Thermodynamic properties of dark solitons

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Localized solutions in the form of a reduction in the field strength superposed on a nonzero background (dark solitons) are considered for nonintegrable nonlinear equations with a repulsive potential. Like ordinary solitons such solutions have attractor properties. These become evident in the evolution of an arbitrary initial distribution. A method of determining the analytical form of the equilibrium distribution consisting of linear waves plus a dark soliton is found using the nonlinear Schrödinger equation and the nonlinear Higgs equation as examples. A sufficient condition for the existence of such an equilibrium is that the dark solitons be nonlinearly stable in the sense of Lyapunov.

1. INTRODUCTION

Particular solutions in the form of dark solitons are of importance in connection with a variety of nonlinear equations having repulsive potentials. Specifically, it is well known[1] that in integrable nonlinear equations the asymptotic form of an arbitrary initial distribution contains only linear waves and solitons. The number and amplitude of the waves and solitons are completely determined by this initial distribution. When the system is nonintegrable, dark solitons (local reductions in the wave intensity) can be in thermodynamic equilibrium with linear waves, similar to ordinary solitons of nonintegrable equations with attractor potentials.[2,3] This means that in the course of its evolution the system forgets its initial distribution, and the asymptotic form can contain only solitons precisely balanced by waves in thermodynamic equilibrium with them. This means that the problem of finding the equilibrium distribution of waves on dark solitons, which is treated in the present work, is not meaningless. A sufficient condition for the existence of such an equilibrium is the nonlinear stability (in the sense of Lyapunov) of the solitons.[4] In nonintegrable systems like the Korteweg–de Vries (KdV) and nonlinear Schrödinger (NLS) equations stable solitons may also exhibit attractor properties. That is, it is thermodynamically favorable (in the sense of leading to an increase in entropy) for waves and solitons to merge and for the amplitudes of the latter to increase.[2,3] The behavior of dark solitons with small-amplitude modulation is similar to that of ordinary solitons:[5] it is thermodynamically favorable for the modulation amplitude of the dark solitons to increase as a result of emission of linear waves. In contrast to ordinary solitons, the increase in the modulation amplitude of dark solitons is bounded, if only because the modulation amplitude itself is bounded. This means that dark solitons with a finite modulation amplitude are the most favored thermodynamically, rather than those with the largest possible modulation amplitude.

This qualitative picture for the behavior of dark solitons is illustrated in the present work for the NLS and Higgs (or Klein–Gordon) equations with quadratic repulsive potentials.

The NLS equation with such a potential (a cubic nonlinearity) can be completely integrated.[1] It follows that, e.g., solutions can be found in the form of linear waves with arbitrary modulation amplitude superposed on a dark soliton that is stable. Because it is completely integrable a system consisting of a dark soliton plus waves cannot reach thermodynamic equilibrium. Such an equilibrium becomes possible only when the potential or the dispersion of the integrable NLS equation changes, as a result of which an infinite number of constants of motion vanish. If these corrections are small, then the analytical form of the usual constants of motion of the dark solitons in waves obtained for the integrable NLS equation changes little, which enables one to find explicitly the thermodynamic equilibrium of linear waves on a dark soliton with an arbitrary modulation amplitude.

The Higgs equation with a cubic nonlinearity is nonintegrable. Dark solitons with the maximum modulation amplitude ("black" solitons) are unstable even in the linear approximation. The stability boundary for dark solitons can be estimated using the Makhan'kov criterion[6] (the modified Vakhitov–Kolokolov condition[7]). Thermodynamic equilibrium with linear waves is therefore possible only for dark solitons with a modulation amplitude less than some limiting value. The Higgs equation linearized about such dark solitons is identical with the linearized equations obtained from analyzing the transverse instability of dark solitons of the NLS equation in Ref. 8. Consequently, even in this case it is possible to find expressions for the thermodynamic equilibrium distribution of waves on dark solitons with a finite modulation amplitude.

