Calculation of the average intensity of scalar waves in a randomly inhomogeneous medium

A. G. Fokin

Moscow Institute of Electronic Technology, 108498 Moscow, Russia

(Submitted 3 November 1994)
Zh. Éksp. Teor. Fiz. **107**, 1122–1134 (April 1995)

The problem of the propagation of scalar waves in an unbounded nonabsorbent randomly inhomogeneous medium (RIM) is solved in the ladder approximation. The evaluation of the damping coefficient (scattering index) $\bar{\gamma}_e$ of the mean field intensity reduces to the solution of an equation similar to the dispersion relation for the mean field. In the long- and shortwave limits, which allow the macroscopic (resulting from the nonuniformity of the medium) spatial dispersion to be disregarded, expressions are found for $\bar{\gamma}_e$ for an RIM that can be described by a normalized binary correlation function $\varphi(\mathbf{r}_1,\mathbf{r}_2)=\exp(-\rho)$, where we have written $a\rho\equiv|\mathbf{r}_1-\mathbf{r}_2|$ and a is the spatial scale of the correlations (three-dimensional and one-dimensional cases are considered). For a one-dimensional RIM with delta-correlated nonuniformities ("white noise") a similar calculation is carried out for the cases of: a) a point source; b) approximation of the coherence function as a product of two plane monochromatic waves.

1. INTRODUCTION

The persistent interest among scientists[1–30] in the problem of wave propagation in a randomly inhomogeneous medium (RIM) is due to the essentially inexhaustible variety of problems which can be formulated in terms of it. The nature of the applications associated with many of them enhances this interest and provides an encouragement to overcome the substantial difficulties that stand in the way of the desired results. It often proves possible to reduce the solution of a wave equation for the original (in general tensor) field to the solution of the scalar wave equation corresponding to it in some approximation.[1,2]

The treatment of waves in an RIM requires that such effects as macroscopic spatial dispersion and attenuation of the wave due to scattering on the inhomogeneities, which are directly related to the structure, be taken into account. To obtain information about the basic dynamical properties of an RIM it turns out to be enough to know the lowest moments of the field in question. However, the rate of attenuation of the mean field gives an exaggerated significance to the energy damping rate.[26,29] This is because the phase fluctuations of the wave, which do not effect the intensity, contribute to the attenuation of the mean field.

In what follows the problem of scalar wave propagation in an unbounded nonabsorbent RIM is considered. To calculate the scattering index a method is developed based on the introduction of the dispersion relation for the mean intensity, which assumes a form similar to the dispersion relation for the mean field in the approximation in which the coherence function B is represented as the product of two plane monochromatic waves. This enables us to use results obtained in the Bourret approximation[16,31] to calculate the corresponding quantities in the ladder approximation.[32]

The proposed method is applied to solve the problem in two cases: a) one- and three-dimensional RIM described by a normalized binary correlation function

$$\varphi(\mathbf{r}_1,\mathbf{r}_2)=\exp(-\rho), \tag{1.1}$$

where we write $a\boldsymbol{\rho}\equiv\mathbf{r}_1-\mathbf{r}_2$ and a is the spatial scale of the correlations; b) a one-dimensional RIM with delta-correlated inhomogeneities ("white noise"). Use of this method permits us to calculate more precisely the analytical properties of the scattering index evaluated from the attenuation of the mean intensity in the known solutions[26,29] for the simplest RIM in the one-dimensional case and to obtain new solutions for three-dimensional RIM.

In Sec. 2 a modification is presented of the familiar results and those we obtained previously, convenient for expounding the new method (references to sources are given where necessary). In Sec. 3 a (very brief) transition is made from the general solutions of Sec. 2 to approximations based on the use of information about the RIM only in terms of binary correlation functions (taking into consideration pairwise interactions between inhomogeneities). We treat the cases of 1) the Bourret approximation;[16,31] and 2) the ladder approximation.[32] In Sec. 4 we treat the dispersion relation for the mean field and the conditions for the applicability of the Bourret approximation. Then, in the approximation where the coherence function B is represented as the product of two monochromatic waves we obtain a dispersion relation for the mean intensity, evaluated in the ladder approximation. In Sec. 5 we consider the special cases mentioned above.

2. EQUATIONS FOR THE FIRST AND SECOND MOMENTS OF THE FIELD AND THE GREEN'S FUNCTION OPERATOR

Consider the scalar monochromatic field $E(\mathbf{r},t)=E(\mathbf{r})e^{-i\omega t}$, described by the inhomogeneous Helmholtz equations

$$\hat{L}E=-f,\quad \hat{L}\equiv(\Delta+k_c^2\tilde{\varepsilon})\hat{I},\quad E\equiv E(\mathbf{r}),\quad \hat{I}F\equiv F, \tag{2.1}$$

$$k_c^2\equiv\varepsilon_c k_0^2,\quad k_0^2c_0^2\equiv\omega^2,\quad \tilde{\varepsilon}\varepsilon_c\equiv\varepsilon,\quad \varepsilon=\varepsilon(\mathbf{r}). \tag{2.2}$$

For the sake of having specific terminology we will call the scalar field ε the dielectric function of the RIM. Then the field E resulting from the source f has the physical meaning of the electric field strength, associated with an inductance D by the relation $D=\varepsilon E$; here c_0 is the speed of light in vacuum. The subscript c labels quantities connected with the comparison medium (see below). The equation corresponding to (2.1) for the Green's function operator $\hat{H}$ takes the form

$$\hat{L}\hat{H}=-\hat{I};\quad E=\hat{H}f,\quad E(\mathbf{r}_1)=\int H(\mathbf{r}_1,\mathbf{r}_2)f(\mathbf{r}_2)d\mathbf{r}_2, \tag{2.3}$$

where $H(\mathbf{r}_1,\mathbf{r}_2)$ is the kernel of the operator $\hat{H}$.

