Hydrodynamic accretion onto black holes

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We consider adiabatic accretion onto a black hole in the hydrodynamic approximation with zero viscosity. For a slowly rotating black hole, the desired solution is a perturbation of a spherically symmetric flow. We obtain an analytic solution for such a perturbation, and find the correction to the accretion rate and the shape of the sonic surface. We take a similar approach to the accretion of matter onto a slowly moving black hole. Our fundamental result is that neither black hole rotation nor slow motion alters the essential nature of accretion. As before, this result can be extended to supersonic velocities in a collisionless flow, with the shape of the sonic surface and the accretion rate remaining virtually unchanged.

1. INTRODUCTION

The accretion of matter onto black holes remains a classic problem of contemporary astrophysics,[1,2] bearing as it does on the related problems of active galactic nuclei and quasars, the mechanism of jets, and the nature of certain galactic x-ray sources.[3,4]

Historically, the first area to be considered was isentropic hydrodynamic flow for a polytropic equation of state (Bondi–Hoyle accretion[5,6]), where it was shown that subsonic flow far from a black hole will inevitably become supersonic, and that the requirement of a smooth traversal of the sonic surface uniquely specifies the accretion rate as a function of two thermodynamic variables, such as the density and temperature of the gas at infinity.

We note immediately that the isentropic approximation provides no way to allow for the interaction of matter with intrinsic radiation. Moreover, it has been shown that the radiative luminosity associated with the adiabatic heating of accreting matter falls far short of the Eddington limit, which also enables one to treat the entropy of the matter as being constant.[2] Systematic discussions of radiative effects in spherically symmetric accretion can be found in Refs. 7–9.

Furthermore, even for adiabatic zero-viscosity flows, solutions have thus far been obtained only in a number of special cases, such as spherically symmetric accretion onto a nonrotating (Schwarzschild) black hole.[1,2,6] In the latter case, the structure of the flow has in fact been determined (the flow is radial), and the existence of integrals of the motion (the Bernoulli integral and the entropy) makes it possible to determine the flow characteristics completely.

In the general case, on the other hand, the flow structure ought to be derivable from the requirement for equilibrium of the streamlines, which reduces to a nonlinear partial differential equation of mixed type—of the form of the Grad–Shafronov equation[10,11]—that changes from elliptic to hyperbolic at the sonic surface. Except for the degenerate case, in which the speed of sound c_s equals the speed of light[12] (and of course the case of spherically symmetric flow), there have been no analytic solutions of this equation. As a result, all analyses thus far have relied either on self-similar solutions[13] or numerical modeling.[14–18]

It would be reasonable to suppose, on the other hand, that a small deviation from spherical symmetry, which for example might result from rotation of the black hole, should not significantly alter the flow structure. One could then look for a solution of the Grad–Shafronov equation that merely entails a small correction to the spherically symmetric solution, and the equation could be linearized in terms of the small perturbation parameter, thereby making a systematic analysis feasible. In particular, as we shall see, one could then easily determine the correction to the accretion rate and shape of the sonic surface.

In Sec. 2 of this paper, we derive the basic equation (equation of equilibrium) for the structure of isentropic hydrodynamic accretion of matter with zero angular momentum onto a rotating (Kerr) black hole. Several exact solutions of the equation are analyzed in Sec. 3. We show in Sec. 4 how the existence of a small parameter enables one to construct a solution for the accretion of matter onto a slowly rotating black hole. In Sec. 5, we develop an approach that can be applied to accretion onto a black hole moving slowly with respect to the ambient medium.

One important and fundamental result is that neither rotation nor slow motion of the black hole substantially alters the nature of accretion near a black hole. As before, this result can be extended to supersonic velocities in a collisionless flow, with the shape of the sonic surface and the accretion rate remaining virtually unchanged. The solution obtained for a moving black hole is essentially identical with the solution previously obtained through numerical modeling.[14,16]

2. BASIC EQUATIONS

We consider the axisymmetric accretion of matter onto a rotating (Kerr) black hole, for which the metric can be conveniently chosen in Boyer–Lindquist coordinates:[19]

$$ds^2 = -\alpha^2 dt^2 + g_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt), \tag{1}$$

where

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$$\alpha=\frac{\rho}{\Sigma}\sqrt{\Delta}, \tag{2}$$

$$\beta^r=\beta^\theta=0, \quad \beta^\varphi=-\omega=-2a\mathscr{M}r/\Sigma^2, \tag{3}$$

(ω is the Lense–Thirring angular velocity),

$$g_{rr}=\rho^2/\Delta, \quad g_{\theta\theta}=\rho^2, \quad g_{\varphi\varphi}=\varpi^2, \tag{4}$$

and

$$\Delta=r^2+a^2-2\mathscr{M}r, \quad \rho^2=r^2+a^2\cos^2\theta,$$

$$\Sigma^2=(r^2+a^2)^2-a^2\Delta\sin^2\theta, \quad \varpi=\frac{\Sigma}{\rho}\sin\theta. \tag{5}$$

Here, as usual, $\mathscr{M}$ and a are the black hole mass and the angular momentum per unit mass, i.e., $a=J/\mathscr{M}$, and we use a system of units in which $c=G=1$. Note that $\alpha=0$ at the black hole horizon.