2. THERMODYNAMIC PROPERTIES OF DARK SOLITONS OF THE NLS EQUATION WITH A REPULSIVE POTENTIAL

With the repulsive potential the NLS equation

$$i\Psi_t' + \Psi_{xx}'' - U(|\Psi|^2)\Psi = 0, \qquad (1)$$

describes the propagation of modulated ion acoustic waves ($U \simeq |\Psi|^2$; see Ref. 9), nonlinear waves in waveguides with a "normal" dependence of the index of refraction on the light intensity [$U \simeq |\Psi|^2 + \alpha|\Psi|^4$ (Ref. 10), $U \simeq |\Psi|^2$ (Ref. 11)], and diffraction in space of a laser beam passing through a

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diffraction grating and through scattering material $[U \simeq |\Psi|^2$ (Ref. 12)]. In the latter case x is the spatial coordinate in the transverse cross section of the beam and t is the spatial coordinate parallel to the beam.

By writing $\Psi = \phi \exp(-i\omega_0 t)$ we can reduce the NLS with a quadratic potential $U \simeq |\Psi|^2$ to the standard form

$$i\phi_t' + \phi_{xx}'' + (\omega_0 - |\phi|^2)\phi = 0, \tag{2}$$

which has dark solitons with velocity v of the form[8]

$$\phi_0(x,t) = v/\sqrt{2} - i\sqrt{\omega_0 - v^2/2}\tanh\left(\sqrt{\omega_0/2 - v^2/4}\zeta\right),$$
$$\zeta \equiv x - vt, \tag{3}$$

satisfying the boundary conditions in the limits $x \to \pm\infty$ given by 1) $|\Psi|^2 \to \omega_0$; 2) $\Psi_x' \to 0$. Linearization of the NLS equation about these dark solitons and nondimensionalization according to

$$\phi = \phi_0 + (R + iI), \quad R = U\exp(-i\omega\tau) + \text{c.c.},$$
$$I = V\exp(-i\omega\tau) + \text{c.c.},$$
$$\mu = \sqrt{\omega_0/2 - v^2/4}, \quad \tau = (\omega_0/2 - v^2/4)t$$

yields the following system of linear equations:

$$R_\tau' - \frac{2}{\sqrt{\alpha-1}}R_\mu' + \frac{4\tanh(\mu)}{\sqrt{\alpha-1}}R + I_{\mu\mu}'' + 2\left(\frac{3}{\cosh^2\mu} - 2\right)I = 0,$$
$$I_\tau' - \frac{2}{\sqrt{\alpha-1}}I_\mu' - \frac{4\tanh\mu}{\sqrt{\alpha-1}}I - R_{\mu\mu}'' - 2\left(\frac{1}{\cosh^2\mu} - \frac{2}{\alpha-1}\right)R = 0, \tag{4}$$

which have an exact solution in the form of linear waves on a dark soliton with an arbitrary modulation amplitude $\alpha = 2\omega_0/v^2$ (Ref. 8):

$$U_k = |\phi_k^\pm|\left(1 + 2\frac{i}{k}\tanh\mu,\right.$$
$$V_k = \frac{|\phi_k^\pm|}{\omega_k^\pm}\left[2k\tanh\mu - i\left(k^2 - \frac{2\omega_k^\pm}{k\sqrt{\alpha-1}} + \frac{2}{\cosh^2\mu}\right)\right] \tag{5}$$
$$\omega_k^\pm = -\frac{2k}{\sqrt{\alpha-1}} \pm k\sqrt{k^2 + 4\alpha/(\alpha-1)}.$$

To find a thermodynamic equilibrium distribution for these waves on a dark soliton we use the Lyapunov functional.[4] The dark soliton $\Psi_0 = \phi_0\exp(-i\omega_0 t)$ is an extremal of the functional formed from ordinary (belonging to the nonintegrable NLS equation) constants of motion:

$$L = E - \omega_0 N - \frac{I}{2}vP \equiv \int |\Psi_x' - \frac{i}{2}v\Psi|^2 + \frac{1}{2}|\Psi|^4 - (\omega_0 - v^2/4)|\Psi|^2 dx,$$

$$E = \int |\Psi_x'|^2 + \frac{1}{2}|\Psi|^4 dx, \quad N = \int |\Psi|^2 dx, \tag{6}$$
$$P = i\int \{\Psi\overset{*}{\Psi}_x' - \overset{*}{\Psi}\Psi_x'\}\, dx,$$

