Theory of continuous quantum measurements of two-mode fields

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(Submitted 7 October 1994; resubmitted 2 February 1995)
Zh. Éksp. Teor. Fiz. **107**, 1422–1463 (May 1995)

We construct a general theory of continuous quantum measurements of two-mode fields. Both measurements of one mode of the field and independent simultaneous measurements of both modes of the field are investigated. Exact expressions are obtained for the basic characteristics of the continuous photodetection process, and are used to develop methods for numerical modeling of this type of random process. The theory is used to analyze temporal evolution during continuous measurements of a field generated by a parametric frequency converter. It is shown that a Fock state can be produced in one mode of the field by carrying out continuous measurements of the other. © *1995 American Institute of Physics.*

1. INTRODUCTION

Over the last few years, the need to describe various experimental situations in which the measurement system and the system on which measurements are carried out interact with one another continuously throughout the entire measurement time has stimulated further elaboration of the theory of continuous quantum measurements. Such situations arise, for example, in photon counting experiments or during observations of so-called "quantum jumps" in investigations of resonance fluorescence from atoms or ions confined in a trap. The standard quantum mechanical theory of measurements, in which the initial state of the system is given and the probability of finding the system in some other state at a later time is calculated, cannot give a complete representation of the temporal evolution of the experimental system in such cases. In the standard theory, it is assumed that measurements are carried out only at definite times at which the experimental system interacts with a detector. Each measurement event irreversibly destroys the preceding state of the system, and the system ends up in a new state. It is assumed that the system evolves freely without interacting with a detector during the time intervals between measurements.

In photon counting experiments, as in certain other experiments, the influence of the detector amounts to more than just the reduction of the state of the system as a result of recording the next count. The detector continues to influence the system during the time intervals between the counts. This is related to the fact that when a detector interacts continuously with the experimental system, information is obtained about the state of the system at each moment of measurement, irrespective of whether a count has been recorded or not. To investigate such situations, the measurement problem must be approached in a different manner. The description of this approach forms the subject matter of the theory of continuous quantum measurements.

A general method for studying continuous quantum measurements was proposed by Srinivas and Davies.[1] In studying the photodetection process for one-mode fields, they represented the random occurrence of photocounts at a detector by a sequence of two elementary events. One event, described by the reduction operator J, consists of detecting a photon, as a result of which the state of the system is reduced to the state with one less photon. The second event is the evolution of the system between two successive counts. This event is described by the evolution operator $S_{t,\tau}$. In introducing this operator, all factors influencing the temporal evolution of the field, including both the interaction with the detector that carries out the continuous measurements and with other elements of the experimental system (sources, reservoirs, and so on), must be taken into account. This representation makes it possible to construct a scheme to describe the continuous photodetection process in such a way that to find the basic characteristics of the process it is only necessary to determine the form of the evolution and reduction operators for any given system. Then, given the initial state of a one-mode field, the temporal evolution of the field during the measurement process can be described.

This approach to the problem of continuous quantum measurements was further elaborated in Ref. 2, and different authors have used it to describe various systems. For example, in Refs. 2 and 3 the temporal evolution of one-mode fields with various initial states was investigated; Ref. 4 describes a situation in which the effect of the environment is taken into account in addition to the interaction between the one-mode field under study and a detector; in Refs. 5 and 6, continuous measurements of one mode of a field generated by a parametric frequency converter were studied. However, the approach used in these papers is highly idealized, since it is based on a description of the interaction between an undamped one-mode field and perfect detector, whose efficiency is 1. To obtain results that describe real experimental situations more completely, a number of refinements must be introduced into the experimental system. One is to take into account imperfections of the system related either to damping of the one-mode field or the fact that the detector efficiency is different from 1, as a result of which some photons which are initially present in the field are lost. This situation is examined in detail in Ref. 7.

Another possible refinement is to study the measurement process in multimode fields. A possible first step in this direction is to construct a theory of continuous photodetection

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in two-mode fields; this is the subject of the present paper. In the present paper we propose a way to describe the temporal evolution of both modes and of the total two-mode field during the measurement of the number of photons in one mode or during simultaneous independent measurements of the number of photons in each moide. The formulation of a measurement scheme in which a separate detector carries out continuous measurements of one mode of the field makes it possible to use the well-studied formalism for describing continuous measurements of one-mode fields to model the continuous measurement of two-mode fields. In addition, by treating the results of independent measurements of both modes of the field as a random process of photon detection by a measurement system consisting of two detectors it is possible to construct a theory of continuous measurements of the total two-mode field.

In Sec. 2, we give a brief exposition of the general approach to the description of continuous measurements of one-mode fields, and present a way to obtain the basic equation of the theory of continuous photodetection—the equation for the reduced density matrix. This method is then used to derive the equations for the reduced density matrices of the fields under study. In Sec. 3, we analyze the theory of continuous measurements of one mode of a two-mode field. In Sec. 4, we use the theory presented here to investigate the temporal evolution of the field generated by a parametric frequency converter, during continuous measurements of the number of photons in the idler mode. In Sec. 5, we construct a theory of continuous measurements of both modes of the field, and methods for modeling this process are developed. Here, the given random process is described from two points of view—as a product of two independent processes measuring the number of photons in each mode of a two-mode field, and as the detection of photons in a two-mode field. Various characteristics describing the temporal evolution of each mode and of the entire field as a whole are determined on the basis of these approaches. In Sec. 6, we use the theory described above to investigate the field generated by a parametric frequency converter. In Sec. 7, we show that a Fock state can be produced in one mode of a given two-mode field by continuously measuring the number of photons in the other mode.

2. THEORY OF THE PHOTODETECTION OF ONE-MODE FIELDS

The investigation of the continuous measurement of one-mode fields is based on the description of interactions between the detection system, assumed to consist of a large number of ideal detectors, and the field being measured. An ideal detector, first studied by Glauber.[8] is a system whose response does not depend on the frequency of the incident radiation over a wide bandwidth, and whose dimensions are small compared to the characteristic spatial variations of the field. An example of such a system is an atom that is ionized upon absorption of a photon. The spectrum of such an atom is displayed in Fig. 1. Once the atom passes into a continuum state when the photon is detected, the atom stays in that state with high probability. Assuming that all atoms of a detector are in the ground state at the beginning of the measurement,

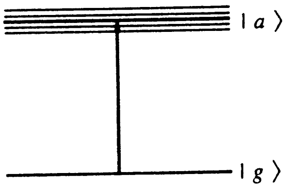

FIG. 1. Energy level scheme of an ideal photon detector.

and determining the number of ionized atoms after the measurement, we can find the number of counts recorded. Moreover, the result of continuous measurements of the number of photons is a random set of photocount occurrence times. The main problem of the theory of continuous measurements is to describe the statistical properties of a given random sequence of points and to determine the characteristics of the radiation being measured on the basis of these properties. To solve the problem, the projection operator $P(\eta)$ is introduced:

$$P(\eta)=\hat{N}\exp\left(-\eta\sum_{j,a} c_a^{+(j)}c_a^{(j)}\right), \tag{1}$$

where c_k^+ and c_k are creation and annihilation operators for the kth atom of the detector, and $\hat{N}$ is the normal-ordering operator, which positions the creation operators to the left of the annihilation operators. The summation extends over all excited states of the jth atom of the detector and over all atoms of the detector.

The most important property of the projection operator $P(\eta)$ is that for $\eta=1$, the operator becomes the projection operator onto the ground state of all atoms of the detector. Then, expanding $P(\eta)$ in a Taylor series near $\eta=1$, we obtain another class of projection operators onto a space with n ionized atoms of the detector, with all other atoms remaining in the ground state:

$$\hat{P}^{(n)}=\frac{1}{n!}\frac{\partial^n\hat{P}(\eta)}{\partial(-\eta)^n}\bigg|_{\eta=1}=\frac{1}{n!}\sum_{\{j_k,a_k\}}\prod_{k=0}^{n} c_{a_k}^{+(j_k)}|0\rangle \times\langle 0|c_{a_k}^{(j_k)}. \tag{2}$$

The average of this operator over the states of the field (f) and detector (d) gives the probability that exactly n ionized atoms of the detector are present at time t:

$$P_{[0,t)}^{(n)}=\mathrm{Tr}_{df}\{\hat{P}^{(n)}\rho(t)\}, \tag{3}$$

where $\rho(t)$ is the density matrix of the system consisting of "field + detector." If it is assumed that photon absorption during detection is irreversible (i.e., the recombination of photoelectrons accompanied by the emission of a photon is neglected), then $P_{[0,t)}^{(n)}$ can also be defined to be the probability that exactly n photons are detected in the field over the time interval $[0,t)$. It is this quantity that is studied in most theories of photodetection. However, an analysis of the continuous measurement process only on the basis of this quantity is incomplete. The description of the statistical properties of the detected radiation is based on determining the probability density $p_{[0,t)}(t_1,t_2,...,t_n)$ for the detection of n photons by time t, with no other counts interspersed. This distribution function is related to $P^{(n)}_{[0,t)}$

$$P^{(n)}_{[0,t)}=\int_0^t dt_n\int_0^{t_n}dt_{n-1}\dots\int_0^{t_2}dt_1 p_{[0,t)}(t_1,t_2,...,t_n) \quad (4)$$

and completely characterizes the measurement process.

To find $p_{[0,t)}(t_1,t_2,...,t_n)$ and $P^{(n)}_{[0,t)}$, it is necessary to solve the equation for the projection operator $\hat{P}^{(n)}$ averaged over the states of the detector:

$$\pi^{(n)}(t)=\mathrm{Tr}_d\{\hat{P}^{(n)}\rho(t)\}, \quad (5)$$

which can be obtained by multiplying von Neumann's equation for $\rho(t)$ by $\hat{P}^{(n)}$ and then averaging:

$$\frac{d\pi^{(n)}(t)}{dt}=-i\hbar\mathrm{Tr}_d\{P^{(n)}[H,\rho(t)]\}. \quad (6)$$

Here, $H=H_f+H_d+V$ is the Hamiltonian of the experimental system: $H_f=\hbar\omega_0 a^+a$ is the Hamiltonian of the one-mode field, $H_d=\Sigma_{j,a}\hbar\omega_a^{(j)}c_a^{+(j)}c_a^{(j)}$ is the Hamiltonian of the detector, and $V=\Sigma_{j,a}(p_a^{(j)}ac_a^{+(j)}+p_a^{*(j)}a^+c_a^{(j)})$ is the interaction Hamiltonian of the one-mode field and detector. The operators a^+ and a are creation and annihilation operators for the field under study. It follows from Eq. (5) that $\pi^{(n)}\times(t)$ can also be interpreted as the reduced density matrix of the field after the detector has recorded n photons.

Expanding the right-hand side of Eq. (6) and using the commutation relations between the field and detector operators $[a,a^+]=1$, $[c_{a'}^{(j')},c_a^{+(j)}]=\delta_{a'a,j'j}$, and $[c_a^{(j)},a^+]=0$, we obtain the following expression for $\pi^{(n)}(t)$:

$$\frac{d\pi^{(n)}(t)}{dt}=-i(H_{\mathrm{eff}}\pi^{(n)}(t)-\pi^{(n)}(t)H_{\mathrm{eff}}^+)+2\lambda a\pi^{(n-1)}\times(t)a^+, \quad (7)$$

where $\lambda=\int_0^t\Sigma_{a,j}p_a^{(j)2}\exp[i\omega_a^j(t-\tau)]d\tau$ is a parameter that specifies the detector efficiency, and $H_{\mathrm{eff}}=(\omega_0-i\lambda)a^+a$.

In deriving this equation, we neglected terms describing the recombination of photoelectrons. The trace of $\pi^{(n)}(t)$ over all possible states of the field gives the probability $P^{(n)}_{[0,t)}$. Then, if the solution of Eq. (7) can be represented as a successive integration of some quantity over time, then the probability density $p_{[0,t)}(t_1,t_2,...,t_n)$ can be determined from Eq. (4). This makes it possible to describe the properties of continuous measurements and construct a method for modeling this random process.

3. CONTINUOUS MEASUREMENTS OF ONE MODE OF A TWO-MODE FIELD

We assume that by time t_0, a field consisting of two modes (a and b) has been produced in a cavity. Let a detector be turned on at time t_0, and let continuous measurements of the number of photons in the mode a be carried out over a long time interval $[t_0, t)$. The Hamiltonian of such a system has the form

$$H=H_a+H_b+H_d+H_{ad}. \quad (8)$$

Here, the symbols a and b denote the mode, H_a and H_b are the Hamiltonians of the individual modes of the field, H_d is the Hamiltonian of the detector, H_{ad} is the Hamiltonian of the interaction between mode a and the detector, and both H_d and H_{ad} are described by the expressions presented above.

Using the method of Sec. 2, the equation for the reduced density matrix of the system after n photons have been detected can be written as follows:

$$\frac{d\pi^{(n)}(t)}{dt}=-i(H_{\mathrm{eff}}\pi^{(n)}(t)-\pi^{(n)}(t)H_{\mathrm{eff}}^+)+2\lambda a\pi^{(n-1)}\times(t)a^+, \quad (9)$$

where

$$H_{\mathrm{eff}}=\omega_a a^+a+\omega_b b^+b-i\lambda a^+a. \quad (10)$$

Here $a(a^+)$ and $b(b^+)$ are annihilation (creation) operators for modes a and b, respectively.

The solution of this equation can easily be written as a recurrence relation:

$$\pi^{(n)}(t)=\int_{t_0}^t S_{t,\tau}J\pi^{(n-1)}(\tau)d\tau, \quad (11)$$

which relates the reduced density matrix of the system after n counts have been recorded to the reduced density matrix of the system after $(n-1)$ counts have been recorded. The evolution operator $S_{t,\tau}$ and the reduction operator J are given, in this case, by the expressions

$$S_{t,\tau}x=\exp[-iH_{\mathrm{eff}}(t-\tau)]x\exp[iH_{\mathrm{eff}}^+(t-\tau)], \quad (12)$$

$$Jx=2\lambda axa^+. \quad (13)$$

Expanding (11), the solution of Eq. (8) can be expressed in terms of the initial density matrix of the field:

$$\pi^{(n)}(t)=\int_{t_0}^t dt_n\int_{t_0}^{t_n}dt_{n-1}\dots\int_{t_0}^{t_2}dt_1 S_{t,t_n}JS_{t_n,t_{n-1}}\times J\dots JS_{t_2,t_1}JS_{t_1,t_0}\rho(t_0). \quad (14)$$

Comparing this solution to (4) and (5), we obtain a consistent interpretation of the photodetection process as a sequence of photocounts occurring at times $t_0<t_1<...<t_{n-1}<t_n$. The solutions (11) and (14) are analogous to the expressions obtained for the reduced density matrix for the case of a continuous measurement of one-mode fields.[2,3,7] The measurement process for one mode of a two-mode field can therefore be represented as a measurement process for a one-mode field with a certain initial state $\rho(t_0)$ that characterizes the actual relationship between the two modes of the field. Then the measurement process can be described by the sequence of events illustrated in Fig. 2, which consist of a sequence of reductions of a two-mode field to a state described by the normalized reduced density matrix, as a result of recording the next count:

$$\rho^{(n)}(t)=\frac{S_{t,t_n}JS_{t_n,t_{n-1}}J\dots JS_{t_2,t_1}JS_{t_1,t_0}\rho(t_0)}{\mathrm{Tr}\{S_{t,t_n}JS_{t_n,t_{n-1}}J\dots JS_{t_2,t_1}JS_{t_1,t_0}\rho(t_0)\}}. \quad (15)$$

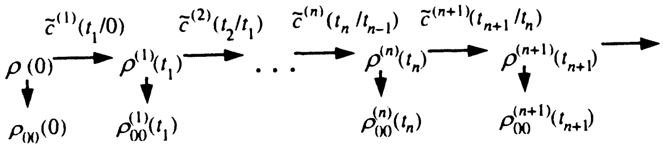

FIG. 2. Sequence of interconnected events describing the continuous measurement process in one mode of a two-mode field.

