Theory of radiation by electrons in a spiral undulator with a longitudinal magnetic field

N. K. Zhevago and V. I. Glebov

Russian "Kurchatov Institute" Research Center, 123182 Moscow, Russia
(Submitted 19 April 1995)
Zh. Éksp. Teor. Fiz. **108**, 493–510 (August 1995)

We develop in this paper a theory for the spontaneous and stimulated emission by relativistic electrons in a magnetic field which is a superposition of the spiral field of an undulator and a uniform longitudinal field. We show that the electron trajectories precess in the transverse plane, while along the axis of the undulator longitudinal oscillations are excited with an amplitude which depends on the magnitude of the longitudinal field. We find in analytical form the frequency and angle distribution of the emitted energy when the electrons move along such trajectories. Using the results for the spectra of the spontaneous emission and the energy and longitudinal quasimomentum conservation laws for the emission and the absorption of a photon by an electron we obtain relatively simple analytical expressions for the amplification coefficient. We give an analysis of the effect of the longitudinal field on the efficiency with which the electron energy is transformed into radiative energy.

1. INTRODUCTION

Free electron lasers in which a spiral undulator with an additional longitudinal magnetic field is used have been studied experimentally in a number of papers[1–3] and in Conde and Bekefi's experiment[3] the longitudinal magnetic field was varied not only in magnitude but also in direction. The presence of the additional longitudinal field leads to a significant change in the power of the emission and the efficiency with which the energy of an electron beam is transformed into electromagnetic radiative energy. The set of nonlinear equations describing the interaction of a beam of relativistic electrons with an external electromagnetic wave in the field of a real undulator with an additional longitudinal magnetic field was formulated in Ref. 4. The authors there took into account that the magnetic field of a real undulator is inhomogeneous in the transverse direction and has an oscillating longitudinal component, while inside the undulator there is a waveguide and the electrons interact with the *TE* or *TM* modes of the electromagnetic radiation. In a more recent paper[5] the effect of the eigenfield of the electron beam was also taken into account in the set of equations. The existing analysis[5–7] of the experimental results of Ref. 3 was performed via numerical integration of the set of nonlinear equations. In contrast to this approach, which is essentially a method for numerically simulating the experiments, in the present paper we shall develop a theory which enables us to find the spectral and angular distribution of the spontaneous emission and the amplification coefficient in a spiral undulator with an additional longitudinal field in analytical form. Although some minor details such as the transverse inhomogeneity of the undulator field and nonlinear effects in the interaction of the electron beam with the accelerated wave are neglected in our calculations, the main effect connected with the important change in the nature of the transverse and the longitudinal motion of the electrons when there is a uniform magnetic field superimposed on the spiral field of the undulator can be analyzed in detail in each stage of the theory developed here.

2. SOLUTION OF THE EQUATIONS OF MOTION AND ANALYSIS OF THE TRAJECTORIES

In order later to calculate the emission spectra we need to find the solution of the electron equation of motion,

$$E\,\frac{d\mathbf{v}}{dt}=e[\mathbf{vH}], \tag{1}$$

where E is the total electron energy ($m=c=1$) in the magnetic field, which is the sum of the undulator field and a longitudinal field,

$$\mathbf{H}(z)=H_\perp(\mathbf{n}_x\cos\omega_w z+\mathbf{n}_y\sin\omega_w z)+H_z\mathbf{n}_z. \tag{2}$$

The quantities $\mathbf{n}_x$, $\mathbf{n}_y$, and $\mathbf{n}_z$ are unit vectors along the coordinate axes; we have written $\omega_w=2\pi/\lambda_w$, where λ_w is the period of the spiral undulator; $H_\perp>0$ is the amplitude of the undulator field, and H_z is the longitudinal magnetic field strength, which is directed along the z-axis. Positive values of H_z correspond to situations when the sense of rotation of the electron in the xy-plane in the field of the undulator (for $H_z=0$) is the opposite to that of the rotation in the uniform field (i.e., for $H_\perp=0$), while negative values of H_z correspond to the same sense of rotation. We neglect the inhomogeneities of the fields in the transverse directions, i.e., $H_\perp$ and H_z are independent of the transverse coordinates x and y.

We introduce linear combinations of the transverse velocity components: $v^{(+)}=v_x+iv_y$, $v^{(-)}=v_x-iv_y$. We can then write the equations of motion in the form

$$\dot v^{(+)}=-i\omega_H v^{(+)}+i\omega_w(p/E)v_z e^{i\omega_w z},$$

$$\dot v^{(-)}=i\omega_H v^{(-)}-i\omega_w(p/E)v_z e^{-i\omega_w z},$$

$$v_z^2=1-E^{-2}-v^{(+)}v^{(-)}, \tag{3}$$

where the dot indicates a time derivative, $\omega_H=eH_z/E$ is the relativistic cyclotron frequency, and $p=eH_\perp/\omega_w$ is the spiral undulator parameter. The last of Eqs. (3) is a consequence of the conservation of the electron energy E in a magnetic field.

 1063-7761/95/080265-10$10.00

We assume that the transverse (with respect to the z-axis) velocity components of the electron always remain small compared to the longitudinal component v_z. Since the estimate $v_\perp \sim p/E$ is valid for the transverse component, the electron energy must be at least relativistic, and unless limited by small undulator parameters p the energy must be ultrarelativistic, $E \gg p$. In that case one can solve the nonlinear set of Eqs. (3) by the method of consecutive approximations in the small parameter $v_\perp/v_z$. In the zeroth approximation we get $v_z \approx 1$, $z \approx t$. We put these values in the first two equations of the set (3). After this we find the solution to the first approximation

$$v^{(+)}(t)=e^{-i\omega_H t}\left[v_0 e^{i\psi}+\frac{p}{E(1+q)}\left(e^{i(\omega_w+\omega_H)t}-1\right)\right]. \qquad (4)$$

Here v_0 is the initial value of the magnitude of the transverse velocity with a direction which makes an angle ψ with the x-axis of the chosen coordinate system, $q=\omega_w/\omega_H$ is the ratio of the cyclotron frequency to the frequency of the spiral undulator. Note that when one changes the direction of the longitudinal field the quantities H_z and ω_H and also q become negative, and the condition $v_\perp \ll v_z$ used when solving the equations of motion according to (4) takes the form $E \gg p/|1+q|$. For negative q close to $q=-1$ this condition imposes more rigorous restrictions on the upper limit of the electron energy than for $q>0$. Using the approximate equation

$$v_z(t)\approx 1-\frac{1}{2}\left[E^{-2}+|v^{(+)}|^2\right], \qquad (5)$$

which follows from the last equation of the set (3) and substituting (4) into (5) we get for the longitudinal velocity component

$$v_z(t)=1-\frac{1}{2E^2}\left\{1+\left(\frac{p}{1+q}\right)^2 \times\left[k^2+1+2k\cos[(\omega_w+\omega_H)t-\alpha]\right]\right\}. \qquad (6)$$