where E, N, and P are the constants of energy, wave number, and momentum, respectively.[1] The phase trajectories of the system consisting of a dark soliton plus waves are located on a hypersurface of the Lyapunov functional L close to the extremal point of L_0, corresponding to a dark soliton only when L has a definite sign (so that the dark soliton is stable). The second variation of the functional L describing linear waves takes the form

$$\frac{1}{2}\delta^2 L = \int \left\{\left|\delta\Psi_x' - \frac{i}{2}v\delta\Psi\right|^2 + \frac{1}{2}(\delta|\Psi|^2)^2 + (|\Psi_0|^2 - \Omega)|\delta\Psi|^2\right\}dx, \quad \Omega \equiv \omega_0 + \frac{v^2}{4}. \tag{7}$$

Hence in order to find the required thermodynamic equilibrium distribution we must substitute into this equation the expressions found above for the linear waves and note that in thermodynamic equilibrium $\frac{1}{2}\delta^2 L$ has the physical meaning of a temperature T. This substitution yields the following equilibrium values of $|\phi_K^\pm|^2$:

$$|\phi_k^\pm|^2 \simeq \frac{T}{\omega_0}\left(1 + \frac{k^2}{2\omega_0} \pm \frac{1}{\sqrt{\alpha}}\sqrt{1 + \frac{k^2}{2\omega_0}}\right)^{-1}. \tag{8}$$

As is to be expected, for $\alpha = 1$ this distribution is the same as that found previously[4] for the distribution of waves on a small-amplitude dark soliton, propagating with nonzero velocity $v = \sqrt{2\omega_0}$. This explains the asymmetry of the distribution ϕ_k^+ for waves moving in the direction of propagation of the dark soliton and the distribution ϕ_k^- for waves moving in the opposite direction. The values $\alpha \to \infty$ correspond to a wave distribution on a black soliton at rest (a dark soliton with the largest possible modulation amplitude). It is clear that in this case the asymmetry of the distributions disappears.

Note that this thermodynamic equilibrium distribution is not a generalized Rayleigh–Jeans distribution ($\simeq T/(\omega_k - \omega_0 - kv)$), since the integrals E_W, Q_W, and P_W of linear waves on a dark soliton are determined by the background ω_0 rather than the modulation amplitude.

The "chemical potentials" in the Lyapunov potential, which plays the role thermodynamically of a free energy, are not arbitrary. It is only with this choice of the coefficients that an equilibrium between the dark soliton and the wave is possible.

3. Thermodynamic properties of dark solitons of the Higgs equation

Consider the one-dimensional Higgs equation with a quadratic potential:

$$-\Psi_{tt}'' + \Psi_{xx}'' + (1 - |\Psi|^2)\Psi = 0, \tag{9}$$

which differs from the NLS equation considered above only in having a higher order derivative with respect to time. Note also that the substitution $t \to iy$ converts this equation to a two-dimensional NLS equation with a repulsive potential. It is not surprising that this equation has solutions in the form of dark solitons, analytically indistinguishable from the dark solitons of the NLS equation. Specifically, by substituting $\Psi = \phi \exp(-i\omega_0 t)$ we can convert the Higgs equation into

$$-\phi''_{tt} + 2i\omega_0\phi'_t + \phi''_{xx} + (1+\omega_0^2 - |\phi|^2)\phi = 0, \tag{10}$$

which has dark solitons moving with velocity V:

$$\phi_0(x,t) = \nu/\sqrt{2} - i\sqrt{1+\omega_0^2 - \nu^2/2}\,\tanh\left(\sqrt{(1+\omega_0^2)/2 - \nu^2/4}\,\zeta\right),$$

$$\zeta = \frac{x - Vt}{\sqrt{1-V^2}}, \quad \nu = \frac{2\omega_0 V}{\sqrt{1-V^2}}, \tag{11}$$

which satisfy the boundary conditions in the limit $x \to \pm\infty$ given by 1) $|\Psi| \to \omega_0$ and 2) $\Psi'_x \to 0$. Since the Higgs equation is invariant under a Lorentz transformation, the second condition uniquely determines the velocity of the dark soliton, just as in the case of the NLS equation.