The detection of each count marks the removal of one photon from mode a, and therefore the reduction of the state of mode a to a state in which there is one less photon. As result of the initial relationship between modes a and b, the information obtained as a result of the measurement leads to additional information about the state of mode b, i.e., the results of the measurement effect a change in the state description of both modes a and b:

$$\rho_a^{(n)}(t_n)=\mathrm{Tr}_b\{\rho^{(n)}(t_n)\}, \tag{16}$$

$$\rho_b^{(n)}(t_n)=\mathrm{Tr}_a\{\rho^{(n)}(t_n)\}. \tag{17}$$

The state of mode a after the next count determines the conditional elementary probability that the next nth count is recorded at time t_n if the preceding count was recorded at time t_{n-1}:

$$\begin{aligned}\tilde{c}^{(n)}(t_n/t_{n-1})&=\mathrm{Tr}\{JS_{t_n,t_{n-1}}\rho^{(n-1)}(t_{n-1})\}\\&=\mathrm{Tr}_a\{JS_{t_n,t_{n-1}}\rho_a^{(n-1)}(t_{n-1})\}.\end{aligned} \tag{18}$$

The continuation of the measurement process after the nth count has been recorded is closely related to the probability that mode a is not in the vacuum state after the $(n-1)$th count:

$$\int_{t_{n-1}}^{\infty}\tilde{c}^{(n)}(t_n/t_{n-1})dt_n=1-\rho_{a\,00}^{(n-1)}(t_{n-1}). \tag{19}$$

The foregoing characteristics of the random photocount detection process—$\rho^{(n)}(t_n)$, $\tilde{c}^{(n)}(t_n/t_{n-1})$, $\rho_{a\,00}^{(n-1)}(t_{n-1})$—completely specify the continuous measurement process for one mode of a two-mode field, and they make it possible to model the process, just as in one-mode fields.

4. CONTINUOUS MEASUREMENTS OF THE IDLER MODE OF A FIELD GENERATED BY A PARAMETRIC FREQUENCY CONVERTER

To illustrate the theory of continuous measurements of one mode of a two-mode field, we consider a measurement of the number of photons in the idler mode of the field generated by a parametric frequency converter.

4.1. Generation of a two-mode field by a parametric frequency converter

Two optical modes—the signal mode and the idler mode—are generated from the vacuum state by a parametric frequency converter. A quantum-mechanical relationship is established between them by the interaction Hamiltonian

$$H_{\mathrm{int}}=\hbar(ka_i^+a_s^++k^*a_ia_s)=\hbar k_0(e^{i\theta}a_i^+a_s^++e^{-i\theta}a_ia_s), \tag{20}$$

where $k=k_0e^{i\theta}$ is the interaction constant, the operators $a(a^+)$ are the creation and annihilation operators of an individual mode, and subscripts s and i denote the signal mode and the idler mode, respectively.

Since the initial state of both fields is the vacuum state, the initial density matrix of the desired state of the field can be represented in the form

$$\rho(0)=|0\rangle_{ss}\langle 0|\otimes|0\rangle_{ii}\langle 0|. \tag{21}$$

The temporal evolution of the initial state of the field is determined by the evolution operator $U(t,0)$:

$$\rho(t)=U(t,0)\rho(0)U^+(0,t), \tag{22}$$

which can be found by solving the Schrödinger equation with initial condition $U(0,0)=1$:

$$\begin{aligned}U(t,0)&=\exp[-ie^{i\theta}\tanh(k_0t)\,a_i^+a_s^+]\\&\times\exp\{-(a_s^+a_s+a_i^+a_i+1)\ln[\cosh(k_0t)]\}\\&\times\exp[ie^{-i\theta}\tanh(k_0t)\,a_ia_s].\end{aligned} \tag{23}$$

Then, by time $t=t_0$, the density operator $\rho(t)$ of the two-mode field will have the form

$$\begin{aligned}\rho(t_0)=&\frac{1}{\cosh^2(k_0t_0)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}[ie^{-i\theta}\tanh(k_0t_0)]^k\\&\times[-ie^{i\theta}\tanh(k_0t_0)]^l|k\rangle_{ss}\langle l|\otimes|k\rangle_{ii}\langle l|.\end{aligned} \tag{24}$$

It follows from this expression that the output fields of the signal and idler modes are correlated, as manifested by the presence of off-diagonal elements in Eq. (24). This correlation is related to the fact that the parametric frequency converter generates photons in pairs.[9,10] The average of (24) over states of the idler mode yields the density operator for the signal mode before the measurement process starts (up to the time t_0):

$$\begin{aligned}\rho_s(t_0)&=\mathrm{Tr}_i\{\rho(t_0)\}\\&=\frac{1}{1+\sinh^2(k_0t_0)}\sum_{k=0}^{\infty}\left(\frac{\sinh^2(k_0t_0)}{1+\sinh^2(k_0t_0)}\right)^k|k\rangle_{ss}\langle k|.\end{aligned} \tag{25}$$

Likewise, the density operator of the idler field is

$$\begin{aligned}\rho_i(t_0)&=\mathrm{Tr}_s\{\rho(t_0)\}\\&=\frac{1}{1+\sinh^2(k_0t_0)}\sum_{k=0}^{\infty}\left(\frac{\sinh^2(k_0t_0)}{1+\sinh^2(k_0t_0)}\right)^k|k\rangle_{ii}\langle k|.\end{aligned} \tag{26}$$

One can see from Eqs. (25) and (26) that the signal and idler fields prior to the measurements are in a thermal state in which the mean number of photons is $\sinh^2(k_0t_0)$, and depends on the generation time t_0.

In summary, the central problem of the theory of continuous quantum measurements can be stated as follows: from vacuum states, a parametric amplifier generates two fields—the signal and idler fields—whose states are correlated up to the time measurements start. A detector is turned on at time t_0 and carries out continuous measurements of the idler field. Below we investigate the temporal evolution of these fields during the detection process, and numerically model the random photon detection process. We note that this problem was also proposed for consideration in Ref. 5 by M. Ueda.

4.2. Temporal evolution of the state of a correlated field during continuous measurements of the idler mode

The continuous reduction of the state of the field generated by a parametric amplifier during continuous photodetection of the idler mode is described in detail in Ref. 5. Here, we briefly discuss the basic results that are required to describe the temporal evolution of the signal and idler modes of the field.

In the time intervals between the nth and $(n+1)$th counts, the state of the correlated field is determined by the normalized reduced density matrix:

$$\rho^{(n)}(t)=[1-\exp[-\lambda(t-t_0)]\ \tanh^2(k_0t_0)]^{n+1}$$
$$\times\sum_{k,l=0}^{\infty}[ie^{-i\theta}\tanh(k_0t_0)]^k$$
$$\times[-ie^{i\theta}\tanh(k_0t_0)]^l\binom{k+n}{k}^{1/2}\binom{l+n}{l}^{1/2}$$
$$\times\exp[-i(\omega_s+\omega_i)(k-l)(t-t_0)$$
$$\times\exp[-(\lambda/2)(k+l)(t-t_0)]|k+n\rangle_{ss}\ \langle l+n|$$
$$\otimes|k\rangle_{ii}\ \langle l|\ , \qquad (27)$$

where $\binom{k+n}{k}$ is a binomial coefficient. It is evident from this expression that the complete correlation between the signal and idler fields that exists prior to the measurements is preserved by the latter. The state of the idler and signal modes is described by identical density matrices. However, the matrix elements of the signal mode are shifted with respect to the idler mode by exactly the number of counts recorded. Thus,

$$\rho_i^{(n)}(t)=[1-\exp[-\lambda(t-t_0)]\tanh^2(k_0t_0)]^{n+1}$$
$$\times\sum_{k=0}^{\infty}\binom{k+n}{k}\{\tanh^2(k_0t_0)$$
$$\times\exp[-\lambda(t-t_0)]\}^k|k\rangle_{ii}\langle k|, \qquad (28)$$

$$\rho_s^{(n)}(t)=[1-\exp[-\lambda(t-t_0)]\tanh^2(k_0t_0)]^{n+1}$$
$$\times\sum_{k=0}^{\infty}\binom{k+n}{k}\{\tanh^2(k_0t_0)$$
$$\times\exp[-\lambda(t-t_0)]\}^k|k+n\rangle_{ss}\langle k+n|, \qquad (29)$$

i.e., because of the strong correlation between modes at the moment the measurement starts, recording n counts in the idler mode indicates that at least n photons are present in the signal mode.

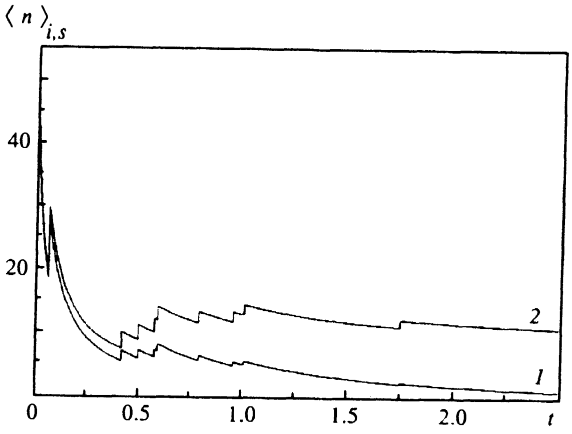

FIG. 3. Temporal evolution of the mean number of photons in the signal mode (1) and idler mode (2) during the measurement process. The mean number of photons in each mode prior to the measurements is 25.

The temporal evolution of the photon statistics of the signal and idler modes will differ while recording the idler photons. The dependence of the mean and variance of the number of photons in the signal and idler modes on the number of detected photons and the time is given by

$$\langle n_s(t)\rangle_n=\frac{n+\exp[-\lambda(t-t_0)]\tanh^2(k_0t_0)}{1-\exp[-\lambda(t-t_0)]\ \tanh^2(k_0t_0)}\ , \qquad (30)$$

$$\langle n_i(t)\rangle_n=\frac{(n+1)\exp[-\lambda(t-t_0)]\tanh^2(k_0t_0)}{1-\exp[-\lambda(t-t_0)]\tanh^2(k_0t_0)}\ , \qquad (31)$$

$$\langle[\Delta n_i(t)]^2\rangle_n=\langle[\Delta n_s(t)]^2\rangle_n$$
$$=\frac{(n+1)\exp[-\lambda(t-t_0)]\tanh^2(k_0t_0)}{[1-\exp[-\lambda(t-t_0)]\tanh^2(k_0t_0)]^2}\ . \qquad (32)$$

Figure 3 displays the temporal evolution of the mean number of photons in the signal (1) and idler modes (2). It is easy to see that in the intervals between counts, the mean number of photons decreases monotonically with time in both the signal and idler modes. When the counts are recorded, the mean number of photons increases abruptly. This jump results

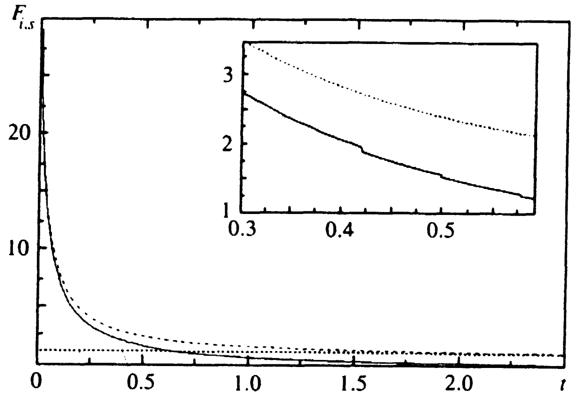

FIG. 4. Temporal evolution of the Fano factor for the signal mode (solid line) and idler mode (dashed line). The dotted line corresponds to the asymptotic Fano factor for the initial mode.

from the super-Poissonian character of the fields in question, and was discussed in Ref. 3. After a count is recorded, the mean number of photons in the idler mode once again starts to decrease monotonically, and with time it drops to zero. This is explained by the fact that as the counts are recorded, photons are systematically removed from the initial field, and if the measurement lasts long enough, the idler mode will end up in the vacuum state. The mean number of photons in the signal mode after a count also decreases monotonically, but to the number of recorded counts and not to zero. This is related to the information that at least n photons are present in the signal mode, which is derived by measuring the idler mode.

In other words, as a result of a measurement, the idler mode is reduced to the vacuum state and the signal mode is reduced to a Fock state in which the number of photons equals the number of recorded counts in the idler mode:

$$\rho_s^{(n)}(t \gg 1/\lambda) = |n\rangle_{ss}\langle n|, \tag{33}$$

$$\rho_i^{(n)}(t \gg 1/\lambda) = |0\rangle_{ii}\langle 0|. \tag{34}$$

This result can also be confirmed by investigating another important characteristic that determines the photon statistics of the fields, the Fano factor:

$$F(t) = \frac{\langle [\Delta n]^2 \rangle_n}{\langle n \rangle_n}. \tag{35}$$

For the signal and idler modes, the Fano factor is

$$F_s(t) = \frac{(n+1)\exp[-\lambda(t-t_0)]\tanh^2(k_0 t_0)}{[1-\exp[-\lambda(t-t_0)]\tanh^2(k_0 t_0)][n+\exp[-\lambda(t-t_0)]\tanh^2(k_0 t_0)]}, \tag{36}$$

$$F_i(t) = \frac{1}{1-\exp[-\lambda(t-t_0)]\tanh^2(k_0 t_0)}. \tag{37}$$

For the idler mode, in contrast to the signal mode, the Fano factor does not depend on the number of recorded counts, and with time it approaches 1 monotonically, in spite of the fact that the mean and variance of the number of photons are discontinuous at the moment the counts are recorded. This is a special property of temporal evolution during continuous measurements of fields initially in a thermal state.[3] The temporal evolution of the Fano factor for the idler mode is displayed in Fig. 4 (dashed curve).

The Fano factor for the signal mode depends not just on time, but on the number of recorded counts as well. In addition, $F_s(t)$ decreases discontinuously at the time a count occurs, and if at least one count is recorded, it approaches zero. The jump in the temporal evolution of the signal Fano factor at the time a count is recorded in the idler mode is related to a decrease in the uncertainty in the description of the signal mode, which in turn is related to the acquisition of additional information about the number of photons. The Fano factor for the signal mode is displayed in Fig. 4 (solid line), whence it follows that during a measurement, the photon statistics of the signal mode change from super-Poissonian at the start of the measurement to sub-Poissonian at the end of the measurement.