The quantities k and α introduced here are connected with the initial velocity through the relation $ke^{i\alpha}=\Delta e^{i\psi}(1+q)-1$ with $\Delta=Ev_0/p$. They can be expressed as follows in terms of the components v_{0x} and v_{0y} of the initial velocity

$$k=|[1-(1+q)(Ev_{0x}/p)]^2+[(1+q)(Ev_{0y}/p)]^2|^{1/2},$$

$$\alpha=\arctan\frac{(1+q)v_{0y}}{(1+q)v_{0x}-p/E}. \qquad (7)$$

Subsequent integration of (4) with the initial conditions $x(0)=y(0)=0$ leads to the following expression for $\rho^{(+)}=x+iy$:

$$\rho^{(+)}(t)=i\,\frac{R_0}{q}\,(1-\Delta e^{i\psi})+i\,\frac{R_0}{1+q}\left(\frac{ke^{i\alpha}}{q}\,e^{-i\omega_H t}-e^{i\omega_w t}\right),$$

where $R_0=p/(E\omega_w)$. The nature of the electron trajectories in the transverse xy-plane becomes clearer if we introduce cylindrical coordinates $\rho(t),\varphi(t)$ for the electron using the relations

$$\rho(t)=|\rho^{(+)}-\rho_c|,\quad \rho_c=i\,\frac{R_0}{q}\,(1-\Delta e^{i\psi}),$$

$$\varphi(t)=\arg(\rho^{(+)}-\rho_c)=\frac{1}{2i}\ln\frac{\rho^{(+)}-\rho_c}{\rho^{(+)*}-\rho_c^*},$$

where $\rho^*=\mathrm{Re}\,\rho-i\mathrm{Im}\,\rho$. As a result we find

$$\rho(t)=R\left\{\left(\frac{k}{q}\right)^2+1-2\,\frac{k}{q}\cos[(\omega_w+\omega_H)t-\alpha]\right\}^{1/2},$$

$$\varphi(t)=\alpha+\frac{\pi}{2}-\omega_H t+\frac{1}{2i}\ln\frac{(k/q)-e^{i(\omega_\rho t-\alpha)}}{(k/q)-e^{-i(\omega_\rho t-\alpha)}} \qquad (8)$$

with $R=R_0/|1+q|$.

In the transverse plane the electron thus carries out radial oscillations with a frequency $\omega_\rho=\omega_w+\omega_H$ equal to the algebraic sum of the undulator frequency ω_w and the cyclotron frequency ω_H. The pericenter $\rho_{\min}$ and the apcenter $\rho_{\max}$ of the orbit are then determined by the relations

$$\rho_{\min}=R\min\left[\,|(k/q)-1|,|(k/q)+1|\right],$$

$$\rho_{\max}=R\max\left[\,|(k/q)-1|,|(k/q)+1|\right]. \qquad (9)$$

The coordinates x_c and y_c of the center about which the radial oscillations take place are connected with the initial transverse velocity through the equations

$$x_c=v_{0y}/\omega_H,\quad y_c=[(p/E)-v_{0x}]/\omega_H.$$

The turning angle of the electron around the center is a quasiperiodic function of the time, i.e., the equation $\varphi(t+T_\rho)=\varphi(t)+\Delta\varphi$ holds, where $T_\rho=2\pi/\omega_\rho$ is the period of the radial oscillations and $\Delta\varphi$ is the precession angle. Apart from an unimportant term which is a multiple of 2π, the latter can be written in the form

$$\Delta\varphi=-2\pi q/(1+q). \qquad (10)$$

For the precession frequency Ω defined, as usual, by the relation $\Omega=\Delta\varphi/T_\rho$ we get the equation

$$\Omega=-\omega_H. \qquad (11)$$

The precession frequency is thus, apart from the sign, the same as the cyclotron frequency. Integrating (6) with the initial condition $z(0)=0$ we find the longitudinal coordinate of the electron as a function of the time in the form

$$z(t)=\langle v_z\rangle t-a\{\sin[(\omega_w+\omega_H)t-\alpha]+\sin\alpha\}. \qquad (12)$$

Here we have introduced the notation

$$\langle v_z\rangle=1-\frac{1}{2E^2}\left[1+\left(\frac{p}{1+q}\right)^2(k^2+1)\right] \qquad (13)$$

for the longitudinal velocity averaged over a period of the radial oscillations and

$$a=\frac{1}{\omega_w}\left(\frac{p}{E}\right)^2\frac{k}{(1+q)^3} \qquad (14)$$

for the amplitude of the longitudinal oscillations. According to (12) the longitudinal oscillations occur with the frequency ω_ρ of the radial oscillations and an amplitude a and the oscillations take place relative to a center which moves uniformly along the z-axis with a velocity $\langle v_z\rangle$.

The special initial conditions for which the initial transverse velocity is directed along the x-axis ($v_{0y}=0$) and has magnitude equal to $v_{0x}=p/E$ correspond to situations such that the electron is in a circular transverse orbit of the undulator until $t=0$ (this is possible if the amplitude of the undulator field grows adiabatically from zero to $H_\perp$ and $v_\perp=0$ at $t=-\infty$) while the longitudinal magnetic field is switched on at time $t=0$. In that case we have $\rho_c=0$. Hence the electron precession occurs relative to the point of injection $x=0$, $y=0$, while the parameter k in Eqs. (8) to (14) becomes equal to $|q|$, the initial phase of the oscillations in (8), and (12) is determined by the equation

$$\alpha=\begin{cases} 0, & q>0, \\ \pi, & q<0, \end{cases}$$

and the pericenter becomes equal to zero. As a result the electron trajectory can be written in a simpler form:

$$\rho(t)=2R\left|\sin\frac{\omega_p}{2}t\right|, \tag{15}$$

$$\varphi(t)=\frac{\omega_w-\omega_H}{2}t-\pi n, \quad nT_p\leqslant t\leqslant(n+1)T_p,$$

$$n=0,1,\ldots,$$

$$z(t)=\langle v_z\rangle t-\tilde{a}\sin\omega_p t, \quad \tilde{a}=\frac{1}{\omega_w}\left(\frac{p}{E}\right)^2\frac{q}{(1+q)^3},$$

$$\langle v_z\rangle=1-\frac{1}{2E^2}\left[1+\frac{p^2(1+q^2)}{(1+q)^2}\right]. \tag{16}$$

If there is no longitudinal field ($q=0$) the trajectory (15) becomes a circle of radius R_0 with center in the point $(0,R_0)$, while for $q=1$ (antiresonance) it becomes the section $x(t)=R_0\sin\omega_w t$, $y(t)=0$. We note that in the more general case when k may differ from $|q|$ the trajectory (8) is in the antiresonance point $q=1$ an ellipse with eccentricity equal to $2\sqrt{k}/(k+1)$.

We can distinguish two other special initial conditions. If the electron enters the field parallel to the undulator axis ($v_0=0$) the coordinates of the precession center are determined by the equations $x_c=0$, $y_c=p/(E\omega_H)$ and the value of the parameter k in Eqs. (6) to (9), (12), and (13) which we found earlier becomes equal to unity. In the other case when the initial transverse velocity satisfies the conditions $v_{0x}=p/[E(1+q)]$, $v_{0y}=0$ the quantity k vanishes and the electron trajectory becomes a circle of the form $\rho(t)=R$, $\varphi(t)=\pi/2+\omega_w t$ with its center in the point $[0, R\,\mathrm{sign}(1+q)]$. This last case is characterized by the fact that the amplitude of the longitudinal oscillations (14) vanishes.

