_\xi B$, $A_\xi B^*$; $A^*_\xi C$, A^*C_ξ;

μ^2: $A^2A^*_{\xi\xi}$, $AA_\xi A^*_\xi$, $A^2_\xi A^*$; $AB^*_{\xi\xi}$,

$A_\xi B^*_\xi$, $A_{\xi\xi}B^*$; ... .

Proceeding from the specific form of Eq. (18) and taking into account only terms of order $\epsilon^3\mu$, we obtain the expansion for $A^{(NL)}_t$ in the form

$$A^{(NL)}_t=(iq_0|A|^2A+\mu\{-q_1|A|^2A_\xi-q_2A^2A^*_\xi+iq_3AB^*_\xi\}$$
$$+O(\mu^2))+O(\epsilon^3) \tag{28}$$

The constant coefficients a_n and q_n need to be defined. All of the expansions for the higher time derivatives can be expressed in terms of the expansion (28) of the first time derivative, for example

$$A_{\xi t}=a_1A_{2\xi}+\mu a_2A_{3\xi}+\mu^3a_3A_{4\xi}+\dots$$
$$+\epsilon^2\mu iq_0(|A|^2A)_\xi+\dots \tag{29}$$

$$A_{tt}=a_1^2A_{2\xi}+2\mu a_1a_2A_{3\xi}+\mu^2(2a_1a_3+a_2^2)A_{4\xi}+\dots$$
$$+i\epsilon^2\mu a_1q_0(|A|^2A)_\xi+\dots\,. \tag{30}$$

The harmonics $B(x,t)$ and $C(x,t)$ are generated by the main wave as a result of the nonlinearity and therefore must be determined as particular solutions of the inhomogeneous equations (19) and (20). Their expansions in the parameter ϵ should begin with terms that are quadratic in A

$$B(x,t)=\left(b_0\int_{-\infty}^{\infty}\frac{|A(\xi,t)|^2}{\xi-x+i0}\,d\xi+O(\mu)\right)+\dots, \tag{31}$$

$$C(x,t)=c_0A^2+\mu c_1AA_\xi+\dots\,. \tag{32}$$

Following expansion (26), we also divide the function $s^{(A)}(x,t)$ into a linear and a nonlinear term: $s^{(A)}=s^{(A)}_1+\epsilon^2s^{(A)}_2$, where

$$s^{(A)}_1=\sum_{n=0}^{\infty}\mu^n\alpha_nA_{n\xi}(\xi,t), \tag{33}$$

$$s_2^{(A)} = -2iq_0|A|^2A + 2\mu\{(q_1 - a_1q_0)|A|^2A_\xi + (q_2 - a_1q_0)A^2A_\xi^* - iq_3AB_\xi^*\}\ . \tag{34}$$

The first few coefficients α_n, according to the expansion (27), have the form

$$\alpha_0 = 0,\quad \alpha_1 = 1 + 2a_1,\quad \alpha_2 = ia_1^2 - 2a_2,$$
$$\alpha_3 = 2ia_1a_2 - 2a_3,\quad \alpha_4 = 2ia_1a_3 + ia_2^2 - 2a_4. \tag{35}$$

Obviously, independently of the nature of the nonlinearity in the equation for the envelope of the main harmonic (18), the undetermined coefficients a_n of expansion (27) are uniquely determined by the equation $s_1^{(A)} = 0$. Consequently, equating all the α_n to zero, we obtain recurrence relations for a_n. The general expression for a_n, taking account of the dispersion law $\omega = \sqrt{k}$ of the waves, can be represented in the form

$$a_n = \frac{i^{n+1}}{n!}\frac{d^n}{dk^n}\sqrt{k}|_{k=1}\ , \tag{36}$$

$$a_0 = 0,\quad a_1 = -\frac{1}{2},\quad a_2 = \frac{i}{8},\quad a_3 = \frac{1}{16}\ldots\ . \tag{37}$$

It is easy to show that for such values of a_n the linear parts of the functions $s_{1\xi}^{(A)}$ and $s_{1\xi\xi}^{(A)}$ vanish:

$$s_{1,\xi}^{(A)} = \left(-i + \mu\frac{\partial}{\partial\xi}\right)s_1^{(A)} = 0,\quad s_{1\xi\xi}^{(A)} = \left(1 - 2i\mu\frac{\partial}{\partial\xi} + \mu^2\frac{\partial^2}{\partial\xi^2}\right)s_1^{(A)} = 0.$$