4.3. Modeling a continuous measurement of the idler mode

Let a measurement start at time t_0, and let n photons be recorded by time t. Then probability density for n counts to occur at times $t_1, t_2, \ldots, t_n$, with no other counts interspersed, takes the form

$$p_{[t_0,t)}(t_1,t_2,\ldots,t_n) = \frac{[2\lambda \tanh^2(k_0 t_0)]^n}{\cosh^2(k_0 t_0)} \exp\left[-2\lambda \sum_{j=1}^{n} (t_j - t_0)\right] \times \frac{n!}{\{1-\exp[-2\lambda(t_n-t_0)]\tanh^2(k_0 t_0)\}^{n+1}}. \tag{38}$$

The conditional probability that the nth count occurs at time t_n if the $(n-1)$th count occurred at time t_{n-1} is

$$\tilde{c}^{(n)}(t_n/t_{n-1}) = 2\lambda \tanh^2(k_0 t_0) n \exp[-2\lambda(t_n-t_0)] \times \frac{\{1-\exp[-2\lambda(t_{n-1}-t_0)]\tanh^2(k_0 t_0)\}^n}{\{1-\exp[-2\lambda(t_n-t_0)]\tanh^2(k_0 t_0)\}^{n+1}}. \tag{39}$$

The possibility that the measurement process terminates after the nth count is recorded is given by the probability that the idler mode is in the vacuum state after the $(n-1)$th count has been recorded:

$$\rho_{i00}^{(n-1)}(t_{n-1}) = \{1-\exp[-2\lambda(t_{n-1}-t_0)]\tanh^2(k_0 t_0)\}^n. \tag{40}$$

It fails to vanish in the general case, and the measurement process after the nth count has been recorded can terminate with probability $\rho_{i00}^{(n-1)}(t_{n-1})$ and continue with probability $1-\rho_{i00}^{(n-1)}(t_{n-1})$. In the limit $t_{n-1} \to \infty$, we obtain $\rho_{i00}^{(n-1)}(t_{n-1}) \approx 1$, i.e., as the measurement time increases, so does the probability that the measurement terminates, and termination is virtually certain after a sufficiently long time. This ensures that the measurement process is bounded in time. The probability $\rho_{i00}^{(n-1)}(t_{n-1})$ also depends on the time at which measurements start. The greater the

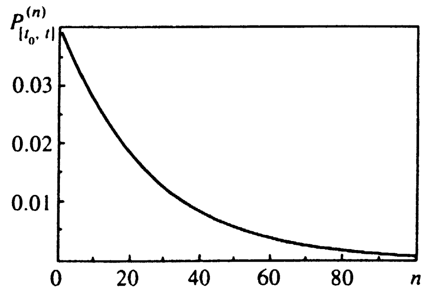

FIG. 5. Distribution $P^{(n)}_{[t_0,t)}$ of the probability that exactly n photons are detected in the idler mode in the time interval $[t_0,t)$. The mean number of photons in the mode prior to the measurements is 25.

value of t_0, the lower the probability of termination at the nth step. This fact can be easily explained, since as t_0 increases, so does the mean number of photons in the idler mode, which equals $\sinh^2(k_0t_0)$.

The probability that exactly n counts are recorded in the time interval $[t_0,t)$ is given by

$$P^{(n)}_{[t_0,t)} = \frac{[\tanh^2(k_0t_0)]^n\{1-\exp[-2\lambda(t-t_0)]\}^n}{\cosh^2(k_0t_0)\{1-\exp[-2\lambda(t-t_0)]\tanh^2(k_0t_0)\}^{n+1}}, \tag{41}$$

whereupon it follows that the distribution $P^{(n)}_{[t_0,t)}$ is a negative binomial distribution and describes a thermal state in which the mean number of photons is $\langle n(t)\rangle=\sinh^2(k_0t_0)\times\{1-\exp[-2\lambda(t-t_0)]\}$. Both $P^{(n)}_{[t_0,t)}$ and $\langle n(t)\rangle$ depend on the duration of the measurement. In the limit $(t-t_0)\to\infty$, $P^{(n)}_{[t_0,t)}\to[\tanh^2(k_0t_0)]^n/\cosh^2(k_0t_0)$, and the number of recorded counts equals the number of photons initially present in the idler mode. A plot of the distribution $P^{(n)}_{[t_0,t)}$ is displayed in Fig. 5. As the measurement time increases, the distribution shifts to the left, and the mean number of recorded counts increases. However, the displacement of the graph eventually stops. This indicates that the mean number of photons detected during the measurement process will be limited.

We modeled the random photocount detection process numerically on the basis of the characteristics obtained, and the results are displayed in Fig. 6.

5. CONTINUOUS INDEPENDENT MEASUREMENTS OF EACH MODE OF A TWO-MODE FIELD

In this section, a theory of continuous quantum measurements is constructed for independent simultaneous measurements of the number of photons in each mode of a two-mode field. We assume, as in Sec. 3, that a two-mode field has been produced in a cavity, and that the two modes can be separated so that independent measurements of the number of photons in each mode can be carried out with two different detectors.

The Hamiltonian of such a system will contain terms beyond those found in the Hamiltonian (8), which describe the second detector and its interaction with the measured mode b. These are given by expressions similar to (3) and (4):

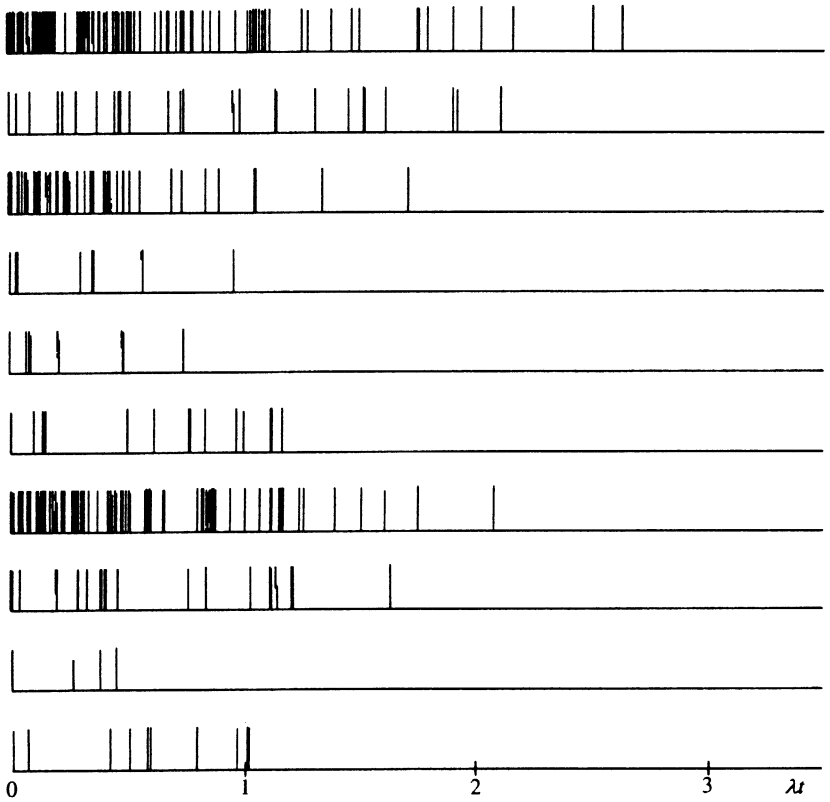

FIG. 6. Random realizations of the continuous measurement of the idler mode of a field generated by a parametric frequency converter. The mean number of photons in the mode prior to the measurements is 25.

$$H=H_a+H_b+H_{d_1}+H_{ad_1}+H_{d_2}+H_{bd_2}. \quad (42)$$

In this formulation of the problem, each detector makes continuous measurements of one mode of the field. This makes it possible to use the familiar formalism in describing the continuous photodetection of one-mode fields to construct a theory of continuous measurements of two-mode fields. We introduce for each independent measurement process the projection operators $P_a^{(n_1)}$ and $P_b^{(n_2)}$ onto spaces in which n_1 counts are recorded in mode a and n_2 counts are recorded in mode b. These are given by expressions given after Eq. (6):

$$\hat{P}_a^{(n_1)}=\frac{1}{n_1!}\sum_{\{j_k,a_k\}}\prod_{k=0}^{n_1} c_{a_k}^{+(j_k)}|0\rangle\langle 0|c_{a_k}^{(j_k)},\quad (43)$$

$$\hat{P}_b^{(n_2)}=\frac{1}{n_2!}\sum_{\{j_k,a_k\}}\prod_{k=0}^{n_2} d_{a_k}^{+(j_k)}|0\rangle\langle 0|d_{a_k}^{(j_k)}.\quad (44)$$

Since the measurements in modes a and b are independent, these operators commute with one another:

$$[P_a^{(n_1)},P_b^{(n_2)}]=0. \quad (45)$$

Recording a count in any mode reduces the overall state of the system. The temporal evolution of the system during the measurement is described by the reduced density matrix $\pi^{(n_1,n_2)}(t)$ after n_1 counts are recorded in mode a and n_2 counts are recorded in mode b. This matrix can be written as

$$\pi^{(n_1,n_2)}(t)=\mathrm{Tr}\{P_a^{(n_1)}P_b^{(n_2)}\rho(t)\}, \quad (46)$$

where the trace extends over the states of both detectors.

Multiplying von Neumann's equation for the density matrix $\rho(t)$ by the projection operators $P_a^{(n_1)}$ and $P_b^{(n_2)}$ and calculating the trace over the states of the detectors, it can be shown that the reduced density matrix satisfies

$$\frac{d\pi^{(n_1,n_2)}(t)}{dt}=-i[H_{\mathrm{eff}}\pi^{(n_1,n_2)}(t)-\pi^{(n_1,n_2)}(t)H_{\mathrm{eff}}^{+}]$$
$$+2\lambda a\pi^{(n_1-1,n_2)}(t)\ a^{+}+2\lambda b\pi^{(n_1,n_2-1)}(t)\ b^{+}. \quad (47)$$

Here,

$$H_{\mathrm{eff}}=\omega_a a^{+}a+\omega_b b^{+}b-i\lambda a^{+}a-i\lambda b^{+}b, \quad (48)$$

where ω_a and ω_b are the frequencies of modes a and b. In the derivation of Eq. (47) it was assumed that both detectors have the same efficiency, as specified by λ.

The solution of the equation for the density matrix can be written as a recurrence relation,

$$\pi^{(n_1,n_2)}(t)=\int_{t_0}^{t}S_{t,\tau}J_a\pi^{(n_1-1,n_2)}(\tau)\ d\tau$$
$$+\int_{t_0}^{t}S_{t,\tau}J_b\pi^{(n_1,n_2-1)}(\tau)\ d\tau, \quad (49)$$

where $S_{t,\tau}$ is the field evolution operator between two successive counts, and is given by (12). In addition, here we have introduced the operators J_a and J_b, which describe the reduction of the corresponding mode, as a result of recording a count, to the state with one less photon:

$$J_a x=2\lambda axa^{+}, \quad (50)$$

$$J_b x=2\lambda bxb^{+}. \quad (51)$$

Equation (49) is important in determining the basic characteristics of continuous independent measurements of each mode of a two-mode field. Its physical interpretation is as follows: The state in which n_1 counts are recorded in mode a and n_2 counts are recorded in mode b can be reached both by recording a count in mode a (first term) and by recording a count in mode b (second term). In addition, the state of the field prior to a count will be different for these two possibilities. It follows from what we have said above that the state with a certain number of counts in the modes can be reached by various paths for which the counts in the modes are recorded in different order, as shown in Fig. 7. To describe the measurement process, here we have introduced the conditional probability $\tilde{c}_a^{(n_1,n_2)}(t_n/t_{n-1})$ that the next count is recorded in mode a at time t_n, if by time t_{n-1} the n_1 counts were recorded in mode a and n_2 counts were recorded in mode b, and similarly for mode b. The method for determining them will be described below when we describe the method for modeling the random process of recording counts.

Let n counts be recorded by time t_n (n_1 in mode a and n_2 in mode b) at times $t_1,t_2,...,t_n$. Then, using the operator relations

$$J_aS_{t,\tau}\,x=\exp[-2\lambda(t-\tau)]S_{t,\tau}J_a x, \quad (52)$$

$$J_bS_{t,\tau}\,x=\exp[-2\lambda(t-\tau)]S_{t,\tau}J_b x, \quad (53)$$

$$[J_a,J_b]=0 \quad (54)$$

and writing out (49), the reduced density matrix of the system can be expressed in terms of the initial state of the field as follows:

$$\pi^{(n_1,n_2)}(t)=C_{n_1+n_2}^{n_1}\int_{t_0}^{t}dt_n\int_{t_0}^{t_n}dt_{n-1}\cdots\int_{t_0}^{t_2}dt_1$$
$$\times\exp[-2\lambda\Sigma_{j=1}^{n}(t_j-t_0)]$$
$$\times S_{t_n,t_0}J_a^{n_1}J_b^{n_2}\rho(t_0). \quad (55)$$

The binomial coefficient $C_{n_1+n_2}^{n_1}$ is the number of ways in which n_1 counts can be recorded in mode a and n_2 counts in mode b.

An important result that follows from the relations (52)–(54) and is reflected in (55) should be noted: to describe the temporal evolution of the density matrix of the system, it is only necessary to know the times at which the counts are recorded—regardless of which count occurred at which time—and the number of counts recorded in each mode. This makes it possible to denote the times at which counts are recorded simply by $t_1,t_2,...,t_n$, where $n=n_1+n_2$.

Traditionally, the main objective of the theory of continuous measurements was to derive expressions for the probability $P_{[t_0,t)}^{(n)}$ that n counts are recorded in the time interval $[t_0,t)$. For the present formulation of the problem, this probability can be expressed as

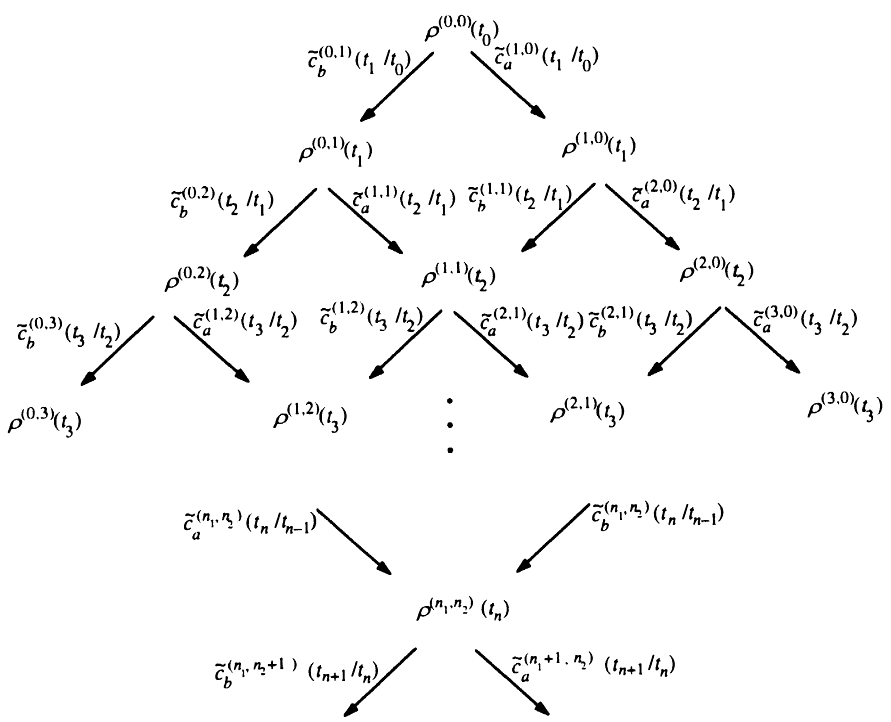

FIG. 7. Possible paths for the evolution of the state of a two-mode field during independent simultaneous measurements of the number of photons in both modes of the field.

$$P_{[t_0,t)}^{(n)}=\sum_{n_1=0}^{n} P_{[t_0,t)}^{(n_1,n_2)}, \qquad (56)$$

where $P_{[t_0,t)}^{(n_1,n_2)}$ is the probability that in the time interval $[t_0,t)$, n_i counts are recorded in mode a and n_2 counts are recorded in mode b. This quantity can be determined in terms of the reduced density matrix of the system, similarly to the case of continuous measurement of a one-mode field:

$$P_{[t_0,t)}^{(n_1,n_2)}=\mathrm{Tr}_f\{\pi^{(n_1,n_2)}(t)\}$$
$$=\frac{\{1-\exp[-2\lambda(t-t_0)]\}^{n_1+n_2}}{(n_1+n_2)!}$$
$$\times C_{n_1+n_2}^{n_1}\mathrm{Tr}\{S_{t,t_0}J_a^{n_1}J_b^{n_2}\rho(t_0)\} . \qquad (57)$$

The trace extends over the states of both modes of the field.