For general initial conditions the orbits (8) and (15) are, depending on the value of the parameter q, precessing rosettes or spirals which, in general, are not closed (Fig. 1). The orbits close after a well defined number of periods of the radial oscillations only when the ratio of the precession frequency to the frequency of the radial oscillations, i.e., the quantity $|q/(1+q)|$, is a rational number. The precession increases the transverse motion of the electrons while, on the other hand, an increase in the longitudinal magnetic field strength for positive q decreases the distance to the center (9) and causes a corresponding compression of the transverse motion. The same picture is observed also for negative values of $q<-1$. Meanwhile, in the range $-1<q<0$ the distance to the center increases sharply as the resonance $q=-1$ is approached. If the center of the initial circular orbit (15) was lying on the undulator axis a significant increase in the possible distance of the electron from the axis under the effect of the longitudinal field may make it necessary to include the dependence of the transverse component $H_\perp$ of the undulator field on the transverse coordinate ρ and also cause a longitudinal component of the field to arise for a real undulator (oscillating with a period λ_w).[8] The condition for the effects to be small when the field is nonuniform or nontransverse in a real undulator is $\omega_w\rho\ll 1$, where ρ is the maximum distance of the electron from the undulator axis due to precession. This implies

$$E\gg\frac{p}{|1+q|},$$

which is the same as the condition, introduced earlier, that $v_\perp$ be small as compared to v_z.

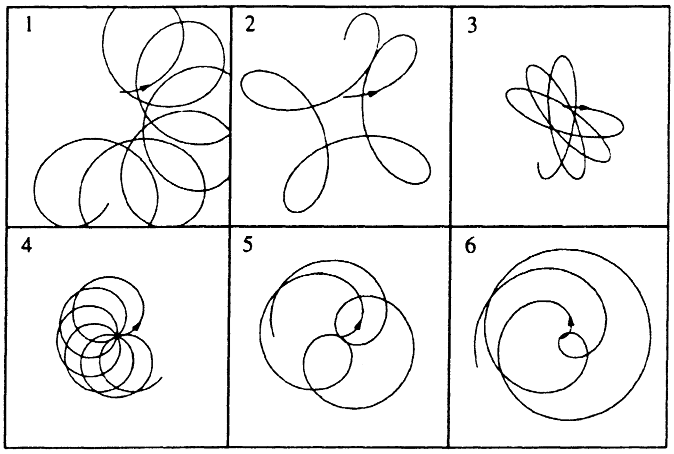

FIG. 1. Typical transverse electron trajectories in a spiral undulator with a longitudinal field for various parameters q and k: *1*: $q=2k/3=0.1$; *2*: $q=2k/3=0.3$; *3*: $q=2k/3=1.2$; *4*: $q=k=-0.1$; *5*: $q=k=-0.3$; *6*: $q=k=-1.6$. The sides of the squares bounding the trajectories are taken equal to $2R_0=2p/(E\omega_w)$ for the trajectories with $q>0$ and equal to $4R_0$ for trajectories with $q<0$. The arrows on the trajectories indicate the direction of the motion from the starting point.

3. DERIVATION OF FORMULAS FOR THE SPECTRAL AND ANGULAR DISTRIBUTION OF THE EMISSION FROM AN UNDULATOR WITH A LONGITUDINAL FIELD

The general formulas for calculating the spectral and angular distribution of the intensity of the emission by ultrarelativistic particles with transverse trajectories which precess with an arbitrary frequency Ω have been obtained before[9] in connection with the analogous problem of the axial channelling of electrons in a crystal. According to Ref. 9 the differential (with respect to the frequency and the solid angle of the emission) intensity of the emission by an electron can be written in the form

$$\frac{d^2I}{d\omega dO}=\frac{e^2\omega^2}{8\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}[|j_{nm}^{(+)}-j_{nm}^{(-)}|^2+|2\theta j_{nm}^{(z)}-(j_{nm}^{(+)}+j_{nm}^{(-)})|^2]\delta[\omega-k_z\langle v_z\rangle-n\omega_\rho-m\Omega]. \quad (17)$$

Here ω is the frequency of the radiation; $k_z=\omega\cos\theta \approx\omega(1-\theta^2/2)$; $dO\approx\theta d\theta\, d\varphi_r$ is the differential solid angle; $\theta\ll 1$ and φ_r are, respectively, the polar and azimuthal angles of the radiation; and n,m are the radial and azimuthal numbers of the harmonics corresponding, respectively, to the frequency ω_ρ of the radial oscillations and the precession frequency Ω. The Fourier components of the current which occur in (17) are determined by the following integrals over the period T_ρ of the radial oscillations

$$j_{nm}^{(\pm)}=\frac{\mp i}{T_\rho}\int_0^{T_\rho}(\dot\rho\pm i\rho\dot\varphi)J_{m\mp1}(\kappa\rho)\exp[im\varphi-i\times(m\Omega+n\omega_\rho)t+i\omega\Delta z]dt,$$

$$j_{nm}^{(z)}=\frac{1}{T_\rho}\int_0^{T_\rho}J_m(\kappa\rho)\exp[im\varphi-i(m\Omega+n\omega_\rho)\times t+i\omega\Delta z]dt, \quad (18)$$

where we have set $\Delta z=z(t)-\langle v_z\rangle t$ and $\kappa\approx\theta\omega$ is the transverse component of the wavevector of the radiation. The angular brackets indicate averaging of the corresponding quantities over a period of the radial oscillations and the dot indicates the time derivatives of the cylindrical coordinates $\rho(t)$ and $\varphi(t)$ of the electron.

It was noted in Ref. 9 that Eq. (17) is the spectral and angular distribution of the intensity of the radiation which is, generally speaking, averaged over the azimuthal angle φ_r since the emission turns out to be axially symmetric only for transverse orbits, which are not closed, with an infinitely large number of radial oscillations, and when deriving (17) we cannot carry out an additional averaging over the azimuthal angle of the radiation. Because of the analogy of the electron trajectories noted above we can use the general results represented by Eqs. (17) and (18), with certain modifications, to calculate the emission spectra from a spiral undulator with a longitudinal field. The Dirac delta-function in (17) corresponds to an infinite electron-field interaction length. If we take into account the finite length $L\gg\lambda_w$ of the undulator we must replace $\delta(\xi)$ by the function

$$f(\xi)=\frac{2}{\pi L}\frac{\sin^2(L\xi/2)}{\xi^2}, \quad (19)$$

and the spectral and angular distribution $d^2W/d\omega dO$ of the energy of the emission by the undulator is then obtained from (17) by multiplying by the electron-field interaction time $t_{int}=L/v_z\approx L$.

Substituting Eqs. (8) for the transverse electron coordinates $\rho(t)$ and $\varphi(t)$ and (12) for the longitudinal coordinate (relative to the uniformly moving center) $\Delta z(t)$ into the general formulas (18) leads, apart from an unimportant phase factor $\exp(-i\omega a\sin\alpha)$, to the following integrals for the Fourier components of the current

$$j_{nm}^{(z)}=\frac{1}{T_\rho}\int_0^{T_\rho}J_m(\kappa\rho)\left(\frac{k/q-e^{i\chi}}{k/q-e^{-i\chi}}\right)^{m/2}e^{-in\omega_\rho t}\times\exp[im(\alpha+\pi/2)-i\omega a\sin\chi]dt,$$

$$j_{nm}^{(\pm)}=\frac{\mp iR\omega_w}{T_\rho}\int_0^{T_\rho}J_{m\mp1}(\kappa\rho)\left(\frac{k/q-e^{i\chi}}{k/q-e^{-i\chi}}\right)^{(m\mp1)/2}\times(k+e^{\pm i\chi})e^{-in\omega_\rho t}\exp[im(\alpha+\pi/2)-i\omega a\sin\chi]dt. \quad (20)$$