The use of the concept of a comparison medium enables us to pass from the differential equation (2.1) to the integral equation

$$E=E_c+\hat{Q}\bar{\varepsilon}'E,\quad \hat{Q}_c\equiv k_c^2\hat{H}_c,\quad F'\equiv F-F_c, \tag{2.4}$$

$$\hat{L}_c\hat{H}_c=-\hat{I},\quad \hat{L}_cE_c=-f;\quad \hat{L}_c\equiv(\Delta+k_c^2)\hat{I}. \tag{2.5}$$

Together with (2.4) it is useful to consider the equation

$$E=\langle E\rangle+\hat{X}E,\quad \hat{X}\equiv\hat{R}\hat{Q}_c\bar{\varepsilon}'$$

$$\hat{R}F\equiv F-\langle F\rangle\equiv F'';\quad \hat{R}^2=\hat{R}. \tag{2.6}$$

The angle brackets denote a statistical average (over an ensemble of realizations). Here $\hat{R}$ is an operator which extracts the random component from the quantity standing to the right of it; it differs from the centering operator,[30] which is denoted by a double prime.

In order to determine the restrictions imposed on the operator $\hat{X}$ of the perturbation, from (2.6) we derive[30,33,34]

$$E=\hat{A}\langle E\rangle,\quad \langle\hat{A}\rangle=\hat{I}, \tag{2.7}$$

$$\hat{A}=(\hat{I}-\hat{X})^{-1}=\sum_0^\infty \hat{X}^n. \tag{2.8}$$

Using the relation[1)]

$$\langle D\rangle=\langle\varepsilon E\rangle\equiv\hat{\varepsilon}_*\langle E\rangle \tag{2.9}$$

to introduce the operator $\hat{\varepsilon}_*$ for the effective dielectric function, from (2.7)–(2.9) we have[33,34]

$$\hat{\varepsilon}_*=\langle\bar{\varepsilon}\hat{A}\rangle=\sum_0^\infty \hat{\varepsilon}_n,\quad \hat{\varepsilon}_n=\langle\bar{\varepsilon}\hat{X}^n\rangle. \tag{2.10}$$

In the case of a regular (or statistically independent of the random field ε) source f the average (2.1) yields by virtue of the definition (2.9)

$$\langle\hat{L}E\rangle\equiv\hat{L}_*\langle E\rangle=-\langle f\rangle,\quad \hat{L}_*=\Delta\hat{I}+k_c^2\hat{\varepsilon}_*. \tag{2.11}$$

The operator $\hat{L}_*$ in (2.11) is called the Dyson operator.[2]

The field of a point source is described in the language of Green's functions (operators). In this case in place of (2.4), (2.6), (2.7), and (2.11) we write

$$\hat{H}=\hat{H}_c+\hat{Q}_c\bar{\varepsilon}'\hat{H},\quad \hat{H}=\langle\hat{H}\rangle+\hat{X}\hat{H},\quad \hat{H}=\hat{A}\langle\hat{H}\rangle, \tag{2.12}$$

$$\hat{L}_*\langle\hat{H}\rangle=-\hat{I},\quad \hat{L}_*=\hat{L}_c+\hat{M},\quad \hat{M}\equiv\langle\hat{L}'\hat{A}\rangle, \tag{2.13}$$

where $\hat{M}$ is the mass operator. Averaging the first of Eqs. (2.12) leads to the Dyson equation[2]

$$\langle\hat{H}\rangle=\hat{H}_c+\hat{H}_c\hat{M}\langle\hat{H}\rangle \tag{2.14}$$

for the average of the Green's function operator $\langle\hat{H}\rangle$.

The comparison medium used as an auxiliary is macroscopically identical (in overall geometry, boundary conditions, and field sources) with the RIM in question and differs from it only in its material properties (dielectric function, density, and so forth) and its internal geometry (the spatial distribution of the inhomogeneities). The operator $\hat{L}_c$ satisfies the usual restrictions imposed on the unperturbed operator: 1) the solution of the problem (2.5) is known; 2) the perturbation operator $\hat{L}'=\hat{L}-\hat{L}_c$ is small in some sense, which ensures the convergence of the perturbation series (2.8) and (2.10). The parameter ε_c is otherwise arbitrary. Since the exact solutions (2.7), (2.10), and (2.12) do not depend on ε_c, this parameter is chosen from considerations of mathematical simplicity of the resulting solutions or the rate of convergence of the iteration procedure in those cases when it is necessary to truncate the series (2.7), (2.10), and (2.12) after just the first two or three terms. We often assume $\varepsilon_c=\langle\varepsilon\rangle$, as a result of which $\varepsilon'=\varepsilon''$ holds. The ladder relation simplifies the calculation of the terms of series (2.10). In studying the general relations it turns out to be convenient to take $\hat{L}_c$ to be the operator $\hat{L}_*$. In the case of (2.12) this yields

$$\hat{\varepsilon}_c\equiv\varepsilon_c\hat{I}\to\hat{\varepsilon}_*\Rightarrow\hat{H}_c\to\hat{H}_*\equiv\langle\hat{H}\rangle,$$

$$\hat{H}''=\hat{H}-\langle\hat{H}\rangle=k_0^2\langle\hat{H}\rangle\hat{R}\hat{\varepsilon}'\hat{H}=k_0^2\langle\hat{H}\rangle\hat{\varepsilon}'\hat{H}. \tag{2.15}$$