We choose a coordinate system identified with a set of so-called reference observers[19] located at a constant distance from the black hole $r=$ const, $\theta=$const, but revolving at the Lense–Thirring angular velocity $dx^\varphi/dt=\omega=-\beta^\varphi$. The spatial metric of the reference observers is the same as the metric g_{ik} in Eq. (4). To determine the structure of the flow, it is then convenient to introduce the stream function $\Phi(r,\theta)$, with

$$\alpha n\mathbf{u}_p=\frac{1}{2\pi\varpi}[\nabla\Phi\times\mathbf{e}_\varphi]. \tag{6}$$

Here n is the density of the medium in the comoving coordinate system, $\mathbf{u}_p$ is the poloidal component of the four-velocity u^i (measured by the reference observers), and $\mathbf{e}_\varphi$ is the toroidal unit vector. Level curves of $\Phi(r,\theta)$ = const directly determine the streamlines of matter, and since $d\Phi=\alpha n\mathbf{u}d\mathbf{S}$, $\Phi(r,\theta)$ is the flux of matter through the surface bounded by $r=$const, $\theta=$const, $0<\varphi<2\pi$. In particular, the total flux through a sphere of radius r is $\Phi(r,\pi)$. Given Eq. (6), the equation of continuity $\nabla(\alpha n\mathbf{u})=0$ is then automatically satisfied. Hereafter, ∇ denotes the covariant derivative in the metric g_{ik} of (4).

A general axisymmetric, isentropic, nonviscous hydrodynamic flow can be characterized by three integrals of the motion, which are constant on the streamlines $\Phi(r,\theta)=$const.[11] The first two follow directly from energy–momentum conservation ($T^{ik}_{;k}=0$) for $i=0$ and $i=\varphi$. These are the Bernoulli integral

$$E(\Phi)=\mu(\alpha\gamma+\omega u_\varphi)$$

and the z component of the angular momentum,

$$L(\Phi)=\mu\omega u_\varphi.$$

Furthermore, for an isentropic flow, the enropy $s=s(\Phi)$ will also be constant. Here $\mu=(P+\varepsilon_p)/n$ is the specific enthalpy (P is the pressure and ε_p is the internal energy) and $\gamma^2=1+u_\varphi^2+\mathbf{u}_p^2$. In the present paper, however, we restrict the discussion to flows with zero angular momentum,

$$L(\Phi)\equiv 0, \tag{7}$$

so that $u_\varphi\equiv 0$. We then have

$$\gamma^2=1+\mathbf{u}_p^2. \tag{8}$$

We emphasize that $u_\varphi\equiv 0$ means that the toroidal velocity of matter with respect to distant observers corresponds directly to revolution at the Lense–Thirring angular velocity ω of (3). Accordingly, the Bernoulli integral can then be rewritten in the form

$$E=\alpha\gamma\mu, \tag{9}$$

so that

$$\mathbf{u}_p^2=\frac{E^2-\alpha^2\mu^2}{\alpha^2\mu^2}. \tag{10}$$

We see that the square of the four-velocity $\mathbf{u}_p^2$ diverges as α^{-2} near the black hole horizon. Finally, we assume in this paper that the integrals of the motion $E(\Phi)$ and $s(\Phi)$ are constant throughout all space, and equal their respective values at infinity. Since $\alpha(\infty)=1$, we have

$$E(\Phi)=\mu_\infty\gamma_\infty, \quad s(\Phi)=s_\infty. \tag{11}$$

For its part, the stream function $\Phi(r,\theta)$ in the current approximation $dE/d\Phi=0$, $ds/d\Phi=0$, $L=0$ satisfies the equation of equilibrium,

$$-\alpha\varpi^2\nabla_k\left(\frac{1}{\alpha\varpi^2}\nabla^k\Phi\right)-\frac{1}{D}\frac{\nabla^i\Phi\nabla^k\Phi\nabla_i\nabla_k\Phi}{(\nabla\Phi)^2}+\frac{1}{2D}$$
$$\times\frac{\nabla_k\varpi^2\nabla^k\Phi}{\varpi^2}-\frac{1}{2D}\frac{\mu^2\nabla_k\alpha^2\nabla^k\Phi}{E^2-\alpha^2\mu^2}=0, \tag{12}$$

where

$$D=-1+\frac{\alpha^2\mu^2}{E^2-\alpha^2\mu^2}\frac{c_s^2}{1-c_s^2}\equiv-1+\frac{1}{\mathbf{u}_p^2}\frac{c_s^2}{1-c_s^2}, \tag{13}$$