We have here introduced the notation $\chi=\omega_\rho t-\alpha$ and, according to (8) we have

$$\rho=R[(k/q-e^{i\chi})(k/q-e^{-i\chi})]^{1/2}.$$

One can evaluate the integrals (20) as follows. We use the Jacobi–Anger formula[10]

$$\exp(-i\omega a\sin\chi)=\sum_{\nu=-\infty}^{\infty}J_\nu(\omega a)e^{-i\nu\chi},$$

and also Gegenbauer's summation formula for Bessel functions[10]

$$J_m(\kappa\rho)\left(\frac{k/q-e^{i\chi}}{k/q-e^{-i\chi}}\right)^{m/2}=\sum_{\mu=-\infty}^{\infty}J_{m+\mu}\left(\frac{k}{q}\kappa\rho\right)J_\mu(\kappa\rho)e^{-i\mu\chi}.$$

After this the term-by-term integration of the series we have obtained is elementary. As a result we get, apart from an unimportant phase factor which is the same for all three components of the current,

$$j_{nm}^{(z)}=\sum_{\nu=-\infty}^{\infty}e^{i\nu\pi}J_\nu(\omega a)J_{m-(n+\nu)}\left(\frac{k}{q}\kappa R\right)J_{n+\nu}(\kappa R),$$

$$j_{nm}^{(\pm)}=R\omega_w\sum_{\nu=-\infty}^{\infty}e^{i\nu\pi}J_\nu(\omega a)\left[J_{m-(n+\nu)}\times\left(\frac{k}{q}\kappa R\right)J_{n+\nu\mp1}(\kappa R)-kJ_{m\mp1-(n+\nu)}\times\left(\frac{k}{q}\kappa R\right)J_{n+\nu}(\kappa R)\right]. \quad (21)$$

In the case considered we obtained for the frequency of the radial oscillations and for the precession frequency the equations $\omega_\rho=\omega_H+\omega_w$, $\Omega=-\omega_H$, so that it is more convenient to change in (17) from summations over the radial and azimuthal harmonics n and m to a summation over the undulator harmonic n and the cyclotron harmonic $n'=n-m$. The spectral and angular density of the emission energy from a spiral undulator with a longitudinal field can then be rewritten in the form

$$\frac{d^2W}{d\omega dO}=\frac{e^2\omega^2}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\left[|\tilde{j}^{(+)}_{nn'}-\tilde{j}^{(-)}_{nn'}|^2+|2\theta\tilde{j}^{(z)}_{nn'}-(\tilde{j}^{(+)}_{nn'}+\tilde{j}^{(-)}_{nn'})|^2\right]\frac{\sin^2(L\xi_{nn'}/2)}{\xi^2_{nn'}}, \tag{22}$$

In accordance with (19) and (21) we have here introduced the notation

$$\xi_{nn'}=\omega-k_z\langle v_z\rangle-n\omega_w-n'\omega_H, \tag{23}$$

$$\tilde{j}^{(z)}_{nn'}=\sum_{\nu=-\infty}^{\infty}J_\nu(\omega a)J_{n+\nu}(\kappa R)J_{n'+\nu}\left(\frac{k}{q}\kappa R\right),$$

$$\tilde{j}^{(\pm)}_{nn'}=R\omega_w\sum_{\nu=-\infty}^{\infty}J_\nu(\omega a)\left[J_{n'+\nu}\left(\frac{k}{q}\kappa R\right)J_{n+\nu\mp 1}(\kappa R)+kJ_{n'+\nu\pm 1}\left(\frac{k}{q}\kappa R\right)J_{n+\nu}(\kappa R)\right]. \tag{24}$$

According to (13) we can write the phase shift $\Phi=\omega-k_z\langle v_z\rangle$, which occurs in (23), in the form

$$\Phi=\frac{\omega}{2}\left[\theta^2+E^{-2}\left(1+p^2\frac{1+k^2}{(1+q)^2}\right)\right]. \tag{25}$$

We bear in mind that, according to the results obtained above, the other quantities occurring in (22) to (25) have, the following form: $\kappa=\omega\theta$, $R_0=p/(E\omega_w)$, $R=R_0/|1+q|$, $p=eH_\perp/\omega_w$, $q=\omega_H/\omega_w$, $\omega_H=eH_z/E$, and the quantities k and a are defined by Eqs. (7) and (14).

4. ANALYSIS OF THE GENERAL EXPRESSION FOR THE SPECTRAL AND ANGULAR DISTRIBUTION OF THE RADIATION

The possible frequencies of the emission at an angle θ to the axis of a spiral undulator with a longitudinal magnetic field are concentrated near the positive zeroes of $\xi_{nn'}$ regarded as a function of the frequency. Denoting these zeroes by $\omega_{nn'}$ we find

$$\omega_{nn'}=\frac{2(n\omega_w+n'\omega_H)}{\theta^2+E^{-2}\left(1+p^2\frac{1+k^2}{(1+q)^2}\right)}. \tag{26}$$

The emission can thus take place both at frequencies which are multiples of the undulator frequency ($n'=0$, $n\neq 0$) or of the cyclotron frequency ($n'\neq 0$, $n=0$) and at combination frequencies ($n'\neq 0$, $n\neq 0$), and in the last case even when $\omega_H>0$ holds one of the numbers n and n' may be negative (only the sum $n\omega_w+n'\omega_H$ must be positive). The Doppler shift of the emitted frequencies $\omega_{nn'}$ relative to the sum $n\omega_w+n'\omega_H$ depends both on the undulator parameter p and on the parameter $q=\omega_H/\omega_w$. The homogeneous spectral line width is, according to (22), determined by the total undulator length

$$\frac{\Delta\omega_{nn'}}{\omega_{nn'}}\simeq\frac{\pi}{L}\frac{1}{(n\omega_w+n'\omega_H)}. \tag{27}$$

However, in real situations there are also inhomogeneous broadening mechanisms. In particular, the spread in energy E or in the square of the initial transverse velocity v_0^2 (angular spread of the beam) in the electron beam leads to a Doppler broadening with a magnitude which one can estimate as the corresponding variation in $\omega_{nn'}$.

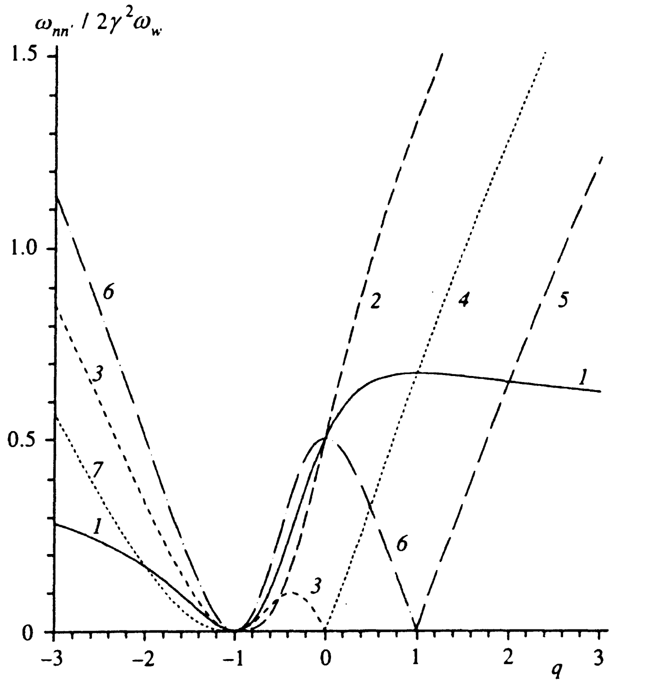

FIG. 2. The frequencies $\omega_{nn'}$ of the radiation at a zero angle θ of the various harmonics (nn') as functions of the ratio $q=\omega_H/\omega_w$ of the cyclotron frequency ω_H to the undulator frequency ω_w. The numbers on the curves correspond to the following harmonics: *1*: (1,0); *2*: (0,1); *3*: (0,−1); *4*: (1,1); *5*: (−1,1); *6*: (1,−1); *7*: (−1,−1).