We introduce the definition of the coherence function B of the field E (Ref. 2):

$$B(\mathbf{r}_2,\mathbf{r}_1)\equiv\langle E^*(\mathbf{r}_2)\otimes E(\mathbf{r}_1)\rangle, \tag{2.16}$$

where * and $\otimes$ respectively denote the complex conjugate and the direct (tensor) product of fields or operators. Using (2.3) we write E^* as

$$E^*(\mathbf{r}_2)=\int H^*(\mathbf{r}_2,\mathbf{r}_4)f^*(\mathbf{r}_4)d\mathbf{r}_4$$

$$\equiv\int f^*(\mathbf{r}_4)H^+(\mathbf{r}_4,\mathbf{r}_2)d\mathbf{r}_4\Rightarrow E^*=f^*\hat{H}^+, \tag{2.17}$$

where $\hat{H}^+$ is the adjoint operator. Using (2.17) we have from (2.16)

$$B=\hat{\Gamma}F,\quad \hat{\Gamma}\equiv\langle\hat{H}^+\otimes\hat{H}\rangle,\quad F\equiv\langle f^*\otimes f\rangle. \tag{2.18}$$

Here $\hat{\Gamma}$ is the coherence operator, whose kernel $\Gamma(\mathbf{r}_4,\mathbf{r}_2;\mathbf{r}_1,\mathbf{r}_3)$ combines with the functions $f^*(\mathbf{r}_4)$ and $f(\mathbf{r}_3)$ of the external coordinates $\mathbf{r}_4$ and $\mathbf{r}_3$. In addition, Eq. (2.18) reflects the statistical independence of the random fields ε and f. In the case of a regular field f the statistical average in the definition of F in (2.18) can be omitted.

By virtue of (2.15) the operator $\hat{\Gamma}$ satisfies the Bethe–Salpeter equation[2,6,32]

$$\langle\hat{H}''^+\otimes\hat{H}''\rangle=\langle\hat{H}^+\rangle\otimes\langle\hat{H}\rangle\hat{K}\hat{\Gamma}, \tag{2.19}$$

where the intensity operator $\hat{K}$ is defined by the relations

$$\langle\hat{Y}^{+}\otimes\hat{Y}\rangle\equiv\hat{K}\hat{\Gamma},\quad \hat{Y}\equiv\hat{R}\hat{L}'\hat{H}=-\hat{R}\hat{L}_{*}\hat{H},$$

which can be reduced by means of (2.12) to the form

$$\hat{K}\langle\hat{A}^{+}\otimes\hat{A}\rangle=\langle\hat{G}^{+}\otimes\hat{G}\rangle,\quad \hat{G}\equiv\hat{R}\hat{L}_{*}\hat{A},\tag{2.20}$$

which is invariant with respect to the choice of the parameter $\hat{\varepsilon}_c$ (or $\hat{L}_c$). Equation (2.19) in the form

$$\hat{\Gamma}=\hat{\Gamma}_{*}+\hat{\Gamma}_{*}\hat{K}\hat{\Gamma},\quad \hat{\Gamma}_{*}\equiv\langle\hat{H}^{+}\rangle\otimes\langle\hat{H}\rangle,\tag{2.21}$$

which resembles (2.14), allows us to expand $\hat{\Gamma}$ in a series

$$\hat{\Gamma}=(\hat{I}_2-\hat{\Gamma}_{*}\hat{K})^{-1}\hat{\Gamma}_{*}=\sum_{0}^{\infty}(\hat{\Gamma}_{*}\hat{K})^{n}\hat{\Gamma}_{*},\quad \hat{I}_2\equiv\hat{I}\otimes\hat{I}.\tag{2.22}$$

It is easy to see that from (2.13) $\hat{\Gamma}_{*}$ satisfies the equation

$$\hat{\mathscr{L}}_{*}\hat{\Gamma}_{*}=\hat{I}_2,\quad \hat{\mathscr{L}}_{*}\equiv\hat{L}_{*}^{+}\otimes\hat{L}_{*},\tag{2.23}$$

analogous to (2.5). Applying the operator $\hat{\mathscr{L}}_{*}$ to both sides of (2.21) and using (2.23), we find

$$\hat{\mathscr{L}}\hat{\Gamma}=\hat{I}_2,\quad \hat{\mathscr{L}}\equiv\hat{\mathscr{L}}_{*}-\hat{K},\tag{2.24}$$

which is similar to (2.13). The equation for the coherence function B that corresponds to (2.24) and (2.18) takes the form

$$\hat{\mathscr{L}}\mathbf{B}=F.\tag{2.25}$$

Equations (2.11) and (2.25) establish the connection between the moments of the field E and the corresponding moments of the source f. For a point source these equations go over to (2.13) and (2.24).

3. THE PAIR-INTERACTION APPROXIMATION

The absence in the general case of complete statistical information about the random field ε, as well as the increase in difficulties of a computational nature as k increases, makes it impossible to sum series like (2.8), (2.10), and (2.22). We therefore often limit ourselves to the pair interactions between inhomogeneities, which is valid when the fluctuations are sufficiently small.

In this approximation for the operator $\hat{A}$ we have from (2.8)

$$\hat{A}=\hat{I}+\hat{Q}_c\bar{\varepsilon}''.\tag{3.1}$$

Substituting (3.1) in (2.10) we find for $\hat{\bar{\varepsilon}}_{*}$ the expression

$$\hat{\bar{\varepsilon}}_{*}\approx\langle\bar{\varepsilon}\rangle\hat{I}+\langle\bar{\varepsilon}''\hat{Q}_c\bar{\varepsilon}''\rangle\equiv\hat{\bar{\varepsilon}}_{BA}.\tag{3.2}$$

The choice $\varepsilon_c=\langle\varepsilon\rangle$ simplifies the mass operator considerably, From (2.13) and (3.2) it is equal in this case to

$$\hat{M}\approx k_c^2\langle\bar{\varepsilon}''\hat{Q}_c\bar{\varepsilon}''\rangle\equiv\hat{M}_{BA}.\tag{3.3}$$

The series expansion of the intensity operator (2.20) begins with the term quadratic in $\bar{\varepsilon}''$. Consequently, in this approximation we have from (3.20)

$$\hat{K}\approx\langle(\hat{L}_c\hat{A}'')^{+}\otimes(L_c\hat{A}'')\rangle\approx k_c^4\langle\bar{\varepsilon}''\hat{I}\otimes\bar{\varepsilon}'\hat{I}\rangle\equiv\hat{K}_{LA},\tag{3.4}$$

where we have taken into account Eqs. (2.4)–(2.6) and the self-adjointness of the operator $\hat{\varepsilon}=\varepsilon\hat{I}$ of the dielectric function.