and

$$c_s^2=\frac{1}{\mu}\left(\frac{\partial P}{\partial n}\right)_s, \tag{14}$$

c_s is the speed of sound. Equation (12) follows directly from the general hydrodynamic equation considered in Ref. 11, which in turn comes from the energy–momentum conservation law $\Phi_{;\lambda}T^{\lambda\nu}_{;\nu}/(\nabla\Phi)^2=0$. It is a quasilinear equation of mixed type—elliptic in the subsonic region $D>0$ and hyperbolic in the supersonic region $D<0$. Equation (12) must naturally be supplemented by the equation of state

$$P=P(n,s). \tag{15}$$

We stress here that the equilibrium equation (12) in fact contains only the stream function $\Phi(r,\theta)$. All other thermodynamic quantities, such as the speed of sound c_s and the specific enthalpy μ must be expressible in terms of the integrals of the motion E and s [Eq. (11)], as well as in terms of the stream function Φ. Indeed, making use of (6) and the explicit expression (9) for the Bernoulli integral E, we can rewrite Eq. (8), $\gamma^2=1+\mathbf{u}_p^2$, in the form[11]

$$M^4=\frac{64\pi^4\varpi^2(E^2-\alpha^2\mu^2)}{(\nabla\Phi)^2}, \tag{16}$$

where

$$M^2=4\pi\mu/n. \quad (17)$$

It is clear from the definition (17) that M^2 is a purely thermodynamic function, so that like the specific enthalpy μ it can be expressed in terms of two thermodynamic variables, such as the speed of sound c_s and the entropy s:

$$M^2=M^2(c_s,s), \quad \mu=\mu(c_s,s). \quad (18)$$

Specifically, at constant entropy,[11]

$$dM^2=-\frac{M^2}{\mu}\frac{1-c_s^2}{c_s^2}\,d\mu. \quad (19)$$

Ultimately, Eq. (16) makes it possible to express all thermodynamic quantities (albeit not explicitly) in terms of the derivative $(\nabla\Phi)^2$, as well as in terms of the two integrals of the motion E and s (11).

Following tradition, we now consider the polytropic equation of state

$$P=kn^{\Gamma}, \quad (20)$$

with $k=k(s)=$ const and $1<\Gamma\leqslant 5/3$ the polytropic index. As is well known,[1,2]

$$\mu=m_p+\frac{\Gamma}{\Gamma-1}\,k(s)n^{\Gamma-1}, \quad (21)$$

m_p is the mass of a single particle, and

$$c_s^2=\frac{\Gamma}{\mu}\,kn^{\Gamma-1}\equiv\frac{1}{\mu}\,(\Gamma-1)(\mu-m_p), \quad (22)$$

so

$$\mu(c_s)=\frac{m_p}{1-c_s^2/(\Gamma-1)}. \quad (23)$$

We will also need an expression for the derivative $dc_s^2/d\mu$:

$$\frac{\mu}{c_s^2}\frac{dc_s^2}{d\mu}=\frac{\Gamma-1-c_s^2}{c_s^2}. \quad (24)$$

As a result, Eqs. (19) and (24) enable us to derive explicit expressions for the thermodynamic quantities M^2 and μ as functions of the speed of sound c_s. For an arbitrary case with $c_s^2<\Gamma-1$, we have

$$\mu(c_s)=\mu_\infty\,\frac{\Gamma-1-c_\infty^2}{\Gamma-1-c_s^2}, \quad (25)$$

$$M^2(c_s)=M_\infty^2\left(\frac{c_s^2}{c_\infty^2}\right)^{-1/\Gamma-1}\left(\frac{\Gamma-1-c_s^2}{\Gamma-1-c_\infty^2}\right)^{(2-\Gamma)/(\Gamma-1)}, \quad (26)$$

where the subscript "∞" again corresponds to the value of the quantity in question at infinity.

Here we note that, as follows directly from the definition (17) (further detail can be found in Ref. 11), the gradient $\nabla_k M^2$ can be written in the form

$$\nabla_k M^2=-M^2\,N_k/D, \quad (27)$$

where

$$-N_k=\frac{\nabla^i\Phi\nabla_i\nabla_k\Phi}{(\nabla\Phi)^2}-\frac{1}{2}\frac{\nabla_k\varpi^2}{\varpi^2}+\frac{1}{2}\frac{\mu^2\nabla_k\alpha^2}{E^2-\alpha^2\mu^2}, \quad (28)$$

and D is given by Eq. (13). The prerequisite for $\nabla_i M^2$ to be regular at the sonic point $D=0$,

$$N_r=0, \quad N_\theta=0$$

leads immediately to an additional relationship that governs the accretion rate. In the case we consider here, with $E=$const and $s=$const, the equilibrium equation (12) can be rewritten more compactly,

$$\alpha\varpi^2\nabla_k\left(\frac{M^2}{\alpha\varpi^2}\,\nabla^k\Phi\right)=0, \quad (29)$$

but this still does not explicitly define the second-order operator.