The dependence on the parameter q of the frequencies $\omega_{nn'}$ emitted at an angle $\theta=0$ for a fixed $p=1$ and for the special initial conditions [see (15)] when $k=|q|$ holds are shown in Fig. 2. The intersection of the curves for several values of q means that the frequency spectrum becomes degenerate. In particular, for $q=1$ the frequencies $\omega_{nn'}$ in (26) depend only on the sum $n+n'$. The degeneracy of the spectrum is connected with the closing of the transverse orbits, mentioned in §2, for those values of q. In the region of the resonance $q=-1$ all frequencies $\omega_{nn'}$ decrease fast; this is connected with the steep increase in the transverse velocity (4) and the corresponding decrease in the longitudinal velocity (13) which determines the Doppler shift. One should bear in mind, however, that in the immediate vicinity of the resonance the conditions for the applicability of the theory developed here may be violated (see §2).

4.1 Limiting cases of motion along a circular spiral

The general equations (22) to (25) which determine the spectral and angular distribution of the emission from a spiral undulator with a longitudinal magnetic field can be significantly simplified in a number of limiting cases. In particular, for $H_z=0$ or $H_\perp=0$ they yield well known results for the emission by electrons in a spiral or in a uniform magnetic field, respectively. Indeed, for $\omega_H=0$ the amplitude of the longitudinal oscillations vanishes together with q, so that in the sums over ν in (24) only the term with $\nu=0$ remains. Assuming that initially the electron is on the circular orbit $k=|q|$ we find

$$\tilde{j}^{(z)}_{nn'}=J_n(\kappa R_0)J_{n'}(\kappa R_0),$$

$$\tilde{j}^{(\pm)}_{nn'}=R_0\omega_w J_{n\mp 1}(\kappa R_0)J_{n'}(\kappa R_0), \qquad (28)$$

$$\xi_{nn'}=\frac{\omega}{2}\left[\theta^2+E^{-2}(1+p^2)\right]-n\omega_w.$$

Substitution of (28) into (22) and use of the identity[10]

$$\sum_{n'=\infty}^{\infty} J^2_{n'}(\kappa R_0)=1$$

and of the recurrence relations between adjacent Bessel functions J_n, $J_{n\pm1}$ leads to the result which is (in the $E\gg1$, $\theta\ll1$ limit) the same as the corresponding result obtained earlier in Ref. 11, for the case of a spiral undulator. In the other limit, $H_\perp=0$, we have $p=0$, $R\to0$, $\alpha=0$ so that in (24) only the term with $\nu=0$ remains, and $k=v_0/(R\omega_w)\to\infty$. As a result we get

$$\tilde{j}^{(z)}_{nn'}=J_{n'}(\kappa v_0/\omega_H)\delta_{n0},\quad \tilde{j}^{(\pm)}_{nn'}=v_0\ J_{n'\pm1}(\kappa v_0/\omega_H)\delta_{n0}, \qquad (29)$$

$$\xi_{0n'}=\frac{\omega}{2}\left(\theta^2+E^{-2}+v_0^2\right)-n'\omega_H,$$

where δ_{n0} is the Kronecker symbol. Substitution of (29) into (22) leads in this case to the well known result from the theory of the emission by ultrarelativistic electrons entering at an angle $\theta_0=v_0$ in a uniform magnetic field and moving along a circular spiral with radius v_0/ω_H.

We noted above that for the special initial conditions corresponding to $k=0$ the transverse trajectories are circular (with radius R) even when neither of the components H_z or $H_\perp$ of the field (2) vanishes. Under the condition $k=0$ the amplitude a of the longitudinal oscillations (see (14)) vanishes and in the sums (24) only the terms with $\nu=0$ and $n'=0$ remain:

$$\tilde{j}^{(z)}_{nn'}=J_n(\kappa R)\delta_{n'0},\quad \tilde{j}^{(\pm)}_{nn'}=R\omega_w J_{n\mp1}(\kappa R)\delta_{n'0}, \qquad (30)$$

$$\xi_{0n}=\frac{\omega}{2}\left[\theta^2+E^{-2}\left(1+\frac{p^2}{(1+q)^2}\right)\right]-n\omega_w.$$

As one should expect, this result is analogous to the case (28) considered above and goes over into it for $q=0$.

4.2 Emission along the undulator axis

The results (22) to (25) can be greatly simplified also for emission at a zero angle θ. In that case the z-component of the current in (22) does not play a role, and since we have $\kappa=0$ for the other components we find

$$\tilde{j}^{(\pm)}_{nn'}=R\omega_w(-1)^n[kJ_n(\omega a)-J_{n\mp1}(\omega a)]\delta_{n',n\mp1}.$$

The spectral and angular density of the emission energy then takes the form

$$\left.\frac{d^2W}{d\omega dO}\right|_{\theta=0}=\frac{e^2\omega^2}{2\pi^2}(R\omega_w)^2\sum_{n=-\infty}^{\infty}\left\{[kJ_n(\omega a)-J_{n-1}(\omega a)]^2\frac{\sin^2(L\xi_{n,n-1}/2)}{\xi^2_{n,n-1}}+[kJ_n(\omega a)-J_{n+1}(\omega a)]^2\frac{\sin^2(L\xi_{n,n+1}/2)}{\xi^2_{n,n+1}}\right\},$$

$$\xi_{n,n\mp1}=\frac{\omega}{2E^2}\left(1+p^2\frac{1+k^2}{(1+q)^2}\right)-n\omega_w-(n\mp1)\omega_H. \qquad (31)$$

We see thus that for $\theta=0$ the numbers of the undulator and the cyclotron harmonics can differ from one another only by unity. An additional analysis shows that for $\theta=0$ the harmonics $n,n'=n-1$ are right-polarized and the harmonics $n,n'=n+1$ are left-polarized. It was noted in §2 that for $q=1$ the transverse orbit is transformed into an ellipse, and for the special initial conditions $k=|q|$ the length of the minor axis of the ellipse is equal to zero, i.e., we have plane transverse oscillations. In antiresonance ($q=1$) the frequencies of the harmonics of order $n,n'=n+1$ are the same as the frequencies of the harmonics of order $n+1,n'=n$. If, moreover, we have $k=|q|$ the spectral and angular distribution of the emission energy along the undulator axis (30) can be written in the form

$$\left.\frac{d^2W}{d\omega dO}\right|_{\theta=0}=e^2\frac{4p^2E^2}{(2+p^2)^2}\sum_{l=1}^{\infty}l^2\left[J_{(l+1/2)}\left(\frac{lp^2}{2(2+p^2)}\right)-J_{(l-1/2)}\left(\frac{lp^2}{2(2+p^2)}\right)\right]^2\frac{\sin^2(N\xi_l)}{\xi_l^2},$$

$$\xi_l=\pi\left[\frac{\omega}{4\omega_w E^2}(2+p^2)-l\right]. \qquad (32)$$

The emission in this case is linearly polarized in the xz-plane of the transverse oscillations. Equation (32) is the same as the analogous one for the case of emission in a plane undulator with a sinusoidal field (see, e.g., Ref. 11); this is due to the plane harmonic transverse oscillations with a frequency $\omega_p/2=\omega_w$ in the two cases compared here.