Calculations of the first moments of the field E and the Green's function operator $\hat{H}$ carried out using the mass operator $\hat{M}_{BA}$ in (3.3) are often called the Bourret approximation.[2,16] The analogous calculations of the second moments of E and $\hat{H}$ using $\hat{K}_{LA}$ given in (3.4) are referred to as the ladder approximation.[2,32]

4. DISPERSION RELATIONS

The passage from the integral equation (2.11) to the corresponding dispersion relation

$$L_{*}(\mathbf{x},q)\equiv q^2\bar{\varepsilon}(\mathbf{x},q)-x^2=0;\quad \mathbf{x}\equiv a\mathbf{k}_{*},\quad q\equiv ak_c\tag{4.1}$$

results from setting the right-hand side of (2.11) equal to zero and substituting the mean field $\langle E\rangle$ in the form of a plane wave

$$\langle E\rangle=E_0\exp(i\mathbf{k}_{*}\mathbf{r})\tag{4.2}$$

$$\mathbf{k}_{*}=k_{*}\mathbf{n},\quad k_{*}\equiv k^{(1)}+ik^{(2)},\quad \mathbf{nn}=1.\tag{4.3}$$

The function $\bar{\varepsilon}_{*}(\mathbf{x},q)$ is the Fourier transform of the kernel $\bar{\varepsilon}_{*}(\mathbf{r},\omega)$ of the operator $\hat{\bar{\varepsilon}}_{*}$, written in dimensionless variables. The parameter a appearing in (4.1) is the spatial scale of the correlations, determined by the spatial dependence of the binary correlation function

$$\langle\bar{\varepsilon}''(\mathbf{r}_1)\bar{\varepsilon}''(\mathbf{r}_2)\rangle\equiv D_{\bar{\varepsilon}}\varphi(\boldsymbol{\rho}),$$
$$D_{\bar{\varepsilon}}\equiv\langle(\bar{\varepsilon}'')^2\rangle,\quad a\boldsymbol{\rho}\equiv\mathbf{r}_1-\mathbf{r}_2\tag{4.4}$$

of the random field $\varepsilon(\mathbf{r})$, assumed here and in what follows to be statistically homogeneous.[2]

The parameters (4.3) of the normal plane wave (4.2) are found by solving the dispersion relation (4.1). In the approximation (3.2) for the function $\bar{\varepsilon}_{*}(\mathbf{x},q)$ we have

$$\bar{\varepsilon}_{BA}(\mathbf{x},q)=\langle\bar{\varepsilon}\rangle+D_{\bar{\varepsilon}}F(\mathbf{x},q),\tag{4.5}$$

$$8\pi^3F(\mathbf{x},q)=\int\varphi(\mathbf{x}-\mathbf{y})Q_c(\mathbf{y},q)d\mathbf{y}.\tag{4.6}$$

Here $\mathbf{y}$ is the dimensionless wave vector, and the Fourier transform $Q_c(\mathbf{y},q)$ of the kernel $\hat{Q}_c$ of (2.4) takes the form

$$Q_c(\mathbf{y},q)=\frac{q^2}{y^2-q^2}.\tag{4.7}$$

The choice of the arbitrary parameter ε_c follows from the nature of the problem being solved (Refs. 2, 5–7, 16, 19, 22, and 28–34). Just as in (3.3) we find the parameter from the condition

$$\langle\bar{\varepsilon}\rangle=1\Rightarrow\varepsilon_c=\langle\varepsilon\rangle,\quad D_{\bar{\varepsilon}}=D_{\varepsilon}\langle\varepsilon\rangle^{-2}\equiv D,\tag{4.8}$$

which gives rise to a simplification in the expressions for $\bar{\varepsilon}_{BA}$ of (4.5) and the roots of Eq. (4.1).

In the present work we use as a criterion for the applicability of the approximation (3.1)–(3.3) the inequality[22]

$$q^{-2}|M_{BA}(\mathbf{x},q)|=D|F(\mathbf{x},q|\leqslant 1.\tag{4.9}$$

The dependence of the function M on the wave vector $\mathbf{x}$ is a manifestation of the spatial dispersion due to the inhomogeneity of the medium,[19,20,22] which can be neglected under the condition[22]

$$\left| \frac{d}{dx^2} M(\mathbf{x},q) \right| \ll 1 \tag{4.10}$$

with the replacement $x \to q$ in the kernel of the mass operator.[5,15,31]

The roots of the dispersion relation (4.1) are determined in the usual way by the dynamical properties of the RIM, obtained from the mean field (4.2). As a quantitative measure of the wave scattering we introduce the dimensionless scattering index

$$a\gamma_* \equiv \bar{\gamma}_* = 2x^{(2)}, \quad x \equiv x^{(1)} + ix^{(2)}, \tag{4.11}$$

where γ_* is the attenuation rate of the mean field intensity.[2,5,15] From the definition[2] the mean intensity J is related to the coherence function B of (2.16) by

$$J(\mathbf{R}) \equiv \mathrm{B}(\mathbf{R},\mathbf{R}) = \langle |E(\mathbf{R})|^2 \rangle, \quad \mathbf{r}_1 = \mathbf{r}_2 \equiv \mathbf{R}. \tag{4.12}$$