Since we limit ourselves to nonlinear terms no higher than cubic, with derivatives no higher than first, all the terms in expressions (21)–(25) containing s fall out, and the expressions themselves substantially simplify

$$\omega_{BA}^{(2)} = -\frac{1}{2}\mu(|A|^2)_\xi,\quad \omega_{AB}^{(2)} = \mu AB_\xi^*\ ,\quad \omega_{AC}^{(2)} = \omega_{CA}^{(2)} = 0,$$

$$\omega_A^{(3)} = \frac{1}{2}|A|^2A + \mu\frac{i}{2}(3|A|^2A_\xi - A^2A_\xi^*).$$

Substituting them in Eq. (18) allows us to determine the remaining undetermined coefficients q_n

$$q_0 = \frac{1}{2},\quad q_1 = \frac{5}{4},\quad q_2 = -\frac{5}{8},\quad q_3 = 1. \tag{38}$$

According to Eq. (28), the parameters ϵ and μ have only a formal character, and in what follows we will set them equal to unity ($\epsilon = \mu = 1$), with the understanding that the power of the amplitude $A(x,t)$ in each term corresponds to the order of that term in ϵ, and the total order of all of the derivatives of the amplitude with respect to ξ, corresponds to the order of the term in μ. The final form of the evolution equation for the amplitude of the envelope of the principal harmonic of nonlinear gravity waves in deep water, with nonlinear dispersion taken into account, has the form

$$i\left(A_t + \frac{1}{2}A_x - \frac{1}{16}A_{3x} + \frac{5}{4}(|A|^2)_xA - \frac{5}{8}A^2A_x^*\right) + \frac{1}{8}A_{2x} + \frac{1}{2}|A|^2A + AB_x^* = 0,$$
$$B_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(|A|^2)_\xi}{\xi - x + i0}\,d\xi. \tag{39}$$

Thus, in the cubic approximation the second harmonic does not contribute to the evolution of $A(x,t)$. The zeroth harmonic gives rise to the single integral term in Eqs. (39) and contributes only to the nonlinear dispersion.

4. MODULATIONAL INSTABILITY

With the aim of isolating the effect of the individual terms, we will analyze the stability of the homogeneous solution for an equation that is more general than Eq. (39)

$$i\left(A_t + \omega' A_x - \frac{\omega'''}{6}A_{3x} + q_1\left|A\right|^2A_x + q_2A^2A_x^*\right) - \frac{\omega''}{2}A_{2x} + q_0|A|^2A + q_3AB_x^* = 0. \tag{40}$$

In the case of gravity waves, the corresponding coefficients are given by Eqs. (37) and (38), the primes denote the derivatives of the dispersion function $\omega(k)$ taken at the point $k=1$.

We introduce a small modulation in the form of a plane wave with wave vector κ and frequency Ω:

$$A + (A_0 + b_+e^{i(\mu x - \nu t)} + b_-e^{-i(\mu x - \nu t)})e^{iq_0A_0^2t}, \tag{41}$$

where $\mu = \kappa/k$ and $\nu = \Omega/\omega$. The instability occurs when $\nu^2 < 0$. Without loss of generality, we will assume that the coefficients A_0 and $b_\pm$ are real. After substituting expression (41) into Eq. (40) and making use of the inequality $|b_\pm| \ll A_0$, we linearize the resulting system of equations in $b_\pm$. Solving the linear problem for the eigenvalues, we obtain a dispersion relation from which the instability condition follows:

$$(\nu - \Delta)^2 = \mu^2\left(\left(\frac{\mu\omega''}{2}\right)^2 - q_3A_0^2\left(\frac{\mu\omega''}{2}\right) + RA_0^2\right) < 0, \tag{42}$$

where

$$\Delta = \mu\left\{\omega' + \frac{1}{6}\mu^2\omega''' + \left(q_1 + \frac{1}{2}q_3\right)A_2^0\right\}, \tag{43}$$

$$R = q_0\omega'' + \left(q_2 + \frac{1}{2}q_3\right)^2A_0^2. \tag{44}$$

The wave vector interval of the modulation for $q_0\omega'' < 0$ follows directly from inequality (42):

$$\mu_- < \mu < \mu_+\ ,$$

where

$$\mu_\pm = \pm\frac{2A_0}{|\omega''|}\sqrt{-\{\omega''q_0 + q_2(q_2 + q_3)A_0^2\}} + \frac{q_3}{\omega''}A_0^2. \tag{45}$$