4.3 Dipole radiation

If the angle through which the electron is deflected by the field (2) is significantly smaller than the effective emission angle $1/E$, i.e., if the inequalities $v_0E\ll1$, $p/|1+q|\ll1$ are satisfied, the arguments of all the Bessel functions in (24) are small compared to unity. We can therefore neglect the retardation of the radiation field within the region of the finite motion of the electron around the uniformly moving (with velocity $\langle v_z\rangle$) center, and the emission therefore has a dipole character. In the dipole limit the nonvanishing Fourier components of the current (24) have the form

$$\tilde{j}^{(z)}_{nn'}=\frac{\kappa R}{2}\left(\frac{k}{q}\delta_{n'1}\delta_{n0}+\delta_{n'0}\delta_{n1}\right), \qquad (33)$$

$$\tilde{j}^{(+)}_{nn'}=kR\omega_w\delta_{n'0}\delta_{n1},\quad \tilde{j}^{(-)}_{nn'}=kR\omega_w\delta_{n'1}\delta_{n0}.$$

In this limit only the first undulator harmonic ($n=1$, $n'=0$) and the first cyclotron harmonic ($n=0$, $n'=1$) are emitted. The partial spectral and angular density of the emission energy $W_{nn'}$ for these harmonics can be written in the relatively simple form

$$\frac{d^2W_{10}}{d\omega dO}=\frac{e^2\omega^2}{(2\pi)^2}\left(\frac{p}{E(1+q)}\right)^2 \times\left[1+\left(\frac{\theta^2-E^{-2}}{\theta^2+E^{-2}}\right)^2\right]\frac{\sin^2(\xi_{10}L/2)}{\xi_{10}^2},$$

$$\xi_{10}=\frac{\omega}{2}\left[\theta^2+E^{-2}\left(1+p^2\frac{1+k^2}{(1+q)^2}\right)\right]-\omega_w, \tag{34}$$

$$\frac{d^2W_{01}}{d\omega dO}=\frac{e^2\omega^2}{(2\pi)^2}\left[\left(\frac{p}{E(1+q)}-v_{0x}\right)^2+v_{0y}^2\right] \times\left[1+\left(\frac{\theta^2-E^{-2}}{\theta^2+E^{-2}}\right)^2\right]\frac{\sin^2(\xi_{01}L/2)}{\xi_{01}^2},$$

$$\xi_{01}=\frac{\omega}{2}\left[\theta^2+E^{-2}\left(1+p^2\frac{1+k^2}{(1+q)^2}\right)\right]-\omega_H. \tag{35}$$

The dipole emission of the cyclotron harmonic (35) vanishes if the initial conditions are such that $v_{0y}=0$, $v_{0x}=p/[E(1+q)]$, i.e., $k=0$ [see also (30)]. On the other hand, if the electron enters parallel to the undulator axis (in which case we have $v_{0x}=v_{0y}=0$) the maxima of the spectral and angular density of the emission of the undulator and cyclotron harmonics are equal in magnitude. It follows from (34) that in the dipole approximation the effect of the longitudinal magnetic field on the emission of the undulator harmonic is reduced to the appearance of the denominator $(1+q)^2$ in (34), which decreases the spectral and angular density for $q>0$ or $q<-2$ but increases it for $-2<q<0$. The longitudinal magnetic field also affects the position of the center of the emission line due to the presence of the same denominator in the expression for ξ_{10}. Although the quantity $[p/(1+q)]^2$ is small, in the dipole limit this shift can be comparable with the line width of the emission.

5. AMPLIFICATION COEFFICIENT AND EFFICIENCY OF THE TRANSFORMATION OF THE ELECTRON ENERGY

So far we have considered the theory of the spontaneous emission by electrons in the field (2). We can find the characteristics of the stimulated emission, when there is not only an electron beam but also a source of external radiation in the undulator, starting from the results of §3 for the characteristics of the spontaneous emission. In particular, if we assume the amplification to be relatively weak and neglect the feedback of the stimulated emission on the dynamics of the electron beam the amplification coefficient $G_{nn'}$ at a frequency close to the frequency $\omega_{nn'}$ of one of the harmonics can be written (see, e.g., Ref. 12)

$$G_{nn'}=\lambda^3\frac{I}{Se}\frac{\partial}{\partial\Phi}\left(\frac{d^2W_{nn'}}{d\omega dO}\right)\Delta\Phi. \tag{36}$$

Here I is the electron current, S is the electron beam cross-section, $\lambda=2\pi/\omega$ is the wavelength of the radiation, $\Delta\Phi$ is the difference between the values of the phase shift Φ [see (25)] for stimulated emission and absorption of a photon by an electron when one takes into account quantum corrections, which can be found using the conservation laws in a quantal discussion of the processes, and $W_{nn'}$ is the partial emission energy of the nn' harmonic.

From a quantum point of view the state of an electron in the field (2) is characterized by the energy $E(p_z,\nu)$ which depends on the longitudinal quasimomentum p_z and the quantum number ν which corresponds to the finite transverse motion of the electron in the longitudinal magnetic field. Because of the translational symmetry of the field (2) along the z-axis the longitudinal quasimomentum p_z is conserved in the emission (absorption) process, apart from a quantity which is a multiple of $2\pi/\lambda_w$, where λ_w is the translation period (period of the spiral undulator). The energy and quasimomentum conservation laws when an electron makes a transition from a state $(p_z^{(i)},\nu^{(i)})$ with energy E_i to a state $(p_z^{(f)},\nu^{(f)})$ with energy E_f, emitting (upper sign) or absorbing (lower sign) a photon with energy ω and longitudinal momentum k_z can be written in the form

$$\Delta E=\pm\omega,\quad \Delta p_z=\pm\left(\frac{2\pi n}{\lambda_w}+k_z\right). \tag{37}$$

Here we put $\hbar=m=c=1$, $\Delta E=E_f-E_i$, $\Delta p_z=p_z^{(f)}-p_z^{(i)}$ and n is an integer corresponding to the multiplicity of the longitudinal momentum transferred to the field. In the quasiclassical limit the relative change in the quantum numbers in the electron transition is small, so that we can restrict ourselves to a few basic terms of the series expansion of ΔE

$$\Delta E\approx\frac{\partial E}{\partial p_z}\Delta p_z+\frac{1}{2}\frac{\partial^2E}{\partial p_z^2}(\Delta p_z)^2+\frac{\partial E}{\partial\nu}\Delta\nu. \tag{38}$$