Hence in the mean-field approximation we have

$$J_*(\mathbf{R}) = \mathrm{B}_*(\mathbf{R},\mathbf{R}) \equiv |\langle E(\mathbf{R}) \rangle|^2. \tag{4.13}$$

Using (4.1)–(4.4), (4.11), and (4.13) we find for J_*

$$J_*(\mathbf{R}) = |E_0|^2 \exp(-\gamma_* \mathbf{n}\mathbf{R}). \tag{4.14}$$

The dimensionless phase and group velocities $\bar{v}_*$ and $\bar{c}_*$ are defined by the relations

$$\bar{v}_* \equiv \frac{q}{(x)^1} = \frac{v_*}{v_c}, \quad \frac{1}{\bar{c}_*} \equiv \frac{dx^{(1)}}{dq} = \frac{v_c}{c_*}, \quad c_0 \equiv v_c \sqrt{\varepsilon_c}, \tag{4.15}$$

where v_* and c_* respectively are the phase and group velocity of a propagating plane scalar wave in the effective medium.

Calculation of the intensity from the mean field (4.13) yields an overestimate $\bar{\gamma}_*$ for the damping rate.[26,29] This is due to the contribution to $\bar{\gamma}_*$ of the wave phase fluctuations that do not affect J. To eliminate this effect we go over from (2.11) to a treatment of Eq. (2.25), which in the approximation (3.4) and in the absence of sources assumes the form

$$F_0(\mathbf{r}_2,\mathbf{r}_1) - K_0(\mathbf{r}_2,\mathbf{r}_1)\mathrm{B}(\mathbf{r}_2,\mathbf{r}_1) = 0,$$

$$F_0 \equiv \hat{\mathscr{L}}_* \mathrm{B}, \quad K_0(\mathbf{r}_2,\mathbf{r}_1) \equiv k_c^4 \langle \tilde{\varepsilon}''(\mathbf{r}_2) \tilde{\varepsilon}''(\mathbf{r}_1) \rangle. \tag{4.16}$$

Just as in the case (4.1)–(4.3), we can choose as the solution of Eq. (4.16) the function

$$\mathrm{B}(\mathbf{r}_2,\mathbf{r}_1) = \mathrm{B}_0 \exp[i(\mathbf{k}_e\mathbf{r}_1 - \mathbf{k}_e^*\mathbf{r}_2)] = \mathrm{B}_0 \exp[\mathbf{n}(i\kappa\mathbf{r} - \gamma_e\mathbf{R})], \tag{4.17}$$

$$\mathbf{k}_e \equiv \mathbf{n}(\kappa + i\gamma_e/2), \quad \mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2, \quad 2\mathbf{R} \equiv \mathbf{r}_1 + \mathbf{r}_2, \tag{4.18}$$

which yields the expression

$$J(\mathbf{R}) = J_0 \exp(-\gamma_e \mathbf{n}\mathbf{R}) \tag{4.19}$$

for the mean intensity (4.12), similar to (4.14).

Substituting (4.17) in (4.16) and converting to dimensionless variables, we can write

$$[|L_*(\mathbf{x},q)|^2 - q^4 D_{\tilde{\varepsilon}} \varphi(\mathbf{r}_1 - \mathbf{r}_2)]\mathrm{B}(\mathbf{r}_2,\mathbf{r}_1) = 0, \tag{4.20}$$

where we have used the notation of Eqs. (4.1) and (4.4). Hence for $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{R}$ we find

$$[|l(\mathbf{z},q)|^2 - D_{\tilde{\varepsilon}}]J(\mathbf{R}) = 0, \quad \mathbf{z} \equiv a\mathbf{k}_e, \tag{4.21}$$

$$-q^2 L_*^{BA}(\mathbf{z},q) \equiv l(\mathbf{z},q) = \bar{z}^2 - \bar{\varepsilon}_{BA}(\mathbf{z},q), \quad q\bar{z} \equiv z. \tag{4.22}$$

The condition for the existence of a nontrivial solution of Eq. (4.21),

$$|l(\mathbf{z},q)|^2 - D_{\tilde{\varepsilon}} = 0, \tag{4.23}$$

which constitutes a dispersion relation for the mean intensity (4.19), is similar to the dispersion relation (4.1) for the mean field (4.2). The roots of Eq. (4.23) determine the energetic scattering index $\bar{\gamma}_e$, related to (4.11) by the inequality $\bar{\gamma}_e \leqslant \bar{\gamma}_*$.

We rewrite (4.23) in the form

$$l(\mathbf{z},q) = \bar{\xi}, \quad \bar{\xi} \equiv \psi\sqrt{D_{\tilde{\varepsilon}}}, \quad \psi \equiv \exp(i\lambda), \tag{4.24}$$

where the phase factor ψ in general is a function of the variables $\mathbf{z}$ and q. When condition (4.10) is satisfied ψ depends only on q. Below we consider the approximation $\lambda = 0$, which automatically takes into account (4.10). In this case from (4.24) and (4.22) we have

$$\bar{\xi} = \sqrt{D_{\tilde{\varepsilon}}} \Rightarrow \bar{z}^2 = \bar{\varepsilon}_{LA} \equiv \bar{\varepsilon}_{BA} + \bar{\xi}. \tag{4.25}$$