The maximum growth rate of the amplitude in the instability region is attained at two values of the modulation wave vector

$$\mu_{1,2}=\pm\frac{A_0}{2|\omega''|}\sqrt{(1.5q_3A_0)^2-8R}+\frac{3}{4\omega''}q_3A_0^2 \quad (46)$$

and is equal to

$$\mathrm{Im}\,\nu_{1,2}=\frac{|\mu_{1,2}|A_0}{\sqrt{2}}\times\sqrt{q_0\omega''+\left\{\left(q_2+\frac{1}{2}q_3\right)^2-\frac{1}{4}\omega''q_2\mu_{1,2}\right\}A_0^2}\,. \quad (47)$$

Thus, in Eq. (40), along with the main terms proportional to ω'' and q_0, which govern the evolution of the amplitude of the envelope in the classical NLS equation,[1] new terms are present which give corrections associated with linear ($\propto\omega'''$) and nonlinear dispersion. It is clear from Eqs. (42)–(44) that the terms proportional to ω''' and q_1 contribute only to the frequency shift. Here the term $\propto\omega'''$ describes the ordinary dispersive spreading of the envelope.[17] It becomes necessary to take it into account in some special cases, e.g., when $\omega''\approx 0$ (Refs. 18 and 19). The term $(q_1+q_3/2)A_0^2$ represents a nonlinear correction to the group velocity of the gravity wave.

The nonlinear term $\propto q_3$ associated with the contribution of the zeroth harmonic (Stokes flow) plays a special role. As follows from Eqs. (46) and (47), it introduces an asymmetry into the conditions of the modulational instability of the perturbations propagating in the forward and backward directions relative to the direction of propagation of the main wave.

In our case of weakly nonlinear gravity waves in deep water ($A_0\ll 1$), according to Eqs. (37) and (38) we have

$$\mu_{\pm}=\pm\sqrt{8}A_0(1\mp\sqrt{2}A_0), \quad (48)$$

$$\mu_{1,2}=\pm 2A_0\left(1\mp\frac{3}{2}A_0\right), \quad (49)$$

$$\mathrm{Im}\,\nu_{1,2}=\frac{1}{2}A_0^2\left(1\mp\frac{3}{2}A_0\right). \quad (50)$$

The second term in parentheses in each equation is a correction to the known expression, first obtained by Benjamin and Feir[4] and is the result of accounting for nonlinear dispersion. The difference in the maximum growth rates for the forward and backward propagating modulation waves is $\propto A_0^3$. And although this difference is small for a broad modulation spectrum, only that fluctuation is manifested which has the maximum growth rate, namely the fluctuation propagating in the direction opposite the direction of propagation of the main wave.

[1] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, London, 1982).
[2] H. C. Yuen and B. M. Lake, *Nonlinear Dynamics of Gravity Waves in Deep Water* [Russian translation] (Mir, Moscow, 1987).
[3] M. J. Lighthill, J. Inst. Math. Appl. **1**, 269 (1965); Proc. Roy. Soc. A. **299**, 28 (1967).
[4] T. B. Benjamin and E. Feir, J. Fluid Mech. **27**, 417 (1967).
[5] V. E. Zakharov, Zh. Prikl. Mekh. Tekh. fiz., No. 2, 86 (1986).
[6] H. Hasimoto and H. Ono, J. Phys. Soc. Japan **33**, 805 (1972).
[7] A. Davey, J. Fluid Mech. **53**, 769 (1972).
[8] H. C. Yuen and B. M. Lake, Phys. Fluids **18**, 956 (1975).
[9] A. Nayfeh, *Perturbation Methods* (Wiley–Interscience, New York, 1973).
[10] T. Taniuti and N. Yajima, J. Math. Phys. **10**, 1369 (1969).
[11] R. S. Johnson, Proc. R. Soc. **A357**, 131 (1977).
[12] T. Kakutani and K. Michihiro, J. Phys. Soc. Japan **52**, 4129 (1983).
[13] V. P. Lukomskiĭ and Yu. V. Sedletskiĭ, Preprint of the Institute of Physics, Ukr. Acad. Sci., No. 11 (1993).
[14] V. P. Lukomskiĭ, JETP Lett. **59**, 622 (1994).
[15] M. V. Keldysh, *Collected Works*, Book 2, *Mechanics* [in Russian] (Nauka, Moscow, 1985), p. 10.
[16] M. V. Fedoroyuk, *The Method of Steepest Descent* [in Russian] (Nauka, Moscow, 1987).
[17] A. Hasegawa and Y. Kodama, Proc. IEEE **69**, 1145 (1981).
[18] S. A. Akhmanov, V. A. Vysloukh, and A. S. Chirkin, Usp. Fiz. Nauk **149**, 449 (1986) [Sov. Phys. Usp. **29**, 642 (1986)].
[19] M. Oikawa, J. Phys. Soc. Japan **62**, 2324 (1993).

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