3. SPHERICALLY SYMMETRIC ACCRETION AND OTHER EXACT SOLUTIONS

We now present the basic relations for spherically symmetric accretion of matter onto a nonrotating (Schwarzschild) black hole, which we will need below. The velocity of the medium at infinity is naturally set to zero, so that $\gamma_\infty=1$. The flow is completely defined by two constants—for example, two of the thermodynamic functions at infinity, s_∞ and μ_∞, which according to (11) also determine the value of the Bernoulli integral E. The stream function Φ is then trivially given by

$$\Phi=\Phi_0(1-\cos\,\theta). \quad (30)$$

It is easily shown that this function (30) is in fact a solution of the equilibrium equation (12) for any constant Φ_0. The actual value of Φ_0 (and the accretion rate $\Phi(\pi)=2\Phi_0$) is dictated by requiring a smooth traversal of the sonic point $D=0$, rather than by solving the equilibrium equation (12).

In point of fact, we may write $D(r_0,c_0)=0$ and $N_r(r_0,c_0)=0$. Hereafter, a subscript 0 refers to a value at the sonic point in spherically symmetric accretion. For the spherically symmetric case in the Schwarzschild metric $a=0$, these take the form

$$D=-1+\frac{\alpha_0^2\mu_0^2}{E^2-\alpha_0^2\mu_0^2}\frac{c_0^2}{1-c_0^2}=0, \quad (31)$$

$$N_r=\frac{2}{r_0}-\frac{1-c_0^2}{c_0^2\alpha_0^2}\frac{\mathscr{M}}{r_0^2}=0. \quad (32)$$

The requirement that $N_\theta(r_0,c_0)=0$ is automatically satisfied in the spherically symmetric case, since $N_\theta\equiv 0$. In addition, we can make use of (16), which by virtue of (30) and (31) at the sonic point takes the form

$$M_0^2=\frac{8\pi^2 r_0^2 E c_0}{|\Phi_0|}. \quad (33)$$

In conjunction with (31)–(33) the definition (9) and the thermodynamic relations (25) and (26), the algebraic equations (31)–(33) immediately enable one to determine all of the fundamental characteristics of the flow. For example, the radius of the sonic surface in the present case ($c_s^2\leqslant 1$) is

$$r_0=\frac{\mathscr{M}}{2}\left(\frac{1}{c_0^2}+3\right), \quad (34)$$

so that when $c_0^2 \ll 1$,

$$r_0 = \frac{\mathcal{M}}{2c_0^2}.$$

The relationship between c_0^2 and c_∞^2 can be obtained by combining (11), (13), and (25). The result is

$$\mu_0^2 = \frac{\mu_\infty^2(1-c_0^2)}{\alpha_0^2},$$

and consequently, by virtue of (34),

$$(\Gamma-1-c_\infty^2)^2 = (\Gamma-1-c_0^2)^2(1+3c_0^2).$$

In the limiting case $c_0^2 \ll 1$, we obtain the well known relations[1,2]

$$c_0^2 = \frac{2}{5-3\Gamma} c_\infty^2, \quad \Gamma \neq 5/3, \tag{35}$$

$$c_0^2 = \frac{2}{3} c_\infty^2, \quad \Gamma \neq 5/3.$$

Similarly,

$$M_0^2 = M_\infty^2 \left(\frac{c_\infty^2}{c_0^2}\right)^{1/(\Gamma-1)} \left(\frac{\Gamma-1-c_0^2}{\Gamma-1-c_\infty^2}\right)^{(2-\Gamma)/(\Gamma-1)}, \tag{36}$$

$$\mu_0 = \mu_\infty \frac{\Gamma-1-c_\infty^2}{\Gamma-1-c_0^2}. \tag{37}$$

Finally, the accretion rate can be written in the form $2m_p\Phi_0$, where

$$\Phi_0 = -\frac{8\pi^2 r_0^2 E c_0}{M_0^2}. \tag{38}$$

Having specified two thermodynamic functions at infinity, for example $c_\infty^2(\mu_\infty, s_\infty)$ and $M_\infty^2(\mu_\infty, s_\infty)$, we can thus use (21), (22), (35)–(37), and (16) to determine the thermodynamic functions $\mu(r)$, $M^2(r)$, and $c_s^2(r)$ at any point in space. At $r \ll r_0$, for example, we obtain

$$\frac{M^2}{M_0^2} \simeq 2\left(\frac{r}{r_0}\right)^{3/2}, \quad \frac{c_s^2}{c_0^2} \simeq \frac{1}{2^{\Gamma-1}}\left(\frac{r}{r_0}\right)^{-(3/2)(\Gamma-1)}. \tag{39}$$

In particular, at the black hole horizon, where $\alpha=0$ and $r=2\mathcal{M}$, we have

$$c_g^2 = \frac{1}{16^{\Gamma-1}} c_0^{5-3\Gamma}, \quad M_g^2 = 16 c_0^3 M_0^2. \tag{40}$$

Consequently, when $c_\infty^2 \ll 1$ (and therefore $c_0^2 \ll 1$), the speed of sound remains small compared to the speed of light all the way out to the black hole horizon.