We then use the Bohr–Sommerfeld correspondence principle:

$$\frac{\partial E}{\partial p_z}=\langle v_z\rangle,\quad \frac{\partial^2E}{\partial p_z^2}=\frac{1-\langle v_z\rangle^2}{E}\approx\frac{2(1-\langle v_z\rangle)}{E},$$
$$\frac{\partial E}{\partial\nu}=\omega_H,\quad \Delta\nu=n', \tag{39}$$

where $\langle v_z\rangle$, ω_H, and n' are the classical quantities introduced earlier: the average longitudinal velocity, the cyclotron frequency, and the number of the cyclotron harmonic, respectively. Taking also into account that in the ultrarelativistic case the relation $k_z\approx\omega(1-\theta^2/2)\gg n\omega_w$ is satisfied we can use (37) to (39) to obtain

$$(\omega-k_z\langle v_z\rangle)\left(1\mp\frac{\omega}{E}\right)=n\omega_w+n'\omega_H,$$

where the quantity on the left-hand side of the equation is the phase shift, taking into account quantum corrections. Hence follows an expression for the difference in the quantum corrections to $\Delta\Phi$ for emission and absorption

$$\Delta\Phi=\frac{2\omega}{E}\Phi. \tag{40}$$

We next consider the case of amplification at the first undulator harmonic ($n=1$, $n'=0$) when radiation with right-handed circular polarization propagates along the undulator axis ($\theta=0$). Using Eqs. (31), (36), and (40) we find the amplification coefficient

$$G_{10}=\frac{I\pi^2}{I_0 S}\lambda_w^{1/2}\lambda^{3/2}\frac{p^2|1+q|}{[(1+q)^2+p^2(1+k^2)]^{3/2}}\times[kJ_1(\alpha)-J_0(\alpha)]^2N^3f(\xi). \tag{41}$$

Here we have used the notation $I_0=mc^3/e$, $\gamma=E/mc^2$, and $N=L/\lambda_w$ is the number of undulator periods,

$$\alpha=\frac{k}{1+q}\frac{2p^2}{(1+q)^2+p^2(1+k^2)},\quad f(\xi)=\frac{d}{d\xi}\frac{\sin^2\xi}{\xi^2},$$

$$\xi=\pi N\left[\frac{\lambda_w}{2\lambda\gamma^2}\left(1+p^2\frac{1+k^2}{(1+q)^2}\right)-1\right]. \tag{42}$$

In what follows we assume $k=|q|$, i.e., initially the electron is on a circular undulator orbit. One sees easily that for $k=|q|=0$ Eq. (41) is the same as the well known result for the usual spiral undulator (see, e.g., Ref. 13). The function $f(\xi)$ has a maximum for $\xi_r\approx-1.30$ with a magnitude equal to 0.54. The amplification coefficient is positive in the range $-\pi<\xi<0$ which includes the principal maximum. We study the amplification coefficient (41) as a function of the magnitude of the longitudinal magnetic field strength $H_\perp$, assuming the other parameters to be fixed. Since the ratio $q=\omega_H/\omega_w$ is proportional to $H_\perp$ it is more convenient to study the q-dependence of G_{10}. We introduce the notation $F(q)=(1+q^2)/(1+q)^2$, and the value q_0 corresponding to the center of the spontaneous emission line with wavelength λ is then determined by the equation

$$F(q_0)=\left[\frac{2\lambda\gamma^2}{\lambda_w}-1\right]\frac{1}{p^2}. \tag{43}$$

The analytical form of the function $F(q)$ yields the relation $F(q)=F(1/q)$, so that there are, in general, two values $q_0^{(1)}$ and $q_0^{(2)}=1/q_0^{(1)}$ satisfying Eq. (43) and in what follows it is sufficient to choose $q_0^{(1)}$ from the range $|q_0^{(1)}|\leq 1$. We can rewrite the variable ξ in the form

$$\xi(q)=\pi N\left[\frac{1+p^2F(q)}{1+p^2F(q_0)}-1\right]. \tag{44}$$

The dependence of the amplification coefficient (41) on the quantity $q>0$ for an undulator with a number of periods equal to $N=50$ and a parameter $p=1$ for different $q_0^{(1)}$ corresponding to different radiation wavelengths in (43) is shown in Fig. 3. If $q_0^{(1)}$ is much smaller than unity one observes essentially a single principal maximum. As $q_0^{(1)}$ increases this maximum broadens and approaches the antiresonance point $q=1$. At the same time another maximum appears above the antiresonance point. These maxima are, generally speaking, separated by a dip at the antiresonance point. However, as $q_0^{(1)}$ increases (and $q_0^{(2)}$ correspondingly decreases) the maxima approach one another and broaden, as a result of which the dip in the antiresonance region ultimately is replaced by a maximum the magnitude of which gradually decreases as $q_0^{(1)}$ approaches unity. The width of the principal maxima is determined by the inequalities

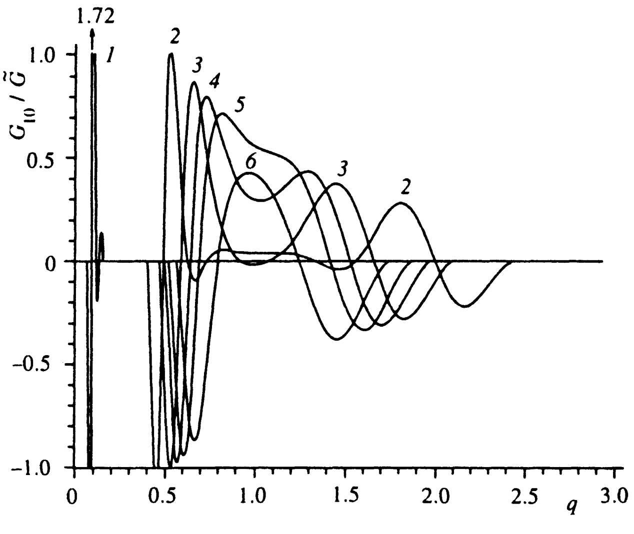

FIG. 3. The amplification coefficient G_{10} at the first harmonic of an undulator with a number $N=50$ of periods and a parameter $p=1$ as a function of the ratio q of the cyclotron frequency ω_H to the undulator frequency ω_w. The curves correspond to different values of q_0 from (43): *1*: $q_0=0.1$; *2*: $q_0=0.5$; *3*: $q_0=0.6$; *4*: $q_0=0.65$; *5*: $q_0=0.7$; *6*: $q_0=0.8$. As the unit in which G_{10} is measured we use the value $\tilde{G}=(I/I_0)\lambda^{3/2}\lambda_w^{1/2}N^3/S$.

$$-\frac{1+p^2F_0}{Np^2}<F(q)-F_0<0, \tag{45}$$

where we have $F_0=F(q_0)$. In the region $0<q<1$ the behavior of $F(q)$ is well approximated by the formula $F(q)\approx 1-q$ and in the region $1<q\lesssim 5$ by the formula $F(q)\approx 0.5+0.056(q-1)$. Hence we get the following estimate for the width of the maxima:

$$-\frac{1+p^2q_0^{(1)}}{Np^2}<q_0^{(1)}-q<0,$$

$$-\frac{1+p^2q_0^{(2)}}{0.056Np^2}<q-q_0^{(2)}<0.$$

For small $p^2\ll 1$ these widths are inversely proportional to Np^2 and the absolute width of the maximum which lies above the antiresonance point is much larger than the width of the maximum below the antiresonance point, as long as these maxima do not overlap.

In the region of negative q corresponding to electrons rotating in the same sense in the spiral field and in the uniform longitudinal field the derivative $F'(q)$ of the function $F(q)$ is relatively large, $F'(q)=2(q-1)/(1+q)^3$, which is connected with the presence of a resonance at the point $q=-1$. As a result the maxima in the negative q re-

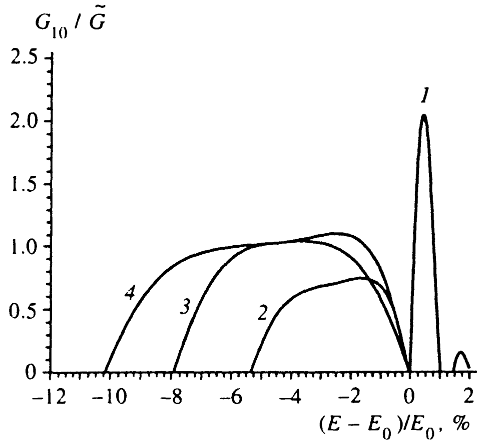

FIG. 4. The quantity G_{10} as a function of the electron energy E in a monochromatic beam. Along the abscissa is plotted the difference $E-E_0$ divided by E_0 (in percent), where E_0 is the energy corresponding to the center of the spontaneous emission line. The curves are constructed for the following pairs of parameters p and $q_0=q(E_0)$: *1*: $p=1.415$, $q_0=0$; *2*: $p=0.082$, $q_0=-1.213$; *3*: $p=0.253$, $q_0=-1.455$; *4*: $p=0.644$, $q_0=-1.760$.

gion, according to Eq. (45), are found to be relatively narrow and are always separated by a relatively broad dip.