The probability $P_{[t_0,t)}^{(n)}$ characterizes this measurement process from the standpoint of the probability density for n counts being recorded in each mode in any combination of the number of recorded counts and in any sequence, i.e., as a random occurrence of photocounts at one of the detectors. The introduction of the probability $P_{[t_0,t)}^{(n_1,n_2)}$ makes it possible to describe in detail the continuous measurement process for the given system, since it specifies the number of counts recorded by each detector.

It follows from what was said above that the continuous measurement process can then be investigated in two ways. One is to represent the given random process as a combination of the two independent measuring processes carried out by each detector. This representation makes it possible to investigate the temporal evolution of each mode of the field and of the entire field as whole, taking into account the sequence in which the photocounts occur at the detectors. To model such a process, it is necessary to introduce new characteristics that describe the measurement process in each mode. On the whole, the characteristics of the continuous measurements in this case will be different from the cases studied previously.

The second method is to consider the measurements to be a random process in which the photons in the two-mode field are detected, irrespective of the mode to which a count refers. The measurement of the number of photons in each mode is an integral part of the general photodetection process in a two-mode field. The problem is to describe the statistical properties of the set of random times at which the photocounts are recorded by a measurement system consisting of two detectors. This description is based on a representation of the measurement process as a random integer photocount detection process $n(t)$ whose laws and characteristics are similar to those of a one-mode field Each method for describing the given measuring process will be discussed in more detail below.

5.1. Measurement of a two-mode field as a set of two independent measurement processes on the number of photons in each mode

This approach to the investigation of the measurement process in a two-mode field is based on a representation of the process as a sequence of events consisting of recording a photocount in one mode or the other. This random sequence can be analyzed by using the standard characteristics of the theory of continuous quantum measurements. In the case at hand, however, besides describing the sequence of events, it is necessary to model the event itself, which is a complicated random photocount detection process. To do so, it is necessary to introduce new characteristics that define the measurement process in each mode, and to develop a new method for constructing random realizations of the occurrence of counts.

Let $n-1$ events occur by time t_{n-1} at times $t_1, t_2, ..., t_{n-1}$. In addition, let n_1 counts be recorded in mode a and n_2 counts in mode b. We study the process of recording photocounts, taking into account the "history" of the occurrence of counts in each detector, and will be interested in describing one realization of the given random process.

The temporal evolution of the two-mode field during a continuous measurement is specified by the normalized reduced density matrix:

$$\rho^{(n_1,n_2)}(t)=\frac{S_{t,t_n}JS_{t_n,t_{n-1}}J\dots JS_{t_2,t_1}JS_{t_1,t_0}\rho(t_0)}{\mathrm{Tr}\{S_{t,t_n}JS_{t_n,t_{n-1}}J\dots JS_{t_2,t_1}JS_{t_1,t_0}\rho(t_0)\}}, \tag{58}$$

where J is the operator of reduction of the field as a result of recording a count in one of the modes:

$$J=\begin{cases} J_a, & \text{if a photon is detected in mode } a, \\ J_b, & \text{if a photon is detected in mode } b. \end{cases} \tag{59}$$

Taking the commutation relations (52)–(54) into consideration, the expression for the density matrix $\rho^{(n_1,n_2)}(t_n)$ is

$$\rho^{(n_1,n_2)}(t)=\frac{S_{t,t_0}J_a^{n_1}J_b^{n_2}\rho(t_0)}{\mathrm{Tr}\{S_{t,t_0}J_a^{n_1}J_b^{n_2}\rho(t_0)\}}. \tag{60}$$

It follows from Eqs. (58)–(60) that the probability of occurrence of the next count does not depend on the times at which the preceding counts occurred, i.e., detection in two modes is a semi-Markovian process, just like the detection process in a one-mode field.[2] It also follows from Eq. (60) that the state of the field at each measurement time can be determined simply from the number of counts recorded in each mode, and does not depend on the sequence in which the photocounts occur. Using the knowledge of the density matrix (60) after a certain number of counts have been recorded in the two modes, we can find the probability for the occurrence of the next (nth) count and the path along which further measurements can proceed. Indeed, the conditional elementary probability $\tilde{c}^{(n)}(t_n/t_{n-1})$ that the nth event occurs at time t_n if the $(n-1)$th event occurred at time t_{n-1} can be determined according to Eq. (49) as the sum of the conditional elementary probabilities that the next count is recorded in one mode or the other:

$$\tilde{c}^{(n)}(t_n/t_{n-1})=\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})+\tilde{c}_b^{(n_1,n_2+1)}(t_n/t_{n-1}). \tag{61}$$

Both $\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})$ and $\tilde{c}_b^{(n_1,n_2+1)}(t_n/t_{n-1})$ can be expressed in terms of the normalized reduced density matrix of the field:

$$\tilde{c}_a^{(n_1+1,n_2)}(t/\tau)=\mathrm{Tr}\{J_aS_{t,\tau}\rho^{(n_1,n_2)}(\tau)\}, \tag{62}$$

$$\tilde{c}_b^{(n_1,n_2+1)}(t/\tau)=\mathrm{Tr}\{J_bS_{t,\tau}\rho^{(n_1,n_2)}(\tau)\}. \tag{63}$$

The probability that the measurement process continues after the occurrence of the $(n-1)$th event is then

$$\int_{t_{n-1}}^{\infty}\tilde{c}^{(n)}(t_n/t_{n-1})\,dt_n = \int_{t_{n-1}}^{\infty}\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})\,dt_n + \int_{t_{n-1}}^{\infty}\tilde{c}_b^{(n_1,n_2+1)}(t_n/t_{n-1})\,dt_n, \tag{64}$$

which in the general case is not equal to unity. It can be shown (see Appendix 1) that there exists a finite probability that the measurement process terminates at the nth step:

$$\int_{t_{n-1}}^{\infty}\tilde{c}^{(n)}(t_n/t_{n-1})dt_n=1-\rho_{0000}^{(n_1,n_2)}(t_{n-1}). \tag{65}$$

Therefore, the measurement process can terminate after the $(n-1)$th step with probability $\rho_{0000}^{(n_1,n_2)}(t_{n-1})$ and continue with probability $1-\rho_{0000}^{(n_1,n_2)}(t_{n-1})$. For two-field measurements, the probability that the measurement process terminates is equal to the probability that after the n_1th count is recorded in mode a and n_2 counts are recorded in mode b, both modes are in the vacuum state.

Next, we construct a method for modeling the nth event, consisting of recording a photocount in one of the detectors. First of all, it is necessary to determine the mode in which the next count is recorded. The probability that a photocount occurs in mode a after the n_1th count is recorded in mode a and n_2 counts are recorded in mode b can be written in the form

$$z_a^{(n_1+1,n_2)}=\frac{\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})}{\tilde{c}^{(n)}(t_n/t_{n-1})}. \tag{66}$$

Similarly, for mode b we have

$$z_b^{(n_1,n_2+1)}=\frac{\tilde{c}_a^{(n_1,n_2+1)}(t_n/t_{n-1})}{\tilde{c}^{(n)}(t_n/t_{n-1})}. \tag{67}$$

Taking a random set of numbers x_a uniformly distributed over the interval [0, 1] to characterize the probability that a count is recorded in mode a, and comparing each in turn to the probability $z_b^{(n_1,n_2+1)}$ that a photocount is recorded in mode b prior to each measurement, it is possible to determine the mode in which the next count is recorded: $x_a > z_b^{(n_1,n_2+1)}$ for a count in mode b, and $x_a < z_b^{(n_1,n_2+1)}$ for a count in mode a.

The modeling of the measurement process in each mode is based on the theory developed for continuous photodetec-

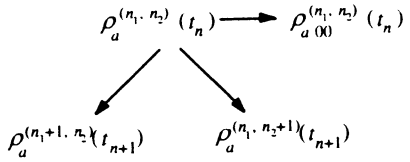

FIG. 8. Possible paths of the evolution of the states of mode a after the n_1th count is recorded in mode a and n_2 counts are recorded in mode b.

tion of one-mode fields. However, there are some peculiarities related to the existence of two modes in which counts can occur. Consider Fig. 8, which depicts the temporal evolution of the state of mode a at the nth measurement step. After the n_1th count is recorded in mode a and n_2 counts are recorded in mode b, mode a is in the state $\rho_a^{(n_1,n_2)}(t_{n-1})$. There are several different possibilities for continuing the random photocount detection process. For example, there is a probability $\rho_{a00}^{(n_1,n_2)}(t_{n-1})$ that the measurement process terminates in mode a. This is the probability that mode a ends up in the vacuum state, while mode b can be in a state with any number of photons:

$$\rho_{a00}^{(n_1,n_2)}(t_{n-1})=\sum_{j=0}^{\infty}\rho_{00,jj}^{(n_1,n_2)}(t_{n-1}). \tag{68}$$

When the measurement process terminates in mode a, the conditional elementary probability that the next count occurs in mode a equals zero. Furthermore, as follows from Eqs. (66) and (67), counts can be recorded only in mode b. Measurements continue until mode b is in the vacuum state. The measurement process for the two-mode field then terminates completely.

Besides the termination probability, there is also a probability $1-\rho_{a00}^{(n_1,n_2)}(t_{n-1})$ that the measurement process continues in mode a. Here, however, in contrast to one-mode fields, there are two possibilities. The first is that a photocount is recorded in mode a, after which the state of the mode will be determined by the density matrix $\rho_a^{(n_1+1,n_2)}\times(t_n)$. The second is that a count is recorded in mode b, after which the state of mode a also changes and will be determined by the density matrix $\rho_a^{(n_1,n_2+1)}(t_n)$. This scheme is also confirmed by numerical modeling. For example, it can be shown (see Appendix 2) that the probability that photocount detection continues in mode a is

$$1-\rho_{a00}^{(n_1,n_2)}(t_{n-1})=\int_{t_{n-1}}^{\infty}\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})\,dt_n+\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{j}{m+j}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1}). \tag{69}$$

The first term on the right-hand side of this expression is the probability that a count is recorded in mode a. The second is the probability that the measurement process does not terminate in mode a, but that a count occurs in mode b. It is the existence of this probability that makes the method of modeling the measurement process for the case of the measurement of the number of counts in one mode of a two-mode field different from the measurement of two-mode fields, since the detection time for the next count is determined only by the probability that a count is recorded in mode a. It should be noted that by indicating the mode in which the count will occur prior to each measurement step, we thereby make a choice of one of the aforementioned possibilities for the continuation of the measurement process in each mode.

A procedure for finding the time of occurrence of the next count can be constructed on the basis of the foregoing. We introduce the normalized probability density that the nth count occurs at time t_n:

$$f_a^{(n_1+1,n_2)}(t_n/t_{n-1})=\frac{\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})}{\int_{t_{n-1}}^{\infty}\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})\,dt_n}. \tag{70}$$

The probability that the next count will occur at a time τ after the preceding count is then

$$\int_{t_{n-1}+\tau}^{\infty}f_a^{(n_1+1,n_2)}(t_n/t_{n-1})\,dt_n=x. \tag{71}$$

The photocount detection process in mode a can be modeled by a random sequence of numbers x uniformly distributed over the interval [0, 1], and the random delay times τ can be obtained from Eq. (71).

This scheme for generating the next count in one mode of a two-mode field is similar for both modes a and b. Applying the scheme successively at each measurement step, we obtain a way to construct random realizations of the stochastic photocount detection process in a two-mode field during independent measurements of the number of photons in each mode.

5.2. Measurements of the number of photons in a two-mode field

We now consider the measurement process in a two-mode field from the standpoint of investigating the probability of finding n photons in the field. We assume that the measurement system that records counts as they occur at a detector contains two detectors that perform continuous measurements in each mode. The photodetection process in the two-mode field can then be represented as a chain of events consisting of successive reductions of the state of the field to a state in which there is one less photon as a result of a photon being detected in one of the modes. This sequence of events can be described by a set of reduced density matrices $\rho^{(n)}(t_n)$, which determine the state of the field being investigated at each measurement time, the conditional elementary probabilities $\tilde{c}^{(n)}(t_n/t_{n-1})$ that the next (nth) count occurs at time t_n if the preceding measurement occurred at time t_{n-1}, and the associated probabilities $\rho_{00}^{(n-1)}(t_{n-1})$ that the measurement process terminates. In other words, we represent the continuous quantum measurement process as a random discrete integer process $n(t)$, whose characteristics and laws are well known.[2] The description of the process is based on determining the probability density $p_{[t_0,t)}(t_1,t_2,...,t_n)$ for exactly n counts occurring at times t_1, t_2, ..., t_n in the time interval $[t_0, t)$, with no other counts interspersed, i.e., if it is possible to find such a quantity for the measurement process under investigation, then the descriptive formalism developed for the random process $n(t)$ can be used.

It is well known that the probability density $p_{[t_0,t)}\times(t_1,t_2,...,t_n)$ is closely related to the probability $P_{[t_0,t)}^{(n)}$ that exactly n counts are recorded in the time interval $[t_0,t)$:

$$P^{(n)}_{[t_0,t)}=\int_{t_0}^{t}dt_n\int_{t_0}^{t_n}dt_{n-1}\dots\int_{t_0}^{t_2}dt_1\ p_{[t_0,t)}(t_1,t_2,\dots,t_n). \quad (72)$$

Using the relation (56) between the probabilities $P^{(n)}_{[t_0,t)}$ and $P^{(n_1,n_2)}_{[t_0,t)}$, and their relation to the reduced density matrix (57) of the system after the n_1th count is recorded in mode a and n_2 counts are recorded in mode b, the probability density $p_{[t_0,t)}(t_1,t_2,\dots,t_n)$ for the two-mode field is given by

$$p_{[t_0,t]}(t_1,t_2,\dots,t_n)=\sum_{n_1=1}^{n}C_n^{n_1}\exp[-2\lambda\Sigma_{j=1}^{n}(t_j-t_0)]\times\mathrm{Tr}\{S_{t,t_0}J_a^{n_1}J_b^{n-n_1}\rho(t_0)\}. \quad (73)$$

It is easy to see that (73) accounts for all possibilities in recording n photons with two independent detectors in a two-mode field, accommodating different photocount detection sequences and different numbers of counts that can be recorded in each mode.