We now note two more cases in which the equilibrium equation (12) can be solved exactly. First and foremost, we can obtain such a solution for dusty matter with $P=0$, for which $c_s^2 \equiv 0$ and therefore $D=-1$. That matter should move in exactly the same way as test particles at rest at infinity that possess no angular momentum with respect to the black hole. It is well known (see Ref. 20, for example) that for an arbitrary black hole rotation rate, such motion can be directed precisely along a radius, thereby yielding streamlines with $\theta=$ const. The density of those streamlines in this pressure-free case is also arbitrary.

It can easily be shown that the arbitrary function $\Phi=\Phi(\theta)$ $(c_s^2=0)$ is, for $D=-1$, a solution of the nonlinear equation of equilibrium at any black hole rotation rate, no matter how high. In fact, since when $c_s^2=0$ we have

$$\mu = \mu_\infty = E = \text{const}$$

throughout all space, the general relation (16) yields

$$M^2(r,\theta) = \frac{8\pi^2 \varpi \rho E\sqrt{1-\alpha^2}}{\partial\Phi/\partial\theta} = \frac{8\pi^2 E \sin\theta}{\partial\Phi/\partial\theta} \times \sqrt{2\mathcal{M}r(a^2+r^2)}. \tag{41}$$

In this last equation, we have made use of the explicit expressions (2), (4), and (5) for the metric coefficients ϖ, ρ, and α. Inserting $M^2(r,\theta)$ from (41) into Eq. (29), we obtain

$$\alpha\varpi^2 \nabla_k\left(\frac{M^2}{\alpha\varpi^2}\nabla^k\Phi\right) \propto \frac{\partial}{\partial\theta}\left[\sqrt{\frac{g_{rr}g_{\varphi\varphi}}{g_{\theta\theta}}}\frac{M^2}{\alpha\varpi^2}\frac{\partial\Phi}{\partial\theta}\right] \propto \frac{\partial}{\partial\theta}\left\{\frac{\sqrt{2\mathcal{M}r(a^2+r^2)}}{\Delta}\right\} \equiv 0.$$

Thus, with $c_s^2=0$ in the stationary case at hand, the accretion rate $\Phi(\pi)$ is completely arbitrary. This should come as no surprise: given that $D=-1$, the flow remains supersonic throughout all space, and therefore no additional constraint on the sonic surface emerges. In the more general case of a nonstationary flow, the accretion rate itself will naturally be time-dependent (see Ref. 1, for example).

Finally, a solution can also be obtained in the case $c_s^2 \equiv 1$, for which, according to (13), we have $D^{-1}=0$. The equilibrium equation (12) then becomes linear,

$$\alpha\varpi^2\nabla_k\left(\frac{1}{\alpha\varpi^2}\nabla^k\Phi\right) = 0,$$

or equivalently

$$\frac{\Delta}{\rho^2}\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{\rho^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\theta}\right) = 0. \tag{42}$$

The solution of this linear equation (42) can be expanded in eigenfunctions $Q_n(\theta)$ of the angular operator

$$L_\theta = \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),$$

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} g_n(r) Q_n(\theta), \tag{43}$$

where

$$Q_0 = 1-\cos\theta,$$

$$Q_n = \frac{2^n n!(n-1)!}{(2n)!}\sin^2\theta P_n'(\cos\theta), \quad n=1,2,\ldots \tag{44}$$

(the P_n are Legendre polynomials). Specifically,

$$Q_1 = \sin^2\theta, \tag{45}$$

$$Q_2=\sin^2\theta\cos\theta. \quad (46)$$

Here the term proportional to $g_0(1-\cos\theta)$ corresponds to the case of spherically symmetric accretion. Far from the black hole, $r\gg\mathscr{M}$, the term proportional to $g_1(r)\sin^2\theta$ can easily be shown by direct recourse to Eq. (42) to take the form

$$\Phi=\Phi_1 r^2\sin^2\theta, \quad (47)$$

which corresponds to a uniform flux of matter, with

$$\Phi_1=\pi n_\infty v_\infty. \quad (48)$$

Therefore, for a black hole moving at velocity v_∞ through a medium with $c_s^2=1$, for example, we must compare the two leading terms in the expansion (43). Thus, we have for a Schwarzschild black hole

$$\Phi=\Phi_0(1-\cos\theta)+\Phi_1(r^2-2\mathscr{M}r)\sin^2\theta, \quad (49)$$

where Φ_1 is given by (48). We see that when $c_s^2=1$, the accretion rate $2\Phi_0$ is also arbitrary, since the flow remains subsonic out to the black hole horizon. A solution equivalent to (49) was first obtained in Ref. 12 for an arbitrary black hole rotation rate.