Using expression (4.5) for $\hat{\varepsilon}_{BA}$ and the notation (2.2), (4.4), and (4.8), we transform (4.25) into

$$\bar{\varepsilon}_{LA} = \langle \bar{\varepsilon} \rangle (1+\xi) + D_{\tilde{\varepsilon}} F(\mathbf{z},q), \quad \xi^2 = D_{\tilde{\varepsilon}} \langle \bar{\varepsilon} \rangle^{-2} \equiv D. \tag{4.26}$$

It is easy to see that in order to simplify the subsequent calculations it is convenient to choose the auxiliary parameter ε_c from the condition

$$\langle \bar{\varepsilon} \rangle (1+\xi) = 1 \Rightarrow \varepsilon_c = \langle \varepsilon \rangle (1+\xi). \tag{4.27}$$

In view of (4.27) expression (4.26) for $\bar{\varepsilon}_{LA}$ is identical in form with $\bar{\varepsilon}_{BA}$ given by (4.5), where ε_c satisfies (4.8). Hence we can use the solution of the dispersion relation (4.1) in the Bourret approximation to find the roots of Eq. (4.26). From (4.27) it follows that we must make the replacements

$$\varepsilon_c = \langle \varepsilon \rangle (1+\xi) \Rightarrow D \to D(1+\xi)^{-2}, \quad \xi = D^{1/2},$$

$$q \to q(1+\xi)^{1/2}, \quad qD^{1/2} \equiv \bar{q} \to \bar{q}(1+\xi)^{-1/2}, \tag{4.28}$$

where we have also used the notation of (2.2) and (4.1).

5. SPECIAL CASES

As an example we consider RIM described by the function $\varphi(\boldsymbol{\rho})$ from (4.4) in the form

$$\varphi(\boldsymbol{\rho}) = \exp(-\rho), \quad \rho \equiv \sqrt{\boldsymbol{\rho}\boldsymbol{\rho}}, \quad a\boldsymbol{\rho} \equiv \mathbf{r}. \tag{5.1}$$

The use of (5.1) in the Bourret approximation (4.5), (4.6) enables us to calculate the parameters (4.11), (4.15) over the whole range of wavelengths.[30] In the case of the function (5.1), whose Fourier transform for three-dimensional RIM is equal to

$$\varphi(\mathbf{y}) = 8\pi(1+y^2)^{-2}, \tag{5.2}$$

the dispersion relation (4.1) has two roots in the upper half of the complex plane. In the long-wavelength ($q<1$) and short-wavelength ($\bar{q}<1$) ranges condition (4.10) is satisfied and only the "real" root x_+, for which $x_+^{(2)} \ll 1$ holds, is important. The second ("virtual") root x_- yields a small and rapidly damped contribution to the average Green's function in these ranges,[2,30] and can be used to make the resulting solution more accurate.[35]

In the Bourret approximation for $\bar{\gamma}_* \equiv 2x_{\mp}^{(2)}$ of Eq. (4.11) we have[30]

$$\bar{\gamma}_*(l) = \frac{2q^4 D}{1+4q^2} \approx 2q^4 D, \quad q<1, \tag{5.3}$$

$$\bar{\gamma}_*(s) = \left(1-\frac{D}{8}\right)\frac{\bar{q}^2}{2}, \quad \bar{q}<1<q. \tag{5.4}$$

Making the replacements (4.28) in (5.3) and (5.4) we find in the ladder approximation (4.25) for $\bar{\gamma}_e$

$$\bar{\gamma}_e(l) = \bar{\gamma}_*(l)(1-4q^2\alpha), \quad \alpha \equiv \frac{\xi}{1+\xi} \approx \xi \equiv \sqrt{D}, \tag{5.5}$$

$$\bar{\gamma}_e(s) = \bar{\gamma}_*(s)[1-\alpha(1-D/4)]. \tag{5.6}$$

For one-dimensional RIM described by the function (5.1) instead of (5.2) we have

$$\varphi(y) = 2(1+y^2)^{-1}. \tag{5.2a}$$

Analogous calculations in the Bourret approximation yield[36]

$$\bar{\gamma}_*(l) = \bar{q}^2, \quad q<1, \tag{5.3a}$$

$$\bar{\gamma}_*(s) = \left(1+\frac{D}{8}\right)\frac{\bar{q}^2}{2}, \quad \bar{q}<1<q. \tag{5.4a}$$

After renormalizing the parameter ε_c in (5.3a) and (5.4a) by means of (4.28) we find in the ladder approximation

$$\bar{\gamma}_e(l) = \bar{\gamma}_*(l)(1-\alpha), \tag{5.5a}$$

$$\bar{\gamma}_e(s) = \bar{\gamma}_*(s)\left[1-\alpha\left(1+\frac{1}{4}D\right)\right]. \tag{5.6a}$$

For RIM with delta-correlated inhomogeneities ("white noise"), when the function $\varphi(\boldsymbol{\rho})$ takes the form

$$\varphi(\boldsymbol{\rho}) = \delta(\boldsymbol{\rho}), \tag{5.7}$$

instead of (5.1), the use of Eq. (4.16) and its Fourier transform (4.20) is complicated by the presence of singularities proportional to (5.7). To overcome these difficulties we must go over to an equation which is intermediate between (2.21) and (4.16), where because of the integration this singularity no longer appears.

Using the operator $\hat{I}\otimes\hat{L}_*$ on both sides of Eq. (2.21) and using the relations (2.3) and (2.18) between E and f and Eq. (2.13), we can write for the coherence function B (Ref. 2)

$$[\hat{I}\otimes\hat{L}_* + \langle\hat{H}^+\rangle\otimes\hat{I}\hat{K}]\mathrm{B} = -\langle E^*\rangle\otimes\langle f\rangle.$$

For a region free of sources ($f=0$) we then have

$$\int L_*(\boldsymbol{\rho}_1-\boldsymbol{\rho}_3)\mathrm{B}(\boldsymbol{\rho}_2,\boldsymbol{\rho}_3)d\boldsymbol{\rho}_3 + q^4 D_{\bar{\varepsilon}}\langle H^*(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)\rangle\mathrm{B}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_1) = 0, \tag{5.8}$$

where we have used dimensionless variables and taken into consideration the relations (3.4), (4.4), and (5.7).