We can now determine the conditional elementary probability that the next count occurs in the two-mode field:

$$\tilde{c}^{(n)}(t_n/t_{n-1})=\frac{p_{[t_0,t_n)}(t_1,t_2,\dots,t_n)}{p_{[t_0,t_{n-1})}(t_1,t_2,\dots,t_{n-1})}=\exp[-2\lambda(t_n-t_0)]\times\frac{\Sigma_{n_1=1}^{n}C_n^{n_1}\ \mathrm{Tr}\{S_{t_n,t_0}J_a^{n_1}J_b^{n-n_1}\rho(t_0)\}}{\Sigma_{n_1=1}^{n-1}C_{n-1}^{n_1}\ \mathrm{Tr}\{S_{t_{n-1},t_0}J_a^{n_1}J_b^{n-n_1-1}\rho(t_0)\}} \quad (74)$$

and the normalized reduced density matrix of the two-mode field after n counts have been recorded:

$$\rho^{(n)}(t_n)=\frac{\Sigma_{n_1=1}^{n}C_n^{n_1}S_{t_n,t_0}J_a^{n_1}J_b^{n-n_1}\rho(t_0)}{\Sigma_{n_1=1}^{n}C_n^{n_1}\mathrm{Tr}\{S_{t_n,t_0}J_a^{n_1}J_b^{n-n_1}\rho(t_0)\}}. \quad (75)$$

Subsequent description of the process of measuring the number of photons in a two-mode field is based on a scheme developed to analyze the random discrete process $n(t)$. The probability that the measurement process terminates at the nth step (after the $(n-1)$th count has been recorded) is

$$\rho_{00}^{(n-1)}(t_{n-1})=1-\int_{t_{n-1}}^{\infty}dt_n\ \tilde{c}^{(n)}(t_n/t_{n-1}). \quad (76)$$

This is the probability that the two-mode field is in the vacuum state after the $(n-1)$th count, i.e., after the $(n-1)$th count has been recorded the field is in the state $\rho^{(n-1)}(t_{n-1})$, and the measurement process can continue with probability $1-\rho_{00}^{(n-1)}(t_{n-1})$ and terminate with probability $\rho_{00}^{(n-1)}(t_{n-1})$. The probability density for the next count in the two-mode field occurring at time t_n is

$$f^{(n)}(t_n/t_{n-1})=\frac{\tilde{c}^{(n)}(t_n/t_{n-1})}{1-\rho_{00}^{(n-1)}(t_{n-1})}. \quad (77)$$

This quantity specifies the probability of finding the next photocount some time τ after the preceding count has been recorded:

$$\int_{t_n+\tau}^{\infty}dt_n\ f^{(n)}(t_n/t_{n-1})=x\in[0,1]. \quad (78)$$

The random delay times $\tau=t_n-t_{n-1}$ can be found by using a set of random numbers x uniformly distributed over the interval [0, 1]. Applying this scheme successively at each measurement step, we obtain a realization of the random photocount detection process in a two-mode field using a measurement system consisting of two detectors.

5.3. Temporal evolution of each mode of the field during continuous independent measurements of the number of photons in both modes of a two-mode field

The foregoing characteristics and laws governing the continuous photodetection of a two-mode field make it possible to investigate the temporal evolution of both the total two-mode field and each mode individually. The statistics of the radiation detected in an individual mode are characterized by the moments of the number of photons in this mode after the n_1 counts is recorded in mode a and n_2 counts are recorded in mode b:

$$\langle n_a^m(t)\rangle^{(n_1,n_2)}=\mathrm{Tr}\{(a^+a)^m\rho^{(n_1,n_2)}(t)\}, \quad (79)$$

$$\langle n_b^m(t)\rangle^{(n_1,n_2)}=\mathrm{Tr}\{(b^+b)^m\rho^{(n_1,n_2)}(t)\}. \quad (80)$$

Setting $m=1$, we find the mean number of photons in the modes after the n_1 counts is recorded in mode a and n_2 counts are recorded in mode b. Using (57), it can be shown that this number is related to the probabilities $P^{(n_1,n_2)}_{[t_0,t)}$:

$$\langle n_a(t)\rangle^{(n_1,n_2)}=\frac{\exp[-2\lambda(t-t_0)]}{1-\exp[-2\lambda(t-t_0)]}(n_1+1)\frac{P^{(n_1+1,n_2)}_{[t_0,t)}}{P^{(n_1,n_2)}_{[t_0,t)}}, \quad (81)$$

$$\langle n_b(t)\rangle^{(n_1,n_2)}=\frac{\exp[-2\lambda(t-t_0)]}{1-\exp[-2\lambda(t-t_0)]}(n_2+1)\frac{P^{(n_1,n_2+1)}_{[t_0,t)}}{P^{(n_1,n_2)}_{[t_0,t)}}. \quad (82)$$

The variance of the number of photons in each mode of the field can be represented in the form

$$\langle[\Delta n_a(t)]^2\rangle^{(n_1,n_2)}=\frac{\exp[-4\lambda(t-t_0)]}{\{1-\exp[-2\lambda(t-t_0)]\}^2}(n_1+1)\times(n+2)\frac{P^{(n_1+2,n_2)}_{[t_0,t)}}{P^{(n_1,n_2)}_{[t_0,t)}}+\langle n_a(t)\rangle^{(n_1,n_2)}-(\langle n_a(t)\rangle^{(n_1,n_2)})^2, \quad (83)$$

$$\langle[\Delta n_b(t)]^2\rangle^{(n_1,n_2)}=\frac{\exp[-4\lambda(t-t_0)]}{\{1-\exp[-2\lambda(t-t_0)]\}^2}(n_2+1)\times(n_2+2)\frac{P^{(n_1,n_2+2)}_{[t_0,t)}}{P^{(n_1,n_2)}_{[t_0,t)}}+\langle n_b(t)\rangle^{(n_1,n_2)}-(\langle n_b(t)\rangle^{(n_1,n_2)})^2. \quad (84)$$

The Fano factor can be written

$$F_a^{(n_1,n_2)}(t)=\langle n_a(t)\rangle^{(n_1+1,n_2)}-\langle n_a(t)\rangle^{(n_1,n_2)}+1, \quad (85)$$

$$F_b^{(n_1,n_2)}(t)=\langle n_b(t)\rangle^{(n_1,n_2+1)}-\langle n_b(t)\rangle^{(n_1,n_2)}+1. \quad (86)$$

It is related to the mean number of photons in the mode before and after the next count is recorded in that mode. Depending on how the mean number of photons in a mode changes after the next count is recorded, the Fano factor can be greater or less than one.[3] thereby yielding different statistics for the detected radiation.

Another characteristic describing the statistical properties of the radiation under study is the clustering parameter. As in the measurement of one-mode fields, this parameter can be introduced on the basis of the elementary conditional probability that the next count is recorded in one of the modes. For example, $\tilde{c}_a^{(n_1,n_2)}(t/t)$ is the transition rate from the state in which the (n_1-1)th count is recorded to the state in which the n_1th count is recorded, or the rate of photocounts in mode a if n_2 counts have been recorded in mode b. The probability of such a transition during a brief time interval Δt is

$$P^a_{n_1,n_1-1}(t)=\tilde{c}_a^{(n_1,n_2)}(t/t)\Delta t, \quad (87)$$

and the clustering parameter for photons in mode a,

$$\eta_a^{(n_1,n_2)}(t_{n_1})=\frac{P^a_{n_1,n_1-1}(t_{n_1})\Delta t}{P^a_{n_1-1,n_1-2}(t_{n_1})\Delta t}=\frac{\tilde{c}_a^{(n_1,n_2)}(t_{n_1}/t_{n_1})}{\tilde{c}_a^{(n_1-1,n_2)}(t_{n_1}/t_{n_1})}, \quad (88)$$

describes the change in the probability that the n_1th count is recorded by time t_{n_1} relative to the probability that the (n_1-1)th count is recorded in mode a if n_2 counts have been recorded in mode b. If $\eta_a^{(n_1,n_2)}(t_{n_1})>1$, we speak of n_1th-order clustering of the counts in mode a. If $\eta_a^{(n_1,n_2)}\times(t_{n_1})<1$, we speak of n_1th-order anticlustering of the counts in mode a. If $\eta_a^{(n_1,n_2)}(t_{n_1})=1$, the photocounts in mode a are independent of one another. A clustering parameter can be introduced similarly for mode b.

Taking into account the relationship between the conditional elementary probabilities $\tilde{c}_a^{(n_1,n_2)}(t_n/t_{n-1})$, $\tilde{c}_b^{(n_1,n_2)}\times(t_n/t_{n-1})$, and $P^{(n_1,n_2)}_{[t_0,t)}$:

$$\tilde{c}_a^{(n_1,n_2)}(t_n/t_{n-1})=2\lambda\frac{\exp[-2\lambda(t_n-t_0)]}{1-\exp[-2\lambda(t_n-t_0)]}n_1\times\frac{P^{(n_1,n_2)}_{[t_0,t_n)}}{P^{(n_1-1,n_2)}_{[t_0,t_{n-1})}}, \quad (89)$$

$$\tilde{c}_b^{(n_1,n_2)}(t_n/t_{n-1})=2\lambda\frac{\exp[-2\lambda(t_n-t_0)]}{1-\exp[-2\lambda(t_n-t_0)]}n_2\times\frac{P^{(n_1,n_2)}_{[t_0,t_n)}}{P^{(n_1,n_2-1)}_{[t_0,t_{n-1})}}, \quad (90)$$

the clustering parameter for each mode of a two-mode field can be represented in the form

$$\eta_a^{(n_1,n_2)}(t_{n_1})=\frac{n_1}{n_1-1}\frac{P^{(n_1,n_2)}_{[t_0,t_{n_1})}P^{(n_1-2,n_2)}_{[t_0,t_{n_1})}}{\left(P^{(n_1-1,n_2)}_{[t_0,t_{n_1-1})}\right)^2}, \quad (91)$$

$$\eta_b^{(n_1,n_2)}(t_{n_1})=\frac{n_2}{n_2-1}\frac{P^{(n_1,n_2)}_{[t_0,t_{n_1})}P^{(n_1,n_2-2)}_{[t_0,t_{n_1})}}{\left(P^{(n_1,n_2-1)}_{[t_0,t_{n_1-1})}\right)^2}. \quad (92)$$

These expressions are similar to the relation obtained for the clustering parameter for one-mode fields.[7] Here, however, in describing the radiation detected in one mode, it is also necessary to take into account the number of photocounts recorded in the other mode, a consequence of correlation between the modes.

In discussing the statistical properties of two-mode fields, it is necessary to consider the cross-correlation of the counts in the different modes. This correlation can be described by introducing the clustering parameter for photons in different modes on the basis of the local joint probability density $h^{(n_1,n_2)}_{[t_0,t)}(t_1,t_2,...,t_n)$ for n_1 counts being recorded in mode a and n_2 counts being recorded in mode b at times $t_1,t_2,...,t_n$ (with other counts possible in between). Such a local joint probability density can be expressed in terms of the Green's matrix $\hat{D}^{(t,t_0)}$ for the kinetic equation for the reduced density matrix $\sigma(t)=\Sigma_{n_1,n_2}\pi^{(n_1,n_2)}(t)$. According to Eq. (47), the Green's matrix is given by

$$\hat{D}^{(t,t_0)}x=S_{t,t_0}\exp\left\{\frac{1-\exp[-2\lambda(t-t_0)]}{2\lambda}(J_a+J_b)\right\}x. \quad (93)$$

Using the commutation properties of the operator $\hat{D}^{(t,t_0)}$ with respect to the operators J_a and J_b, which are the analogs of the properties of the operator $S_{t,\tau}$ (52)–(53), and the relation

$$\mathrm{Tr}\{\hat{D}^{(t,t_0)}x\}=\mathrm{Tr}\{x\}, \quad (94)$$

we find that

$$h^{(n_1,n_2)}_{[t_0,t)}(t_1,t_2,...,t_n)$$

$$=\mathrm{Tr}\{\hat{D}^{(t,t_0)}J\hat{D}^{(t_n,t_0)}...J\hat{D}^{(t_1,t_0)}\rho(t_0)\}$$

$$=\exp[-2\lambda\Sigma^n_{j=1}(t_j-t_0)]\mathrm{Tr}\{J^{n_1}_aJ^{n_2}_b\rho(t_0)\}, \qquad (95)$$

where

$$J=\begin{cases}J_a, & \text{if a photon is detected in mode } a,\\ J_b, & \text{if a photon is detected in mode } b.\end{cases}$$

In the case of two independent modes, the quantity $h^{(n_1,n_2)}_{[t_0,t)}\times(t_1,t_2,...,t_n)$ factorizes into two factors, each of which determines the joint probability density for recording at least n_1 (n_2) counts in mode a (b) in the time interval $[t_0,t)$ at times $t_{1a},t_{2a},...,t_{n_1a}$ ($t_{1b},t_{2b},...,t_{n_2b}$):

$$h^a_{[t_0,t)}(t_{1a},t_{2a},...,t_{n_1a})=\exp[-2\lambda\Sigma^{n_1}_{j=1}(t_{ja}-t_0)]\times\mathrm{Tr}\{J^{n_1}_a\rho_a(t_0)\}, \qquad (96)$$

$$h^b_{[t_0,t)}(t_{1b},t_{2b},...,t_{n_2b})=\exp[-2\lambda\Sigma^{n_2}_{j=1}(t_{jb}-t_0)]\times\mathrm{Tr}\{J^{n_2}_b\rho_b(t_0)\}. \qquad (97)$$

The ratio of the probability density $h^{(n_1,n_2)}_{[t_0,t)}(t_1,t_2,...,t_n)$ for the two-model field under study to the analogous quantity for two initially independent modes of the field yields the clustering parameter for the modes of the field, which characterizes the difference between the distribution of counts in measurements of two-mode correlated fields from those in measurements of two independent modes:

$$g^{(n_1,n_2)}=\frac{h^{(n_1,n_2)}_{[t_0,t)}(t_1,t_2,...,t_n)}{h^a_{[t_0,t)}(t_{1a},t_{2a},...,t_{n_1a})h^b_{[t_0,t)}(t_{1b},t_{2b},...,t_{n_2b})}=\frac{\mathrm{Tr}\{J^{n_1}_aJ^{n_2}_b\rho(t_0)\}}{\mathrm{Tr}\{J^{n_1}_a\rho_a(t_0)\}\mathrm{Tr}\{J^{n_2}_b\rho_b(t_0)\}}. \qquad (98)$$

When $g^{(n_1,n_2)}>1$, there is statistical clustering of photocounts in the different modes of a two-mode field. This shows up as an increase in the probability that a count will be recorded in mode a after a photon has been detected in mode b. When $g^{(n_1,n_2)}<1$, the photocounts have a tendency toward anticlustering, and recording a count in one mode diminishes the probability that a count will be recorded in the other. When $g^{(n_1,n_2)}=1$, the photocounts in the two modes are independent. The quantity $g^{(n_1,n_2)}$ does not depend on time, and it defines the statistical properties of the random photocount detection process over the entire measurement time.

6. CONTINUOUS MEASUREMENTS OF A TWO-MODE FIELD GENERATED BY A PARAMETRIC FREQUENCY CONVERTER

6.1. Independent simultaneous measurements of the number of photons in the signal and idler modes of the field

We now use the theory presented above to describe continuous measurements of the field generated by a parametric frequency converter (see Sec. 4.1). Let two detectors that independently measure the number of photons in the two modes of the field be turned on at time t_0. The goal is to study the temporal evolution during the measurement process of both the complete two-mode field and each mode individually.

Suppose that by time t_n, n_1 counts have been recorded in the idler mode and n_2 in the signal mode. The state of the two-mode system can be described by the normalized reduced density matrix

$$\rho^{(n_1,n_2)}(t)=\left\{\sum_{k,l=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(k+l-n_1-n_2)t']\beta^k\beta^{*l}\sqrt{\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}}\sqrt{\frac{l!}{(l-n_1)!}\frac{l!}{(l-n_2)!}}|k-n_1\rangle_{ss}\langle l-n_1|\oplus|k-n_2\rangle_{ii}\langle l-n_2|\right\}\times\left\{\sum_{k=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)t'][\tanh^2(k_0t_0)]^k\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}\right\}^{-1}, \qquad (99)$$

where $\beta=-ie^{i\theta}\tanh(k_0t_0)$ and $t'=t-t_0$.