4. ACCRETION ONTO A SLOWLY ROTATING BLACK HOLE

We now consider accretion onto a slowly rotating black hole for which $a\ll\mathscr{M}$. Let

$$\varepsilon=a/\mathscr{M}, \quad (50)$$

then $\varepsilon\ll 1$. We can thus assume that the thermodynamic functions of the medium at infinity, μ_∞ and s_∞, are the same as the characteristics of the medium considered in the case of spherically symmetric accretion.

We see from the definitions (4) and (5) that when $\varepsilon\ll 1$, the corrections to the metric coefficient g_{ik} and thus to the equilibrium equation (12) are of order ε^2. It is therefore reasonable to seek a solution of the equilibrium equation (12) that is a small correction to the spherically symmetric solution of (30),

$$\Phi=\Phi_0[1-\cos\theta+\varepsilon^2 f(r,\theta)], \quad (51)$$

where as before, Φ_0 is given by (38). Inserting (51) into (12) and collecting terms proportional to ε^2 and a^2, we obtain

$$-\varepsilon^2\alpha^2 D\frac{\partial^2 f}{\partial r^2}-\frac{\varepsilon^2}{\rho^2}(D+1)\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right)+\varepsilon^2\alpha^2 N_r\frac{\partial f}{\partial r}=\frac{a^2}{r^4}\left(1-\frac{2\mathscr{M}}{r}\right)\times\left(1-2\frac{\mu^2}{E^2-\alpha^2\mu^2}\frac{\mathscr{M}}{r}\right)\sin^2\theta\cos\theta, \quad (52)$$

where

$$N_r=\frac{2}{r}-\frac{\mu^2}{E^2-\alpha^2\mu^2}\frac{\mathscr{M}}{r^2}. \quad (53)$$

Here we see that μ, D, and N_r are to be evaluated using the unperturbed stream function $\Phi=\Phi_0(1-\cos\theta)$ in the Schwarzschild metric $a=0$. Thus, in particular, $\alpha^2=1-2\mathscr{M}/r$, $\rho=r$, while the functions D and N_r do not depend on the angular variable θ. The singularity in (52) then coincides with the sonic surface $r=r_0$ [Eq. (34)] given by the spherically symmetric solution with the same values of μ and s at infinity. According to (31) and (32), we therefore find that $D(r_0)=0$ and $N_r(r_0)=0$ to zeroth order in ε^2.

We note above all that by virtue of Eq. (13),

$$D+1=\frac{\alpha^2\mu^2}{E^2-\alpha^2\mu^2}\frac{c_s^2}{1-c_s^2}, \quad (54)$$

so the relativistic factor α^2 appears in all terms of Eq. (52). We can thus conclude that the linearized equilibrium equation (52) gives rise to no singularities at the black hole horizon. On the other hand, we see that the angular operator in Eq. (52) is the same as the one in Eq. (42), and thus we can seek a solution of (52) in the form

$$f(r,\theta)=\sum_{n=0}^{\infty} g_n(r)Q_n(\theta), \quad (55)$$

where the eigenfunctions $Q_n(\theta)$ are given, as before, by Eqs. (44)–(46). Since $Q_n(\pi)=0$ $(n\neq 0)$, the total accretion rate will be determined solely by the zeroth harmonic $g_0(1-\cos\theta)$.

Substituting the expansion (55) into Eq. (52), we now obtain a set of ordinary differential equations for the radial functions $g_n(r)$:

$$-D\frac{d^2 g_n}{dr^2}+N_r\frac{dg_n}{dr}-C_n\frac{\mu^2}{E^2-\alpha^2\mu^2}\frac{c_s^2}{1-c_s^2}\frac{g_n}{r^2}=0, \quad n\neq 2, \quad (56)$$

$$-D\frac{d^2 g_2}{dr^2}+N_r\frac{dg_2}{dr}+6\frac{\mu^2}{E^2-\alpha^2\mu^2}\frac{c_s^2}{1-c_s^2}\frac{g_2}{r^2}=\frac{\mathscr{M}^2}{r^4}\left(1-2\frac{\mu^2}{E^2-\alpha^2\mu^2}\frac{\mathscr{M}}{r}\right), \quad n=2. \quad (57)$$

In Eq. (56), the C_n are eigenvalues of the operator L_θ ($C_0=0$, $C_1=-2$,...). As we have already noted, all quantities in (56) and (57) such as $D(r)$, $N_r(r)$, $\mu(r)$, $c_s^2(r)$, and $\alpha(r)$ should be taken from the spherically symmetric solution. For the polytropic equation of state (20), these can easily be determined using the algebraic relations (16), (25), (26), and (30).