Substituting the function B in the form (4.17) in (5.8) and going over from B to J given by (4.19) for $\boldsymbol{\rho}_1=\boldsymbol{\rho}_2$, we find

$$[L_*(\mathbf{z},q) + q^4 D\langle H^*(0)\rangle]J_0 = 0. \tag{5.9}$$

Here we have used the notation (4.21) and (4.8), and $L_*(\mathbf{z},q)$ is evaluated using Eq. (4.1) in the Bourret approximation (4.5).

It is easy to see that $\langle H^*(0)\rangle$ has a finite value only in the simplest case of a one-dimensional medium, when we have

$$H_c(\boldsymbol{\rho}) \to H_c(\rho) = \frac{i}{2q}\exp(iq\rho), \quad a\rho \equiv |\mathbf{r}\mathbf{e}_3|, \tag{5.10}$$

where $\mathbf{e}_3$ is the unit vector parallel to the third Cartesian axis. Using (5.7), (5.10), and (4.8) with the Bourret approximation (4.5), (4.6) we find

$$\bar{\varepsilon}_{BA}(\mathbf{x},q) \to \bar{\varepsilon}_{BA} = 1 + \frac{i}{2}qD. \tag{5.11}$$

Since $\bar{\varepsilon}_{BA}$ is independent of the wave vector, an RIM described by the function (5.7) has no macroscopic spatial dispersion (resulting from the inhomogeneity of the medium). As a result, the average Green's function calculated using (5.11) can be represented in the form

$$\langle H(\rho)\rangle = \frac{i}{2x}\exp(ix\rho), \quad x \equiv q\bar{\varepsilon}_{BA}^{1/2}. \tag{5.12}$$

From (5.9) we find using (5.11) and (5.12) and the notation (4.22), (4.25) that the wave number $\bar{z}$ is equal to

$$\bar{z}^2 = \bar{\varepsilon}_{LA} = \bar{\varepsilon}_{BA} + \left(i\frac{q^2 D}{2x}\right)^* = 1 + i\frac{qD}{2}\left[1-\left(\frac{1}{\sqrt{\bar{\varepsilon}_{BA}^*}}\right)\right]. \tag{5.13}$$

These values of x in (5.12) and (5.11) and of z in (5.13) enable us to find $\bar{\gamma}_*$ and $\bar{\gamma}_e$. Calculation using Eq. (4.11) yields

$$\bar{\gamma}_* = \tfrac{1}{2}q^2 D \tag{5.14}$$

in the Bourret approximation (3.2), (3.3) and

$$\bar{\gamma}_e = \tfrac{3}{64}q^4 D \tag{5.15}$$

in the ladder approximation (3.4) respectively. In both cases, by virtue of (4.9) the inequality $qD \ll 1$ holds, which together with the restriction $\bar{q}<1$ of Eq. (5.4), which ensures that $\bar{\gamma}_*$ is small, gives rise to the inequalities

$$\bar{\gamma}_e \ll \bar{\gamma}_* \ll 1. \tag{5.16}$$

Let us now consider the case of a point source in RIM described by function (5.7). Returning to Eq. (2.21), we have in the ladder approximation

$$j(\mathbf{R}) = j_*(\mathbf{R}) + q^4 D\int j_*(\mathbf{R}-\mathbf{R}_1)j(\mathbf{R}_1)d\mathbf{R}_1,$$

$$j(\mathbf{R}) \equiv \langle|H(\boldsymbol{\rho})|^2\rangle, \quad j_*(\mathbf{R}) \equiv |\langle H(\boldsymbol{\rho})\rangle|^2, \quad \varepsilon_c = \langle\varepsilon\rangle. \tag{5.17}$$

Here $H(\boldsymbol{\rho})$ and $\langle H(\boldsymbol{\rho})\rangle$ are the kernels of the operators $\hat{H}$ and $\langle\hat{H}\rangle$ respectively, written in dimensionless variables. Going over to $\mathbf{k}$ space we find from (5.17)

$$j(\mathbf{K})=j_*(\mathbf{K})+q^4Dj_*(\mathbf{K})j(\mathbf{K}). \quad (5.18)$$

In the case of a one-dimensional medium the definitions (5.17) and (4.11) imply

$$j_*(\rho)=\frac{1}{4|x|^2}\exp(-\bar{\gamma}_*\rho), \quad (5.19)$$

which for the Fourier transform $j_*(K)$ yields

$$j_*(K)=\frac{1}{2|x|^2}\frac{\bar{\gamma}_*}{K^2+\bar{\gamma}_*^2}. \quad (5.20)$$

After substituting (5.20) in (5.18) we have

$$j(K)=\frac{j_*(K)}{1-q^4Dj_*(K)}=\frac{1}{2|z|^2}\frac{\bar{\gamma}_e}{K^2+\bar{\gamma}_e}$$

$$\bar{\gamma}_e^2\equiv\bar{\gamma}_*^2-\frac{q^2\bar{\gamma}_*}{2|\bar{\varepsilon}_{BA}|}, \quad \bar{\gamma}_*|z|^2=\bar{\gamma}_e|x|^2. \quad (5.21)$$

By virtue of (5.14) the energetic scattering index $\bar{\gamma}_e$ of (5.21) takes the form

$$\bar{\gamma}_e=\bar{\gamma}_*\sqrt{1-|\bar{\varepsilon}_{BA}|^{-1}}\approx\frac{\sqrt{2}}{8}q^3D^2 \quad (5.22)$$

and satisfies the inequalities (5.16). The expression for $\bar{\gamma}_e$, like (5.21), was found previously by Brekhovskikh.[29]

In conclusion we note a way in which the solutions (5.15) and (5.22) differ. In calculating variables which are linear in the field (the Bourret approximation) we describe the field $\langle E\rangle$ of (4.2) and the average Green's function $\langle H\rangle$ of (5.12) by means of the same parameter $\bar{\gamma}_*$. But in the ladder approximation the quantities (4.19) and (5.17), which are quadratic in the field, are studied. The relation between these is not as simple as in the first case.