It follows from this expression that the intermode correlation at the start of measurements remains unchanged during the photodetection process, regardless of the number of counts recorded in each mode. The state of the signal and idler modes after n_1 counts have been recorded in the idler mode and n_2 counts have been recorded in the signal mode is described by the reduced density matrices $\rho^{(n_1,n_2)}_i(t)$ and $\rho^{(n_1,n_2)}_s(t)$, respectively, whose diagonal matrix elements have the form

$$\rho_{imm}^{(n_1,n_2)}(t)=\begin{cases}\dfrac{\exp[-2\lambda(2m+n_1-n_2)(t-t_0)][{}^2(k_0t_0)]^{m+n_1}\dfrac{(m+n_1)!(m+n_1)!}{m!(m+n_1-n_2)!}}{\sum\limits_{k=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t-t_0)][\tanh^2(k_0t_0)]^k\dfrac{k!}{(k-n_1)!}\dfrac{k!}{(k-n_2)!}}, & m\geqslant n_2-n_1,\\ 0, \quad m<n_2-n_1, \end{cases} \tag{100}$$

$$\rho_{sjj}^{(n_1,n_2)}(t)=\begin{cases}\dfrac{\exp[-2\lambda(2j+n_2-n_1)(t-t_0)][\tanh^2(k_0t_0)]^{j+n_2}\dfrac{(j+n_2)!(j+n_2)!}{j!(j+n_2-n_1)!}}{\sum\limits_{k=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t-t_0)][\tanh^2(k_0t_0)]^k\dfrac{k!}{(k-n_1)!}\dfrac{k!}{(k-n_2)!}}, & j\geqslant n_1-n_2.\\ 0, \quad j<n_1-n_2. \end{cases} \tag{101}$$

Equations (100) and (101) reflect the law of photon conservation in each mode. Since photons are created in pairs in a parametric amplifier, the number of photons in the signal and idler modes is the same when measurements begin. During the measurement process, the number of photons in the modes decreases because photons are removed from the field as the counts are recorded. However, the sum of the recorded counts and the photons remaining in a mode must remain constant during the measurement process, and must be the same for the two modes:

$$m+n_1=j+n_2. \tag{102}$$

If the number of counts recorded in the idler mode is greater than in the signal mode ($n_1>n_2$), then at least n_1-n_2 photons remain in the signal mode, and the probability of observing the signal mode in a state with fewer than n_1-n_2 photons is 0.

The state of the field at any measurement time determines the conditional probability $\tilde{c}^{(n)}(t_n/t_{n-1})$ that the next (nth) event occurs at time t_n if the preceding event occurred at time t_{n-1}, as well as the probability that the measurement process terminates at the nth step. This equals the probability that after the n_1th count has been recorded in the idler mode and n_2 counts have been recorded in the signal mode, the field ends up in the vacuum state:

$$\rho_{0000}^{(n_1,n_2)}(t_{n-1})=\begin{cases}\dfrac{(2\lambda)^n\left(\dfrac{n}{2}!\right)^2[\tanh^2(k_0t_0)]^{n/2}}{\sum\limits_{k=n/2}^{\infty}\exp[-2\lambda(2k-n)(t_{n-1}-t_0)][\tanh^2(k_0t_0)]^k\left(\dfrac{k!}{(k-n/2)!}\right)^2}, & n_1=n_2=n/2,\\ 0, \quad n_1\neq n_2. \end{cases} \tag{103}$$

Thus, the measurements can terminate only if the same number of photons has been detected in the two modes. Upon reaching a state with an equal number of counts in the modes after the nth count, the measurement process can continue with probability $1-\rho_{0000}^{(n_1,n_2)}(t_{n-1})$ and terminate with probability $\rho_{0000}^{(n_1,n_2)}(t_{n-1})$. If $n_1\neq n_2$, however, the measurement process must necessarily continue, at least until the same number of counts is recorded in both modes.

The occurrence of the next count in the idler mode is determined by the conditional probability $\tilde{c}_i^{(n_1+1,n_2)}(t_n/t_{n-1})$ that the next count occurs in the idler mode at time t_n if the preceding count occurred at time t_{n-1}, with n_1 counts having been recorded in the idler mode and n_2 counts in the signal mode:

$$\tilde{c}_i^{(n_1+1,n_2)}(t_n/t_{n-1})=2\lambda\left\{\sum_{k=\max\{n_1+1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t_n-t_0)][{}^2(k_0t_0)]^k\frac{k!}{(k-n_1+1)!}\frac{k!}{(k-n_2)!}\right\}$$
$$\times\left\{\sum_{k=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t_{n-1}-t_0)][\tanh^2(k_0t_0)]^k\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}\right\}^{-1}. \tag{104}$$

The analogous probability for the signal mode takes the form

$$\tilde{c}_s^{(n_1,n_2+1)}(t_n/t_{n-1})=2\lambda\left\{\sum_{k=\max\{n_1,n_2+1\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t_n-t_0)][\tanh^2(k_0t_0)]^k\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2+1)!}\right\}$$
$$\times\left\{\sum_{k=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t_{n-1}-t_0)][\tanh^2(k_0t_0)]^k\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}\right\}^{-1}. \tag{105}$$

The probability that the measurement process terminates in one of the modes is given by the probability of observing a given mode in the vacuum state after the next count is recorded, the number of counts in the other mode being arbitrary:

$$\rho_{i00}^{(n_1,n_2)}(t_{n-1})=\begin{cases}\dfrac{\exp[-2\lambda(n_1-n_2)(t_{n-1}-t_0)][\tanh^2(k_0t_0)]^{n_1}\dfrac{(n_1!)^2}{(n_1-n_2)!}}{\sum\limits_{k=n_1}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t_{n-1}-t_0)][\tanh^2(k_0t_0)]^{k}\dfrac{k!}{(k-n_1)!}\dfrac{k!}{(k-n_2)!}}, & n_1\geqslant n_2,\\ 0, \quad n_1<n_2, \end{cases} \tag{106}$$

$$\rho_{s00}^{(n_1,n_2)}(t_{n-1})=\begin{cases}\dfrac{\exp[-2\lambda(n_2-n_1)(t_{n-1}-t_0)][\tanh^2(k_0t_0)]^{n_2}\dfrac{(n_2!)^2}{(n_2-n_1)!}}{\sum\limits_{k=n_2}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t_{n-1}-t_0)][\tanh^2(k_0t_0)]^{k}\dfrac{k!}{(k-n_1)!}\dfrac{k!}{(k-n_2)!}}, & n_2\geqslant n_1,\\ 0, \quad n_2<n_1. \end{cases} \tag{107}$$

Note that if the number of photons detected in one mode is greater than the number detected in the other, there will be a nonvanishing probability that the measurement process terminates in this mode. But, as soon as the number of counts in the other mode exceeds the number of counts in the given mode, the probability of termination drops to zero.

It may seem that there is a contradiction here. For example, there exists a probability that the measurement process terminates in the idler mode when $n_1 \neq n_2$. For example, let the measurements in the idler mode terminate after the n_1th count is recorded ($n_1 \geqslant n_2$). This means that exactly n_1 counts will have been recorded in the idler mode. In the signal mode, however, the measurements will continue, and as long as $n_1 > n_2$, the termination probability will vanish. When the number of recorded counts is the same in the two modes, Eq. (100) tells us that the probability of the measurement process terminating in the signal mode is different from unity, i.e., there is some probability $1-\rho_{s00}^{(n_1,n_2)}\times(t_{n-1})$ that the measurement process continues in the signal mode, and therefore the number of signal counts that can be recorded is greater than the number of idler counts. As a result, the probability that the measurement process terminates in the idler mode vanishes and the measurements must continue.

This contradiction goes away if it is borne in mind that termination of the measurement process in one of the modes means that this mode (the idler mode, in the case described above) is found precisely in the vacuum state and $\rho_{i00}^{(n_1,n_2)}\times(t_{n-1})=1$. In (69), the probability that the measurement process continues in the second mode then consists of just one term, which determines the probability that the next count occurs in the given mode. The second term vanishes, since we have established that a count cannot occur in the first mode. The probability that the measurement process will continue in the second mode is therefore

$$\int_{t_{n-1}}^{\infty}\tilde{c}_s^{(n_1,n_2+1)}(t_n/t_{n-1})dt_n=1-\rho_{s00}^{(n_1,n_2)}(t_{n-1}). \tag{108}$$

For the signal mode it takes the form

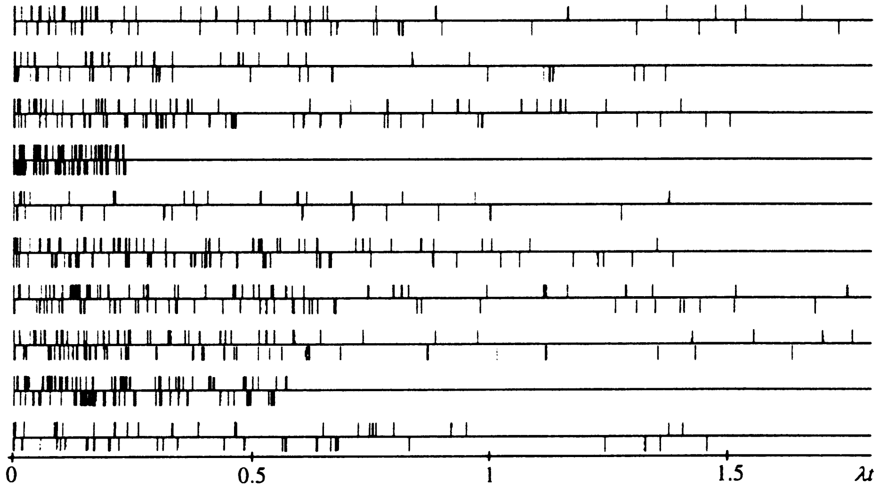

FIG. 9. Random realizations of continuous independent measurements of the number of photons in each mode of a field generated by a parametric frequency converter. The top bars mark the times of the counts in the idler mode and the bottom bars mark the times in the signal mode. The mean number of photons in each mode prior to the measurements is 25.

$$\rho_{s00}^{(n_1,n_2)}(t_{n-1})=\begin{cases}0, & n_1>n_2,\\ 1, & n_2\geqslant n_2.\end{cases} \quad (109)$$

It is easy to see that when the number of recorded countsis-the same in the two modes, the measurement process also terminates in the second (signal) mode.

The foregoing behavior leads to the following conclusion: the information obtained in the course of the measurements on the state of one mode dictates the result that must be obtained with a measurement of the second mode.

The quantities presented above make it possible to numerically model the continuous photodetection of a two-mode field generated by a parametric frequency converter (Fig. 9), as described in Sec. 5.1.

The probability density $P_{[t_0,t)}^{(n_1,n_2)}$ for the n_1th count being recorded in the idler mode and n_2 counts being recorded in the signal mode is

$$P_{[t_0,t)}^{(n_1,n_2)}=\frac{[1-\exp[-2\lambda(t-t_0]]^{(n_1+n_2)}}{n_1!n_2!\mathrm{ch}^2(k_0t_0)} \times\sum_{k=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t-t_0)] \times[\tanh^2(k_0t_0)]^k\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}, \quad (110)$$

and is presented in Fig. 10. For short measurement times $t-t_0\ll 1/\lambda$, the probability of recording a different number of counts in the modes is different from zero (Fig. 10a). This is related to the fact that over such a measurement time interval, not all photons present in the modes can be recorded. Therefore, at the end of the measurement time interval, the modes can end up in states with a different number of photons, and thus [see Eq. (102)] the number of detected photons can also differ. When the measurement time is sufficiently long $t-t_0\gg 1/\lambda$ (Fig. 10b), however, all photons initially present in the modes should be detected during the measurement process, i.e., the same number of counts should be recorded in both modes.

6.2. Temporal evolution of the signal and idler modes during the measurement process

The mth-order moments of the number of photons in each mode after the n_1th count has been recorded in the idler mode and n_2 counts have been recorded in the signal mode take the form

$$\langle n_i^m(t)\rangle^{(n_1,n_2)}=2\lambda\left\{\sum_{k=\max\{n_1+1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t-t_0)][\tanh^2(k_0t_0)]^k(k-n_1)^m\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}\right\} \times\left\{\sum_{k=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t-t_0)][\tanh^2(k_0t_0)]^k\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}\right\}^{-1}, \quad (111)$$

$$\langle n_s^m(t)\rangle^{(n_1,n_2)}=2\lambda\left\{\sum_{k=\max\{n_1,n_2+1\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t-t_0)][\tanh^2(k_0t_0)]^k(k-n_2)^m\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}\right\} \times\left\{\sum_{k=\max\{n_1,n_2\}}^{\infty}\exp[-2\lambda(2k-n_1-n_2)(t-t_0)][\tanh^2(k_0t_0)]^k\frac{k!}{(k-n_1)!}\frac{k!}{(k-n_2)!}\right\}^{-1}. \quad (112)$$

Having determined the first two moments from these expressions, we can investigate the temporal evolution of the mean number of photons in the modes and the Fano factor for each mode during the measurement process. Figure 11 displays the time dependence of the mean number of photons in the signal (curve *1*) and idler (curve *2*) modes. One can see from these plots that the mean number of photons in the modes increases rapidly at the time the counts are recorded, and decreases monotonically in the time intervals between the counts. As we have already mentioned, such jumps are a consequence of the sub-Poissonian character of the field in the modes. For example, Figs. 12a and 12c display the initial distribution of the photons in the modes. Although the mean number of photons in the modes is 25, the probability that the field is in the ground state is high. Suppose a count is recorded in the signal mode at time t_0. The photon distribution in the modes changes abruptly (Fig. 12b, d), and the probability of observing a mode in the vacuum state vanishes. (This happens in the signal mode because a count was recorded and therefore at least one photon was present in the mode. It happens in the idler mode because there is a correlation between the number of photons in the modes.) The redistribution of the probabilities of observing the field in one state or the other results in an abrupt change in the mean number of photons in the modes:

$$\langle n_i(t_0)\rangle^{(0,1)}=51, \quad (113)$$

$$\langle n_s(t_0)\rangle^{(0,1)}=50. \quad (114)$$

The difference of one photon is related to the fact that one photon is removed from the signal mode when a count is recorded.

It turns out that the temporal evolution of the mean number of photons in the modes between counts depends on the number of counts recorded in each mode. To understand this evolution, we employ the following relations:

$$\langle n_i(t)\rangle^{(n_1,n_2)}\big|_{t-t_{n_1+n_2}\to\infty}\to\begin{bmatrix} n_2-n_1, & n_1<n_2, \\ 0, & n_1\geqslant n_2, \end{bmatrix} \tag{115}$$

$$\langle n_s(t)\rangle^{(n_1,n_2)}\big|_{t-t_{n_1+n_2}\to\infty}\to\begin{bmatrix} n_1-n_2, & n_2<n_1, \\ 0, & n_2\geqslant n_1, \end{bmatrix} \tag{116}$$

i.e., if after the n_1th count is recorded in the idler mode and n_2 counts are recorded in the signal mode, the next count was not recorded for a long time, the mode in which the most counts are recorded would approach the vacuum state (signal mode in Fig. 11) and the second mode would approach the state in which the number of photons equals the difference between the counts recorded in the two modes. This is a result of the conservation of photons in the modes:

$$\langle n_i(t)\rangle^{(n_1,n_2)}-\langle n_s(t)\rangle^{(n_1,n_2)}=n_2-n_1. \tag{117}$$

As soon as the next count is recorded, the mean number of photons in the modes increases discontinuously. As described above, the mean number of photons then once again decays monotonically.

In Sec. 6.2, it was determined that during a prolonged measurement of the field generated by a parametric frequency converter, the same number of counts should be recorded in both modes. For long measurement times, the mean number of photons in the modes must therefore be the same and must approach zero, in connection with the fact that all photons initially present in the modes are detected.