For example, with $c_s^2\ll 1$, $\mu\simeq m_p$, and $\Gamma\neq 5/3$, Eq. (57) in dimensionless variables

$$y=(r_0^2/\mathscr{M}^2)g_2, \quad x=r/r_0, \quad u=M^2/M_0^2, \quad v=c_s^2/c_0^2$$

becomes

$$\left(1-\frac{x^4 v}{u^2}\right)\frac{d^2y}{dx^2}+2\left(\frac{1}{x}-\frac{x^2}{u^2}\right)\frac{dy}{dx}+6\frac{x^2 v}{u^2}y-\frac{1}{x^4}+\frac{4}{xu^2}=0, \quad (58)$$

where according to (26),

$$v=u^{-(\Gamma-1)},$$

and by virtue of (27) and (28), the dimensionless quantity $u(x)$ satisfies

$$\frac{du}{dx}=2\,\frac{u}{x}\,\frac{u^2-x^3}{u^2-x^4v},$$

with[2]

$$u(1)=1,\quad \left.\frac{du}{dx}\right|_{x=1}=\frac{4-\sqrt{10-6\Gamma}}{\Gamma+1}.$$

We now determine the boundary conditions on Eqs. (56) and (57). Most importantly, we see immediately from the two equations that the requirement that $\Phi(r,\theta)$ be regular at the sonic point $r=r_0$ leads to the following conditions on the radial functions g_n:

$$g_2(r_0)=-\frac{1}{2}\frac{\mathcal{M}^2}{r_0^2}\,\alpha_0^2,\quad g_n(r_0)=0,\quad n\neq 0,2. \tag{59}$$

It can easily be verified that Eqs. (58) and (59) follow directly from the regularity condition $N_\theta(r_0)=0$. We see that for $c_\infty^2\ll 1$ (and hence for $r_0\gg\mathcal{M}$), the correction to the spherically symmetric solution at the sonic point has, apart from a multiplier ε^2, an additional small factor $(\mathcal{M}/r_0)^2$. As for the radial function g_0 (which by virtue of Eq. (58) for $n=0$ should not depend on the radius, $g_0=\text{const}$), the vanishing of the eigenvalue C_0 means that it cannot be determined from Eq. (56).

Accordingly, the prerequisite for the solution to be regular at infinity can be written in the form

$$\lim_{r\to\infty} g_n(r)=0,\quad n\neq 0. \tag{60}$$

Since as is easily verified, the two asymptotic solutions of Eqs. (56) and (57) are

$$g_n^{(1)}\propto r^{n+1},\qquad g_n^{(2)}\propto r^{-n}, \tag{61}$$

[cf. (47)], so that regularity at infinity requires that we take the decreasing solution $\propto r^{-n}$. Equations (58)–(60) also make it possible to completely determine the radial functions $g_n(r)$ for $n\neq 0$.

More than anything else, Eqs. (59) and (60) show that all of the radial functions except g_0 and $g_2(r)$ vanish. The latter $g_2(r)$ must be chosen in such a way as to yield a decreasing solution $\propto r^{-2}$ as $r\to\infty$. An analysis of Eqs. (57) and (58) shows that at distances $r\ll r_0$,

$$g_2(r)\propto r^{(1-3\Gamma)/2}.$$

The radial function therefore takes the convenient form

$$g_2(r)=-G(r)\,\frac{\mathcal{M}^2}{r_0^2}\left(\frac{r}{r_0}\right)^{(1-3\Gamma)/2}, \tag{62}$$

where, according to (59)

$$G(r_0)=\ \alpha_0^2/2. \tag{63}$$

In Fig. 1, we show the behavior of the function $G(r)$ obtained by numerically solving Eq. (58) with boundary conditions (59) and (60) for the case $c_s^2\ll 1$, and for various

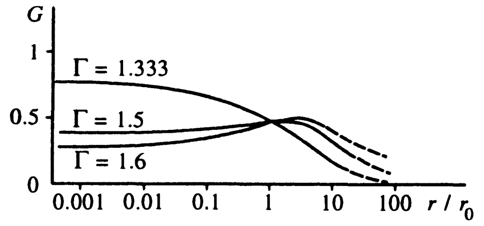

FIG. 1. The behavior of $G(r)$ for various values of Γ.

values of the polytropic index Γ. We see that on the whole, $g_2(r)$ retains its power-law behavior over the full range of radii, $2\mathcal{M}<r<r_0$. Accordingly, the derivative $g_2'(r_0)$ takes the convenient form

$$g_2'(r_0)=k_2(\Gamma)\ \mathcal{M}^2/r_0^3, \tag{64}$$

where the numerical coefficient $k_2(\Gamma)$, which is of order unity, also depends on the polytropic index Γ. Table I lists some values of $k_2(\Gamma)$.

We note finally that, as is clear from Eq. (62), $g_2(2\mathcal{M})\simeq(\mathcal{M}/r_0)^{(5-3\Gamma)/2}$. The largest perturbation of the stream function $\Phi_0(r)$ therefore takes place near the black hole horizon, where the correction, however, is at most of order $\sim\varepsilon^2(\mathcal{M}/r_0)^{(5-3\Gamma)/2}$. Hence, the perturbation remains small over all space outside the black hole.