We now consider the behavior of the amplification coefficient as a function of the deviation of the electron energy E from the value E_0 corresponding to the center of the spontaneous emission line. Since q is inversely proportional to the energy we can write γ in (42) in the form $\gamma=q_0/q$, where q_0 corresponds to the value of q for $E=E_0$. As a result the variable ξ can be written in the form

$$\xi(q)=\pi N\left[\left(\frac{q}{q_0}\right)^2\frac{1+p^2F(q)}{1+p^2F(q_0)}-1\right], \qquad (46)$$

where we have $q=q_0E_0/E$. The dependence of the amplification coefficient G_{10} on the relative shift in energy $(E-E_0)/E_0$ for an undulator with $N=50$ is illustrated by a series of graphs in Fig. 4, each of which corresponds to well defined values of q_0 and the parameter p of the undulator. If there is no longitudinal field ($q=0$) the width of the curve *1* is determined by the quantity $1/2N\approx10^{-2}$. However, in relatively strong longitudinal fields corresponding to $q<-1$ the width of the curves increases and the values of the amplification coefficient then remain relatively high. This means that the requirement of an electron beam energy spread is weakened. If, however, we consider a spread in the electron energy as the result of energy losses to stimulated emission, the width of the curves in Fig. 4 determines a possible efficiency of the transformation of the electron energy into radiative energy. The efficiency can thus be enhanced if $q<0$ holds, i.e., if the direction of the longitudinal field corresponds to pumping of the transverse oscillations. However, we noted earlier that when the field is in the opposite direction ($q>0$), the function $F(q)$ depends very little on $q(E)$ and the width of curves such as those shown in Fig. 4 remains practically unchanged relative to the usual spiral undulator, $q=0$, since it is basically determined by the factor $(q/q_0)^2$ in Eq. (46). Therefore, in the linear approximation on which the considerations in the present paper are based no significant increase in the efficiency is observed when the sense of rotation of electrons in the spiral field is opposite to that in the longitudinal field.

CONCLUDING REMARKS

We have shown in the present paper that the transverse motion of relativistic electrons in a spiral undulator in the presence of an additional longitudinal field is characterized in the general case by two frequencies, the frequency ω_ρ of the radial oscillations and the precession frequency Ω. They are linearly connected with the undulator frequency ω_w and the relativistic cyclotron frequency ω_H. This enabled us to obtain relatively simple expressions for the spectral and angular energy distribution of the spontaneous emission and the amplification coefficient. The method can be applied also in the more complicated case when the emission occurs not in an open space but in a waveguide, e.g., in the experiment of Ref. 3 with millimeter radiation. However, in the case of a waveguide it is necessary to use the results obtained when the radiation field is expanded in waveguide modes instead of Eqs. (17) and (18), which correspond to the expansion of the radiation field in free cylindrical waves. In that case the possible radiation frequencies are determined as before by the equation $\xi_{nn'}(\omega,k_z)=0$ [see (23)] in which now the frequency ω and the z-component of the wavevector are connected by the dispersion relation corresponding to a well defined waveguide mode.

The analysis of the amplification coefficient at the basic undulator harmonic, given above, shows the possible existence of a dip in the vicinity of the antiresonance ($q=1$), which may serve as an explanation of the corresponding dip in the power of the stimulated emission observed in the experiment of Ref. 3. Moreover, the linear theory does not a significantly increase the efficiency with which the electron energy is transformed into radiative energy for positive values of q, as observed in the experiment. This apparent contradiction may be explained by the influence of nonlinear effects, in particular, the dependence of the field of a real undulator on the transverse coordinates, which was neglected in our calculations. A numerical simulation[6] of the experiment of Ref. 3 showed that the field variations decrease the efficiency relative to the idealized case of a uniform field. However, the effect of the inhomogeneity decreases with increasing q, since the transverse orbits shrink in that case, as we showed above. This behavior of the orbits may serve as a qualitative explanation of the relative increase in the efficiency for $q>0$. The corresponding quantitative theory is connected with going beyond the framework of the approximation for the undulator field which was used in the present paper.

Since the relativistic cyclotron frequency ω_H decreases with electron energy, small values $q\ll1$ are characteristic for relatively high energies $E\gg1$ (since values $q\approx1$ would correspond to values of the longitudinal field strength which are too high). In that case ($q\ll1$) the main difference between the amplification coefficient Eq. (41) and the amplification coefficient in the usual spiral undulator is the shift of the resonance, which is determined by the quantity $p^2F(q) \approx p^2(1-2q)$ in Eq. (42) for ξ. The dependence of the resonance on the longitudinal field can in principle be used to enhance the efficiency of an undulator when the parameter p is not too small if one uses a field whose strength gradually increases along the undulator axis to compensate for the decrease in γ^2 in (42) due to radiation losses.

The work was done with support of the Russian Fund for Fundamental Research (Project No 95-2-04457A).

[1] S. H. Gold, D. L. Hardesty, A. K. Kinkead *et al.*, Phys. Rev. Lett. **52**, 1218 (1984).
[2] J. Fajans, J. S. Wurtele, G. Bekefi *et al.*, Phys. Rev. Lett. **57**, 579 (1986).
[3] M. E. Conde and G. Bekefi, Phys. Rev. Lett. **67**, 3082 (1991).
[4] A. K. Ganguly and H. P. Freund, Phys. Rev. A **33**, 2275 (1985); Phys. Fluids **31**, 387 (1988).
[5] H. P. Freund, R. H. Jackson, and D. E. Pershing, Nucl. Instr. Meth. A **341**, 259 (1994).
[6] A. Bourdier, V. A. Bazylev, P. Gouard *et al.*, Nucl. Instr. Meth. A **341**, 250 (1994).
[7] G. Renz and G. Spiller, Nucl. Instr. Meth. A **341**, ABS99 (1994).
[8] P. Diament, Phys. Rev. A **23**, 2537 (1981).
[9] N. K. Zhevago and M. Kh. Khokonov, Zh. Éksp. Teor. Fiz. **87**, 56 (1984) [Sov. Phys. JETP **60**, 33 (1984)].
[10] A. Erdélyi (ed.), *Higher Transcendental Functions* (California Institute of Technology H. Bateman MS Project), Vol. 2, McGraw-Hill, New York (1954).
[11] D. F. Alferov, Yu. A. Bashmakov, and E. G. Bessonov, Proc. Lebedev Inst. **80**, 100 (1975).
[12] V. A. Bazylev and N. K. Zhevago, *Radiation of Fast Particles in Matter and in External Fields* [in Russian] Nauka, Moscow (1987).
[13] C. Pellegrini, Nucl. Instr. Meth. **177**, 227 (1980).

Translated by D. ter Haar