The Higgs equation has four conventional constants of motion:[13] the energy E (invariance with respect to t), the charge Q (invariance with respect to phase rotation), the momentum P (invariance with respect to x), and the rotation M (invariance with respect to Lorentz transformations). Of these only the first three can be used to construct a Lyapunov functional L with definite sign:

$$L = E + \omega_0 Q + VP \equiv \int |\Psi'_t + i\omega_0\Psi + V\Psi'_x|^2 + \frac{1}{2}|\Psi|^4 - (1+\omega_0^2)|\Psi|^2 + (1-V^2)|\Psi'_x|^2 - i\omega_0(\Psi\overset{*}{\Psi}{}'_x - \overset{*}{\Psi}\Psi'_x)dx,$$

$$E = \int \{|\Psi'_t|^2 + |\Psi'_x|^2 + \tfrac{1}{2}|\Psi|^4 - |\Psi|^2 dx,$$

$$Q = i\int \{\Psi^*\Psi'_t - \Psi\Psi^{*\prime}_t\}dx, \tag{12}$$

$$P = \int \Psi^{*\prime}_t\Psi'_x - \Psi'_t\Psi^{*\prime}_x dx.$$

The extremal of this functional is given by the dark solitons considered above. To determine whether a thermodynamic equilibrium distribution is possible for a system consisting of a dark soliton plus waves and its form it is therefore necessary only to find the solutions of the Higgs equation linearized about a dark soliton and to substitute them into the expression for the second variation $\delta^2 L$ of the Lyapunov functional,

$$\frac{1}{2}\delta^2 L = \int \{|\delta\phi'_t + V\delta\phi'_x|^2 + |\delta\phi'_x|^2(1-V^2) + (|\phi_0|^2 - (1+\omega_0^2))|\delta\phi|^2 + \tfrac{1}{2}(\phi_0\delta\overset{*}{\phi} + \overset{*}{\phi}_0\delta\phi)^2 - i\omega_0 V(\delta\phi\delta\overset{*}{\phi}{}'_x - \delta\overset{*}{\phi}\delta\phi'_x)\}dx. \tag{13}$$

Aside from notation and the normalization, the linearized system of equations is exactly the same as the system considered in Ref. 8, derived in order to determine the transverse instability of dark solitons of the NLS equation:

$$-\lambda^2 U + \tilde{L}U + \frac{2i\omega_0\lambda}{\sqrt{1-V^2}}\sigma U = 0, \quad \delta\phi = R + iI,$$

$$R, I \simeq \exp(\lambda y), \quad y = \frac{t - Vx}{\sqrt{1-V^2}}, \tag{14}$$

$$\zeta = \frac{x - Vt}{\sqrt{1-V^2}}, \quad U = \begin{pmatrix} R \\ I \end{pmatrix}, \quad \sigma = \begin{pmatrix} o & i \\ -i & 0 \end{pmatrix},$$

where the operator $\tilde{L}(\zeta)$ was introduced in Ref. 8. It was in fact shown in Ref. 8 using perturbation theory, starting with dark solitons having some finite modulation amplitude, that this system has solutions with real values of λ, the linear growth rate for the instability of the dark solitons.

This can be verified most easily for a black soliton ($\phi_0 = -i\sqrt{1+\omega_0^2}\tanh(\sqrt{(1+\omega_0^2)/2x})$). The operator $\tilde{L}$ in this case becomes diagonal.[8] After nondimensionalizing ($\zeta = \sqrt{(1+\omega_0^2)/2x}$, $\gamma = \lambda/\sqrt{1+\omega_0^2}$) we find

$$-\gamma^2 R - \frac{2\omega_0\gamma}{\sqrt{1+\omega_0^2}} I - \hat{L}_0 R = 0, \quad \hat{L}_0 = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{\cosh^2 x},$$

$$-\gamma^2 I + \frac{2\omega_0\gamma}{\sqrt{1+\omega_0^2}} R - \hat{L}_1 I = 0, \tag{15}$$

$$\hat{L}_1 = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{3}{\cosh^2 x} + 2.$$