The Fano factor given by Eqs. (85), (86), (111), and (112) for the two modes of the field depends on the number of recorded photons. Its temporal evolution is displayed in Fig. 13 (curve *1* for the signal mode, curve *2* for the idler mode). As one can see, whenever a photon is detected, the Fano factor for both modes undergoes a jump, which can be either up or down. This depends on the mode in which the next count occurred: the Fano factor increases for the mode in which the count is recorded, and accordingly decreases for the other mode (Fig. 13a). The existence and magnitude of this jump are related to various factors—the time elapsed after the preceding count was recorded, the number of pho-

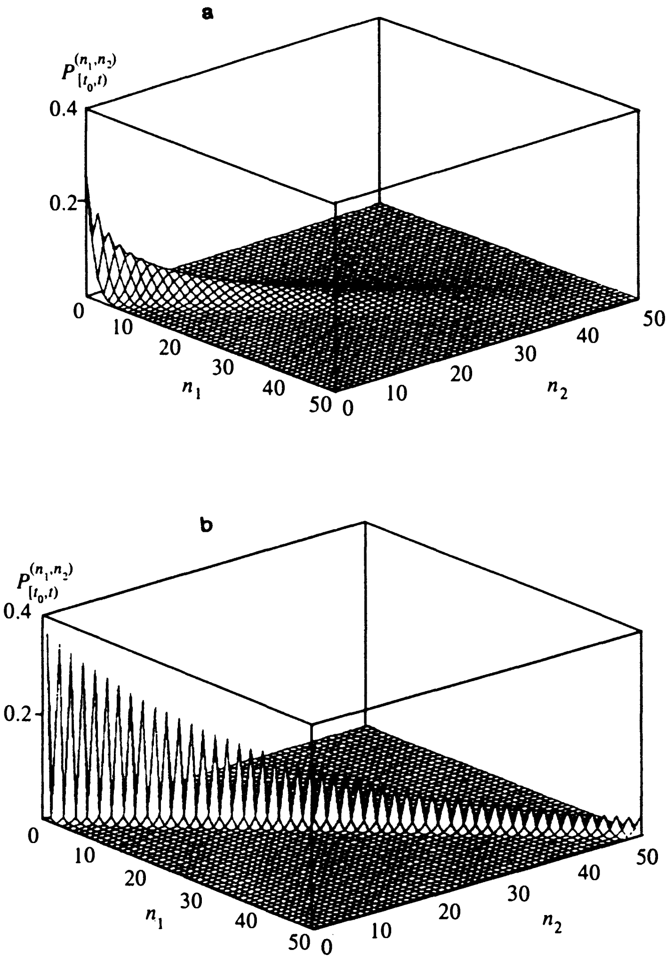

FIG. 10. Distribution $P^{(n_1,n_2)}_{[t_0,t)}$ of the probability that over the time interval $[t_0,t)$ the n_1th count is recorded in the idler mode and n_2 counts are recorded in the signal mode for measurement times (a) $\lambda(t-t_0)\leqslant 1$, and (b) $\lambda(t-t_0)\gg 1$. The mean number of photons in each mode prior to the measurements start is 25.

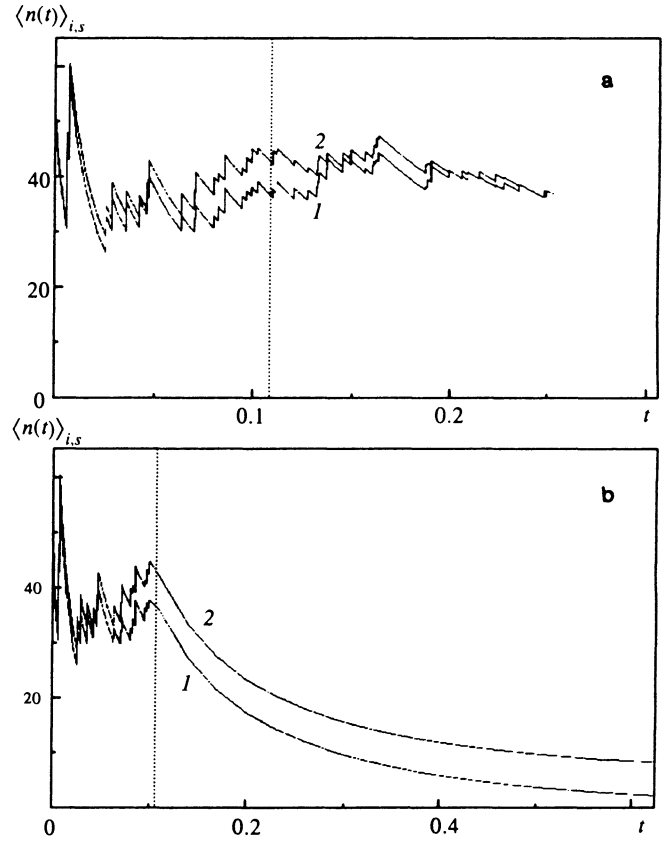

FIG. 11. (a) Temporal evolution of the mean number of photons in the signal mode (*1*) and idler mode (*2*) during independent measurements of the number of photons in each. (b) Temporal evolution of the mean number of photons in the modes when no counts were recorded for either mode during the long time $t-t_{n_1+n_2}\to\infty$. The time $t_{n_1+n_2}$ is denoted in the figure by a vertical dotted line. The mean number of photons in each mode prior to the measurements is 25.

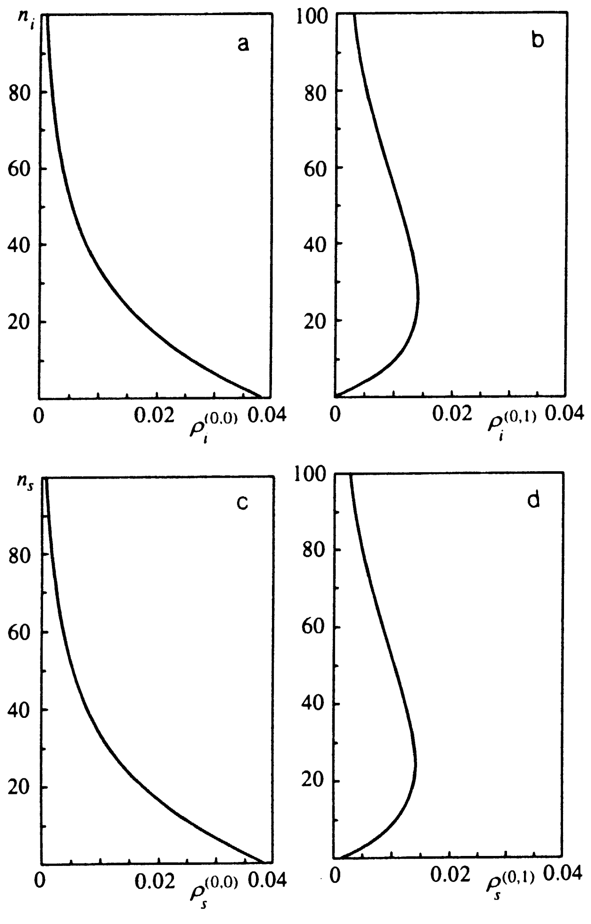

FIG. 12. Distribution of the number of photons in the modes prior to the measurements ((a) in the idler mode and (c) in the signal mode) and after a count is recorded in the signal mode at time t_0 ((b) in the idler mode and (d) in the signal mode). The mean number of photons in each mode prior to the measurements is 25.

tons detected in each mode, and so on. This dependence warrants further study.

The evolution of the Fano factor between counts is governed by the number of photons detected in each mode. For the mode in which the most photons were detected, it is always greater than unity (and in the limit of a long time between successive counts $t_n - t_{n-1} \gg 1/\lambda$, it approaches unity), i.e., the state of the given mode is super-Poissonian. For the mode with fewer recorded counts, the Fano factor evolves to zero for $t_n - t_{n-1} \gg 1/\lambda$, thereby dictating the sub-Poissonian character of the field in this mode. This behavior is shown in Fig. 13b, which displays the temporal evolution of the Fano factor after 13 photons have been detected in the signal mode and 6 photons have been detected in the idler mode, if no counts were recorded for a long time. The Fano factor of the idler mode approaches zero with time. This confirms the result obtained above that the field will be in a Fock state.

As we have established, however, for an infinitely long measurement time, all photons initially present in the field should be measured, and the number of recorded counts in each mode should be the same. Therefore, the temporal evolution of the Fano factor changes during measurements in such a manner that it ultimately tends to unity for both modes, thereby enforcing the super-Poissonian statistics of the initial state of the field in the modes.

The clustering parameter for photons in the signal and idler modes takes the form

$$g^{(n_1,n_2)}=\frac{\sum\limits_{k=\max\{n_1,n_2\}}^{\infty}[\tanh^2(k_0t_0)]^k\,\frac{k!}{(k-n_1)!}\,\frac{k!}{(k-n_2)!}}{(\langle n_i(t_0)\rangle)^{n_1}n_1!(\langle n_s(t_0)\rangle)^{n_2}n_2!}. \qquad (118)$$

As one can see from the plot in Fig. 14, it is greater than 1 for all $n_1, n_2 > 0$, and it increases with the number of counts recorded in each mode. The photons in the different modes of the field generated by a parametric frequency converter thus tend to cluster. This explains the many coincident detection times in the two modes in the simulated random photocount detection process presented in Fig. 9.

6.3. Measurement of the number of photons in a field generated by a parametric frequency converter

Viewing the process described above as a method for measuring the number of photons in the two-mode field generated by a parametric frequency converter, we can put the probability density for exactly n photons being detected at times $t_1, t_2, \dots, t_n$ in the form

$$p_{[t_0,t)}(t_1,t_2\dots,t_n)=\frac{[2\lambda\,\tanh(k_0t_0)]^n n!}{2\cosh^2(k_0t_0)}\times\exp\Big[-2\lambda\Sigma_{j=1}^{n}(t_j-t_0)\Big]\times\left\{\frac{1}{[1+\tanh(k_0t_0)e^{-2\lambda(t-t_0)}]^{n+1}}+\frac{(-1)^n}{[1-\tanh(k_0t_0)e^{-2\lambda(t-t_0)}]^{n+1}}\right\}. \qquad (119)$$

Using Eq. (72), the probability of recording n photons over the time interval $[t_0, t)$ is then

$$P^{(n)}_{[t_0,t)}=\frac{[(1-e^{-2\lambda(t-t_0)})\tanh(k_0t_0)]^n}{2\cosh^2(k_0t_0)}\times\left\{\frac{1}{[1+\tanh(k_0t_0)e^{-2\lambda(t-t_0)}]^{n+1}}+\frac{(-1)^n}{[1-\tanh(k_0t_0)e^{-2\lambda(t-t_0)}]^{n+1}}\right\}. \qquad (120)$$

This distribution is displayed in Fig. 15. It is clearly periodic, which is responsible for the fact that the probability of detecting an even number of photons is different from the probability of detecting an odd number. This derives from the fact that the same number of photons are initially present in the two modes of the field, and therefore the total number of photons in the field is even. The nonvanishing probability of detecting an odd number of photons in a two-mode field at measurement times $\lambda(t-t_0)\leqslant 1$ (Fig. 15a) results from all

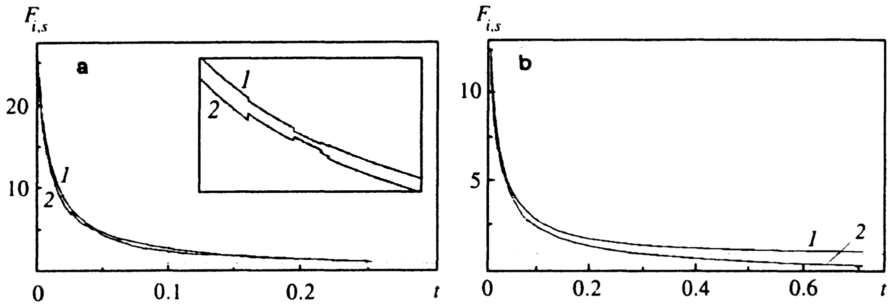

FIG. 13. (a) Temporal evolution of the Fano factor for the signal mode (1) and idler mode (2) during independent measurements of the number of photons in each mode. (b) Temporal evolution of the Fano factor for the case when no counts are recorded in either mode over a long time $t-t_{n_1+n_2}\to\infty$. The time $t_{n_1+n_2}$ is denoted in the figure by the vertical dotted line. The mean number of photons in each mode prior to the measurements is 25.

photons present in the field not having been detected by this time (assuming that the efficiency of the detector equals 1). For long measurement times $\lambda(t-t_0)\gg 1$, this probability approaches zero (Fig. 15b).

The quantities $p_{[t_0,t)}(t_1,t_2,...,t_n)$ and $P^{(n)}_{[t_0,t)}$ completely determine the measurement process for the number of photons in a field. They can be used as a basis for numerical modeling the random process of measuring the number of photons in a two-mode field by the method proposed in Sec. 5.3. However, there is no specific need to solve this problem if modeling results are available for each measurement channel. If we combine these results, which are presented in Fig. 9, taking into account only the temporal sequence of the counts in the modes, i.e., studying the ordering of points irrespective of the mode in which the counts occur, we obtain a numerical realization of the random photocount detection process in the field generated by a arametric frequency converter.

7. PRODUCTION OF A FOCK STATE OF THE FIELD IN ONE MODE BY CONTINUOUSLY MEASURING THE NUMBER OF PHOTONS IN THE OTHER MODE

The characteristics of the continuous measurement process described in Secs. 4 and 6, for one or two modes of a field generated by a parametric frequency converter, lead to the conclusion that it is possible to produce a state with a definite, precisely known number of photons in one of the modes. This conjecture is related to a property of the initial state of the field, in which the initial states of the signal and idler modes are the same and are highly correlated. Because of this property, the information obtained from measurements of the state of one mode directly determines the state of the other, as described above. If exactly n photons were detected in the idler mode of a two-mode field and the mode was found to be in the vacuum state, the signal mode should then also contain precisely n photons, i.e., it should be in a Fock state. We prove this proposition on the basis of the characteristics of the continuous measurement process presented above.

Let n photons be recorded in the idler mode at time t, after which the measurement process terminates. As follows from Eq. (27), this event can occur with probability $\rho^{(n)}_{i00}\times(t)=[1-\exp[-2\lambda(t_n-t_0)]\tanh^2(k_0t)]^{n+1}$. Therefore, if it occurred with certainty, the density matrix of the two-mode field will be, according to Eq. (27),

$$\rho^{(n)}(t_n)=|n\rangle_{ss}\langle n|\otimes|0\rangle_{ii}\langle 0|. \tag{121}$$

It follows then that the signal mode is in a state with a precisely determined number of photons, equal to the number of counts recorded in the idler mode:

$$\rho^{(n)}_s(t_n)=\mathrm{Tr}_i\{\rho^{(n)}(t_n)\}=|n\rangle_{ss}\langle n|. \tag{122}$$

This result is important both from the standpoint that it determines the possibility of producing a Fock state in one mode of the field by independent measurements in the other, and that the proposed method can be interpreted as the nondestructive measurement of the signal mode of a two-mode field.