[1)]The definition of $\hat{\varepsilon}_*$ holds when the random field ε has sufficiently general properties. However, when ε and f are statistically related, the explicit form of $\hat{\varepsilon}_*$ depends on the behavior of the random field f. In the present work we neglect the statistical action of f on ε. Then the dependence of the field E on f is manifested only by virtue of Eq. (2.1), and ε and f are completely decoupled.

[1] L. M. Brekhovskikh, *Waves in Layered Media*, 2nd ed., Academic, New York (1980).
[2] S. M. Rytov, Yu. A. Kravtsov, and V. I. Tatarskii, *Introduction to Statistical Radiophysics*, Vol. 2, Springer, New York (1989).
[3] V. I. Klyatskin, *The Method of Imbedding in Wave Propagation Theory* [in Russian], Nauka, Moscow (1986).
[4] V. E. Ostashev, *Sound Propagation in Moving Media* [in Russian], Nauka, Moscow (1992).
[5] Yu. A. Ryzhov and V. V. Tamoikin, Izv. Vuzov. Radiofiz. **13**, 356 (1970).
[6] Yu. N. Barabanenkov, Yu. A. Kratsov, S. M. Rytov, and V. I. Tatarskii, Usp. Fiz. Nauk **102**, No. 1, 13 (1970) [Sov. Phys. Usp. **13**, 551 (1971)].
[7] Yu. N. Barabanenkov, Usp. Fiz. Nauk **117, No. 1**, 49 (1975) [Sov. Phys. Usp. **18**, 673 (1975)].
[8] S. A. Gredeskul and V. D. Freilikher, Usp. Fiz. Nauk **160**, No. 2, 239 (1990) [Sov. Phys. Usp. **33**, 134 (1990)].
[9] V. I. Klyatskin and A. I. Saichev, Usp. Fiz. Nauk **162**, No. 3, 161 (1992) [Sov. Phys. Usp. **35**, 231 (1992)].
[10] C. L. Pekeris Phys. Rev. **71**, 268 (1947).
[11] W. P. Mason and H. J. McSkimin, J. Appl. Phys. **19**, 940 (1948).
[12] H. B. Huntington, J. Acoust. Soc. Am. **22**, 362 (1950).
[13] I. M. Lifshitz and G. D. Perkhomovskiĭ, Zh. Éksp. Teor. Fiz. **20**, 175 (1950)
[14] S. I. Pekar, Zh. Éksp. Teor. Fiz. **33**, 1022 (1957) [Sov. Phys. JETP **6**, 785 (1957)].
[15] E. A. Kaner, Izv. Vuzov. Radiofiz. **2**, 827 (1959).
[16] R. C. Bourret, Can. J. Phys. **40**, 782 (1962).
[17] F. C. Karal, Jr., and J. B. Keller, J. Math. Phys. **5**, 537 (1964).
[18] E. E. Salpeter and S. B. Treiman, J. Geophys. Res. No. 69, 869 (1964).
[19] Yu. A. Ryzhov, V. V. Tamoikin, and V. I. Tatarskii, Zh. Éksp. Teor. Fiz. **48**, 656 (1965) [Sov. Phys. JETP **21**, 433 (1965)].
[20] Yu. A. Ryzhov, Izv. Vuzov. Radiofiz. **9**, 39 (1966).
[21] J. B. Keller and F. C. Karal, Jr., J. Math. Phys. **7**, 661 (1966).
[22] V. M. Finkel'berg, Zh. Éksp. Teor. Fiz. **53**, 401 (1967) [Sov. Phys. JETP **26**, 268 (1968)].
[23] K. M. Watson, J. Math. Phys. **10**, 688 (1969).
[24] L. L. Rokhlin, Akust. Zh. **18**, 90 (1972) [Sov. Phys. Acoustics **18**, 71 (1972)].
[25] F. E. Stanke and G. S. Kino, J. Acoust. Soc. Am. **75**, 665 (1984).
[26] V. L. Brekhovskikh, Zh. Éksp. Teor. Fiz. **89**, 2013 (1935).
[27] B. Shapiro, Phys. Rev. Lett. **57**, 2168 (1986).
[28] A. G. Fokin and T. D. Shermergor, Izv. Vuzov. Radiofiz. **32**, 176 (1989).
[29] V. L. Brekhovskikh, Izv. Vuzov. Radiofiz. **33**, 680 (1990).
[30] A. G. Fokin, Zh. Éksp. Teor. Fiz. **101**(1), 67 (1992) [Sov. Phys. JETP **74**, 36 (1992)].
[31] R. C. Bourret, Nuovo Cimento **26**, No. 1, 1 (1962).
[32] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[33] A. G. Fokin, Phys. Status Solidi (b) **119**, 741 (1983).
[34] A. G. Fokin, Zh. Éksp. Teor. Fiz. **104**, 3178 (1993) [JETP **77**, 492 (1993)].
[35] L. A. Apresyan, Izv. Vuzov. Radiofiz. **17**, 165 (1974).
[36] A. G. Fokin, Izv. Vuzov. Radiofiz. **37**, 407 (1994).

Translated by David L. Book