After eliminating R we find in analogy with Ref. 8

$$\frac{4\omega_0^2\gamma^2}{1+\omega_0^2} I + (\hat{L}_0 + \gamma^2)(\hat{L}_1 + \gamma^2) I = 0. \tag{16}$$

In contrast to $\hat{L}_1$, the operator $\hat{L}_0$ does not have a definite sign, so that for moderately large ω_0 solutions of this equation can be found for which γ is real. For example, for $\omega_0 = 0$ there is an exact solution $R = 1/\cosh(x/\sqrt{2})$, $\gamma = 1/\sqrt{2}$ (see Ref. 14). This means that for not too large ω_0 (see below for an estimate) a black soliton of the Higgs equation is linearly unstable and cannot be an attractor. Using the Makhan'kov criterion[6] (the modified Vakhitov–Kolokolov condition[7]) we can estimate the critical modulation amplitude of a dark soliton, above which it becomes unstable. Specifically, the total charge $Q(\alpha)$ of a dark soliton as a function of the modulation amplitude $\alpha = 2(1+\omega_0^2)/\nu^2$ ceases to be monotonic for $\alpha^* = 2(1+\omega_0^2)/(1-\omega_0^2)$. Hence we see that in all probability for $\omega_0 \geqslant 1$ a dark soliton also becomes stable, which was already noted qualitatively in Ref. 14.

As in Ref. 4, we can analytically estimate the thermodynamic distribution of waves on a stable dark soliton with a finite modulation amplitude. For this we substitute into $\delta^2 L$ the asymptotic expression ϕ_0 for linear waves on a dark soliton (to be precise, superposed on a constant background $\sqrt{1+\omega_0^2}$):

$$\delta\phi=|\phi_k^{\pm}|\left[\cos(\omega_k^{\pm}t-kx)+i\,\frac{2\omega_0\omega_k^{\pm}}{\omega_k^{\pm 2}-k^2}\right.$$

$$\left.\times\sin(\omega_k^{\pm}t-kx)\right]\frac{\phi_0}{|\phi_0|},$$

$$\omega_k^{\pm 2}=(1+k^2+3\omega_0^2)\left[1\pm\sqrt{1-k^2\,\frac{k^2+2(1+\omega_0^2)}{(1+k^2+3\omega_0^2)^2}}\right]. \tag{17}$$

This substitution yields the desired thermodynamic equilibrium distribution:

$$|\phi_k^{\pm}|^2\simeq\frac{(\omega_k^{\pm}-kV)\omega_k^{\pm}}{(\omega_k^{\pm 2}-k^2)k}\,[(\omega_k^{\pm 2}-k^2)+4k^2\omega_0^2]. \tag{18}$$

This expression has no singularities, since the wave phase velocity is larger than the velocity of the dark soliton. For solitons with a small modulation amplitude in the limit $(k,\omega_0,\omega_0/k)\rightarrow 0$ we find

$$|\phi_k^+|^2\simeq T/k,\quad |\phi_k^-|^2\simeq Tk. \tag{19}$$

As in the case of the NLS equation, the distributions become asymmetrical when dark solitons with a small modulation amplitude move with nonzero velocity $V\simeq 1$.

4. CONCLUSION

We have presented a way of determining the thermodynamic equilibrium distribution of waves on dark solitons arising asymptotically as a result of the evolution of nonlinear nonintegrable equations (such as the NLS equation and the Higgs equation). It is in principal applicable for multidimensional systems as well. These distributions are not a modified form of the Rayleigh–Jeans distribution $[\simeq T/(\omega_k-\omega_0-kV)]$, since they contain information about the dark soliton itself (equilibrium is possible only when the waves and solitons have equal "chemical potentials" for a specified modulation amplitude). We have also shown that it is not always thermodynamically favorable for the modulation amplitude of the dark soliton to increase: dark solitons can occur with a finite modulation amplitude less than the maximum value, just as in media with a saturating nonlinearity. In contrast to the collapse of unstable solitons, instability of dark solitons with a large modulation amplitude probably only decreases the modulation amplitude (with absorption of waves and/or an increase in the number of dark solitons) until stability is attained.

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