8. CONCLUSIONS

The basic goal of this work was to investigate theoretically the continuous measurement process in two-mode

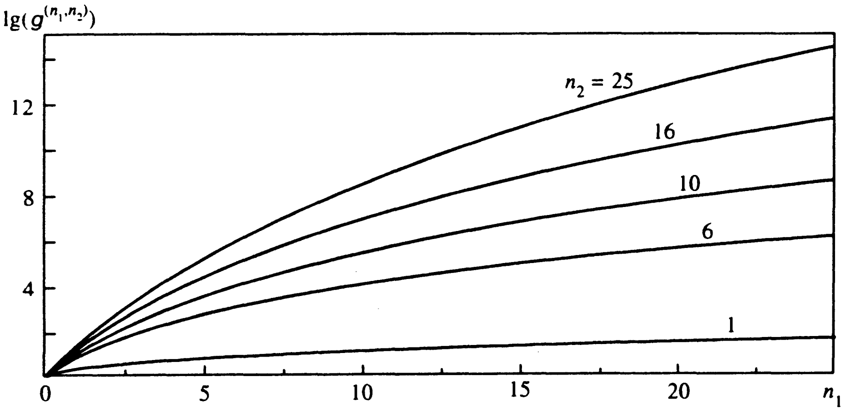

FIG. 14. Clustering parameter for photons of the signal and idler modes as a function of the number n_1 of photons detected in the idler mode with a fixed number n_2 of photons of detected in the signal mode.

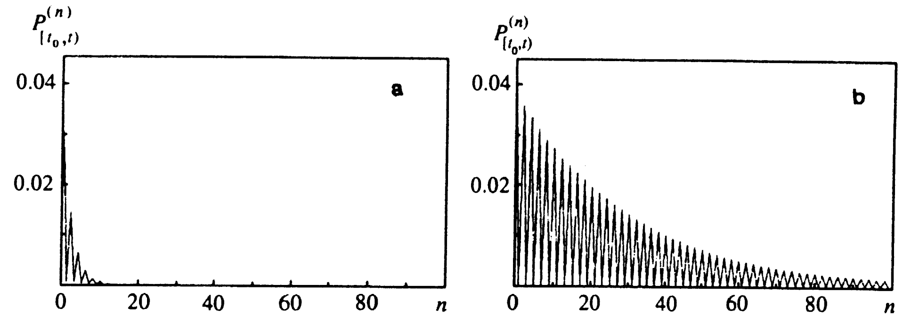

FIG. 15. Distribution $P^{(n)}_{[t_0,t)}$ of the probability that during continuous measurements in each mode over a time interval $[t_0,t)$ exactly n counts are recorded in the two-mode field generated by a parametric frequency converter for measurement times (a) $\lambda(t-t_0)\leqslant 1$, and (b) $\lambda(t-t_0)\gg 1$. The mean number of photons in each mode prior to the measurements is 25.

fields, included correlated fields. Two detection schemes were considered, involving one and two detectors. The photodetection process was modeled in terms of two elementary events: recording a count in one mode or the other, and evolution of the system between successive counts. The two-mode nature of the field and the use of two detectors result in a number of new features compared to the previously developed theory of continuous photodetection of one-mode fields:

1. The evolution operator between counts now allows for the presence of a second mode in the system, and for measurements of both modes of the field it also allows for the second detector.

2. Recording a photocount is determined by the reduction operators that refer to the mode of the field in which the count occurred.

3. The scheme employed previously for modeling the random sequence of photocounts for one-mode fields can be used to investigate continuous measurements in *one* mode of a two-mode field (for example, mode a), since all characteristics of this random process are similar to the characteristics that govern the measurement process for a one-mode field with the initial state $\mathrm{Tr}_b\{\rho(t_0)\}$, and do not depend on the state of mode b, which is not measured. However, the temporal evolution of the second mode of the field and of the two-mode field as a whole can be reconstructed by using the realization of the photocounts in the measured mode and taking the initial correlation between the modes into consideration.

4. To describe simultaneous measurements of both modes of the field, the modeling scheme must be altered somewhat, because the measurement process can now evolve along a new, additional path. An analysis of the temporal evolution of an individual mode shows that, besides the probability of termination of the measurement process ($\rho_{a00}^{(n_1,n_2)}(t_{n-1})$) and the probability of recording the next count in this mode ($\int_{t_{n-1}}^{\infty}\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})\,dt_n$), which were introduced previously in the investigation of one-mode fields, it also characterizes the probability that the measurement process does not terminate in a given mode but that the count occurs in the other mode $[\sum_{m=1}^{\infty}\sum_{j=1}^{\infty} j/(m+j)\,\rho_{mmjj}^{(n_1,n_2)}(t_{n-1})]$ and, accordingly, the state of the first mode changes. The existence of this probability is determined by the existence initially of a quantum correlation between the modes. The sum of the probabilities, $\int_{t_{n-1}}^{\infty}\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})\,dt_n+\sum_{m=1}^{\infty}\sum_{j=1}^{\infty} j/(m+j)\,\rho_{mmjj}^{(n_1,n_2)}\times(t_{n-1})$ is the probability that the measurement process will *continue* in an individual mode. The existence of three (not two, as in measurements of one-mode fields or one mode of a two-mode field) different paths for the development of each mode of the field after the next count has been recorded makes the modeling and interpretation of this random process more omplicated.

The theory proposed for investigation of the continuous photodetection process in a field generated by a parametric frequency converter has made it possible to study explicitly the quantum properties and distinctive features of this process. The main property determining the temporal evolution of each mode and of the field as a whole during photodetection is that the initial intermode correlation of the number of photons is preserved during the measurement process. The result is that the information obtained by photodetection of one mode predetermines either the result of measurement of the other mode (if both modes are measured) or the state to which the second mode reduces (if one mode is measured). For example, the total correlation of the states of the signal mode and the results of measurement of the idler mode suggested the possibility of creating in one mode of the field generated by a parametric frequency converter a state with a precisely determined number of photons (the Fock state) by measuring the number of photons in the other mode. Theoretical calculations confirmed this conjecture. We note that this result is equivalent to the one presented in Ref. 6 for a detector whose efficiency equals 1 and the condition that the parametric generation of the field terminates by time t_0.

In addition, it was shown that in the measurement process for both modes of the field generated by a parametric frequency converter, states should be detected with the same number of photons in each mode (Fig. 10), and therefore with an even number of photons in the field (Fig. 14). In other words, by making independent measurements of the number of photons in the modes whose initial states are thermal and by recording the thermal statistics in each measurement channel, we nonetheless have a nonthermal state of the two-mode field as a whole. The probability distribution for recording the n_1th count in the idler mode and n_2 counts in the signal mode is periodic, and it reflects the fact that the probability that an even number of photons will be recorded in the field differs from the probability that an odd number of photons will be detected. Such an oscillatory form for the distribution $P^{(n_1,n_2)}_{[t_0,t)}$ agrees with the result presented in Ref. 11, and explained from the standpoint of interference in a four-dimensional phase space.

In conclusion, we note that the theory proposed in this paper for the continuous measurement process in two-mode fields not only formalizes a general approach to the investigation of the temporal evolution of two-mode fields during photodetection, it also provides a foundation for further elaboration of the theory of continuous measurements—the analysis of continuous measurements of multimode fields.

Financial support for this work was provided, in part, by the International Science Foundation (Grant No. RWJ000).

APPENDIX 1

The conditional elementary probability density for recording the next count in the measurement process for a two-mode field can be determined as follows [Eqs. (61)–(63)]:

$$\tilde{c}^{(n)}(t_n/t_{n-1})=\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})+\tilde{c}_b^{(n_1,n_2+1)}(t_n/t_{n-1})=\mathrm{Tr}\{J_aS_{t_n t_{n-1}}\rho^{(n_1,n_2)}(t_{n-1})\}+\mathrm{Tr}\{J_bS_{t_n,t_{n-1}}\rho^{(n_1,n_2)}(t_{n-1})\}. \tag{A1.1}$$

We examine in greater detail the different representations of the expression $\mathrm{Tr}\{S_{t_n,t_{n-1}}x\}$:

$$\begin{aligned}\mathrm{Tr}\{S_{t_n,t_{n-1}}x\}&=\mathrm{Tr}\{\exp[-i(\omega_a-i\lambda)a^+a(t_n-t_{n-1})]\exp[-i(\omega_b-i\lambda)b^+b(t_n-t_{n-1})]\\&\quad\times\exp[i(\omega_a+i\lambda)a^+a(t_n-t_{n-1})]\exp[i(\omega_b+i\lambda)b^+b(t_n-t_{n-1})]\}\\&=\mathrm{Tr}\{\exp[-2\lambda a^+a(t_n-t_{n-1})]\exp[-2\lambda b^+b(t_n-t_{n-1})]x\}\\&=\mathrm{Tr}\left\{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!m!}(\exp[-2\lambda(t_n-t_{n-1})]-1)^{k+m}a^{+k}a^kb^{+k}b^kx\right\}\\&=\mathrm{Tr}\left\{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!m!}(\exp[-2\lambda(t_n-t_{n-1})]-1)^{k+m}a^kb^kxb^{+k}a^{+k}\right\}.\end{aligned}$$

Using the definition of the reduction operators (50)–(51), this relation can be written in the form

$$\mathrm{Tr}\{S_{t_n,t_{n-1}}x\}=\mathrm{Tr}\left\{\exp\left[\frac{J_a}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right]\exp\left(\frac{J_b}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right)x\right\}. \tag{A1.2}$$

Then

$$\begin{aligned}\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})&=\mathrm{Tr}\{J_aS_{t_n,t_{n-1}}\rho^{(n_1,n_2)}(t_{n-1})\}=\exp[-2\lambda(t_n-t_{n-1})]\mathrm{Tr}\{S_{t_n,t_{n-1}}J_a\rho^{(n_1,n_2)(t_{n-1})}\}\\&=\exp[2\lambda(t_n-t_{n-1})]\mathrm{Tr}\left\{\exp\left[\frac{J_a}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right]\right.\\&\quad\left.\times\exp\left[\frac{J_b}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})-1)]J_a\rho^{(n_1,n_2)}(t_{n-1})\right.\right\}.\end{aligned} \tag{A1.3}$$

Similarly,

$$\begin{aligned}\tilde{c}_b^{(n_1,n_2+1)}(t_n/t_{n-1})&=\exp[-2\lambda(t_n-t_{n-1})]\mathrm{Tr}\left\{\exp\left(\frac{J_a}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right)\right.\\&\quad\left.\times\exp\left[\frac{J_b}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right]J_b\rho^{(n_1,n_2)}(t_{n-1})\right\}.\end{aligned} \tag{A1.4}$$

We substitute the relations obtained above into Eq. (A1.1):

$$\begin{aligned}\tilde{c}^{(n)}(t_n/t_{n-1})&=\exp[-2\lambda(t_n-t_{n-1})]\mathrm{Tr}\left\{\exp\left[\frac{J_a}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right]\exp\left[\frac{J_b}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right](J_a\right.\\&\quad\left.+J_b)\rho^{(n_1,n_2)}(t_{n-1})\right\}\\&=-\frac{d}{dt_n}\mathrm{Tr}\left\{\exp\left[\frac{J_a}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right]\exp\left[\frac{J_b}{2\lambda}(\exp[-2\lambda(t_n-t_{n-1})]-1)\right]\rho^{(n_1,n_2)}(t_{n-1})\right\}.\end{aligned} \tag{A1.5}$$

Hence,

$$\tilde{c}^{(n)}(t_n/t_{n-1})=-\frac{d}{dt_n}\operatorname{Tr}\{S_{t_n,t_{n-1}}\rho^{(n_1,n_2)}(t_{n-1})\}. \tag{A1.6}$$

The probability that the measurement process continues after the nth count has been recorded, defined as $\int_{t_{n-1}}^{\infty}\tilde{c}^{(n)}\times(t_n/t_{n-1})\,dt_n$, can then be easily transformed:

$$\int_{t_{n-1}}^{\infty}\tilde{c}^{(n)}(t_n/t_{n-1})dt_n=-\int_{t_{n-1}}^{\infty}\frac{d}{dt_n}\operatorname{Tr}\{S_{t_n,t_{n-1}}\rho^{(n_1,n_2)}(t_{n-1})\}dt_n$$

$$=-\int_{t_{n-1}}^{\infty}\frac{d}{dt_n}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}{}_b\langle m|_a\langle j|S_{t_n,t_{n-1}}\rho^{(n_1,n_2)}(t_{n-1})|j\rangle_a|m\rangle_b dt_n=\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\exp[-2\lambda(m+j)(t_n$$

$$-t_{n-1})]\rho_{mmjj}^{(n_1,n_2)}(t_{n-1})|_{t_{n-1}}^{\infty}=\sum_{m=0}^{\infty}\sum_{\substack{j=0\\ m\neq j\neq 0}}^{\infty}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1}). \tag{A1.7}$$

Using the completeness property

$$\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1})=1. \tag{A1.8}$$

The expression (A1.7) can be written in the form

$$\int_{t_{n-1}}^{\infty}\tilde{c}^{(n)}(t_n/t_{n-1})=1-\rho_{0000}^{(n_1,n_2)}(t_{n-1}), \tag{A1.9}$$

as was to be proved.

APPENDIX 2

The probability that the next count is recorded in mode a is

$$\int_{t_{n-1}}^{\infty}\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})dt_n=\int_{t_{n-1}}^{\infty}\operatorname{Tr}\{J_aS_{t_n,t_{n-1}}\rho^{(n_1,n_2)}(t_{n-1})\}dt_n. \tag{A2.1}$$

Substituting into this relation the expressions (16) and (50) for the evolution and reduction operators and carrying out the cyclic permutation under the trace, we obtain

$$\int_{t_{n-1}}^{\infty}\tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})dt_n=2\lambda\int_{t_{n-1}}^{\infty}\operatorname{Tr}\Big\{\exp[-2\lambda a^+a(t_n-t_{n-1})]\exp[-2\lambda b^+b(t_n-t_{n-1}]a^+a\rho^{(n_1,n_2)}(t_{n-1})\}dt_n$$

$$=2\lambda\int_{t_{n-1}}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}{}_b\langle m|_a\langle j|\exp[-2\lambda(m+j)(t_n-t_{n-1})]m\rho^{(n_1,n_2)}(t_{n-1})|j\rangle_a|m\rangle_b dt_n$$

$$=\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}\frac{m}{m+j}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1})=\sum_{m=1}^{\infty}\rho_{mm00}^{(n_1,n_2)}(t_{n-1})+\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{m}{m+j}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1})$$

$$=\sum_{m=1}^{\infty}\rho_{mm00}^{(n_1,n_2)}(t_{n-1})+\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1})-\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{j}{m+j}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1}). \tag{A2.2}$$

Using the completeness property (A1.8), the second term of this expression can be rewritten:

$$\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1})=1-\rho_{0000}^{(n_1,n_2)}(t_{n-1})-\sum_{m=1}^{\infty}\rho_{mm00}^{(n_1,n_2)}(t_{n-1})-\sum_{j=1}^{\infty}\rho_{00jj}^{(n_1,n_2)}(t_{n-1}). \tag{A2.3}$$

Thus

$$\int_{t_{n-1}}^{\infty} \tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})dt_n = 1-\rho_{0000}^{(n_1,n_2)}(t_{n-1}) - \sum_{j=1}^{\infty} \rho_{00jj}^{(n_1,n_2)}(t_{n-1}) - \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} \frac{j}{m+j}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1}) = 1-\rho_{a00}^{(n_1,n_2)}(t_{n-1})$$

$$-\sum_{m=0}^{\infty}\sum_{j=1}^{\infty} \frac{j}{m+j}\rho_{mm\,jj}^{(n_1,n_2)}(t_{n-1}). \tag{A2.4}$$

Hence we find that the probability that the measurement process continues in mode a after the n_1th count is recorded in mode a and n_2 counts are recorded in mode b is

$$1-\rho_{0000}^{(n_1,n_2)}(t_{n-1}) = \int_{t_{n-1}}^{\infty} \tilde{c}_a^{(n_1+1,n_2)}(t_n/t_{n-1})dt_n + \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} \frac{j}{m+j}\rho_{mmjj}^{(n_1,n_2)}(t_{n-1}).$$

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Translated by M. E. Alferieff