We now determine the constant g_0, which governs the correction to the accretion rate. To do so, we consider in more detail the behavior of the solution near the sonic surface $r=r_*(\theta)$, whose radius, by virtue of our assumption that the thermodynamic functions match at infinity, ought to differ from r_0 by a quantity of order $\sim\varepsilon^2$. We can therefore write

$$r_*(\theta)=r_0[1+\varepsilon^2 d(\theta)]. \tag{65}$$

It would be reasonable to suppose that the thermodynamic functions $M_*^2(\theta)$, $\mu_*(\theta)$, and $c_*^2(\theta)$ at the sonic point differ from the corresponding values M_0^2, μ_0, and c_0^2 given by (35)–(37) for the spherically symmetric solution by a quantity of order ε^2 as well. Hence, we can write

$$c_*^2(\theta)=c_0^2[1+\varepsilon^2 b(\theta)], \tag{66}$$

$$M_*^2(\theta)=M_0^2[1+\varepsilon^2 q(\theta)], \tag{67}$$

$$\mu_*(\theta)=\mu_0[1+\varepsilon^2 p(\theta)]. \tag{68}$$

To determine the four dimensionless functions $d(\theta)$, $q(\theta)$, $p(\theta)$, and $b(\theta)$, it is then sufficient to make use of the two thermodynamic relations (19) and (24), which yield

$$p+\frac{c_0^2}{1-c_0^2}\,q=0, \tag{69}$$

TABLE I.

Γ	1.01	1.1	1.2	1.333	1.5	1.6
$k_2(\Gamma)$	0.858	0.858	0.859	0.861	0.872	0.895

$$b-\frac{\Gamma-1-c_0^2}{c_0^2}p=0, \quad (70)$$

plus the exact relations $D(r_*)=0$, $N_r(r_*)=0$, in which we need to expand terms out to quantities of order ε^2. The end result is

$$\frac{c_0^2}{1-c_0^2}b+\frac{2\mathcal{M}}{r_0\alpha_0^2}d+2p=-2\frac{\mathcal{M}^3}{r_0^3}\left(\frac{1}{\alpha_0^2}-\sin^2\theta\right), \quad (71)$$

$$\frac{2}{1-c_0^2}b+\frac{2}{\alpha_0^2}d=\frac{r_0}{\sin\theta}\frac{\partial^2 f}{\partial r\partial\theta}\bigg|_{r=r_0}-\frac{\mathcal{M}^2}{r_0^2}\left(\frac{4\mathcal{M}}{r_0\alpha_0^2}-1+5\cos^2\theta+9\frac{\mathcal{M}}{r_0}\sin^2\theta\right). \quad (72)$$

Since $g_0=\text{const}$ here, the only harmonic that enters into the derivative

$$\frac{\partial^2 f}{\partial r\partial\theta}\bigg|_{r=r_0}$$

is $g_2(r)\sin^2\theta\cos\theta$. From (64), we then have

$$\frac{r_0}{\sin\theta}\frac{\partial^2 f}{\partial r\partial\theta}\bigg|_{r=r_0}=\frac{\mathcal{M}^2}{r_0^2}k_2(\Gamma)(3\cos^2\theta-1).$$

To summarize, we can write the solution of Eqs. (69)–(72) with $c_s\ll 1$ ($\mathcal{M}/r_0\ll 1$) in the form

$$b(\theta)=\frac{4\mathcal{M}^2(\Gamma-1)(1-k_2)}{D_1r_0^2}(3\cos^2\theta-1), \quad (73)$$

$$d(\theta)=-\frac{\mathcal{M}^2}{D_1r_0^2}[(10-6\Gamma)\cos^2\theta+(1-k_2)(\Gamma+1)\times(3\cos^2\theta-1)], \quad (74)$$

$$p(\theta)=\frac{2\mathcal{M}^3(1-k_2)}{D_1r_0^3}(3\cos^2\theta-1), \quad (75)$$

$$q(\theta)=-\frac{4\mathcal{M}^2(1-k_2)}{D_1r_0^2}(3\cos^2\theta-1), \quad (76)$$

where

$$D_1=10-6\Gamma+18c_0^2\simeq\begin{cases}10-6\Gamma, & \Gamma\neq 5/3\\ 12c_\infty, & \Gamma=5/3.\end{cases} \quad (77)$$

For $\Gamma=5/3$ ($k_2(5/3)\simeq 1$ from Table I), we see that a more accurate expansion in $\mathcal{M}/r_0$ is required.

By way of example, Fig. 2 shows the shape of the sonic surface for $\Gamma=4/3$, $\varepsilon^2\mathcal{M}^2/r_0^2=0.1$. In actual fact, when $\mathcal{M}\ll r_0$, an additional small factor $\mathcal{M}^2/r_0^2$ enters into the expression for d in (74) [as it does for $g_2(r_0)$], so that the sonic surface differs little from a sphere of radius r_0.

As for g_0, we actually require a fifth relationship, which can be derived from (16). Expanding it near the sonic surface to order $\sim\varepsilon^2$, we obtain

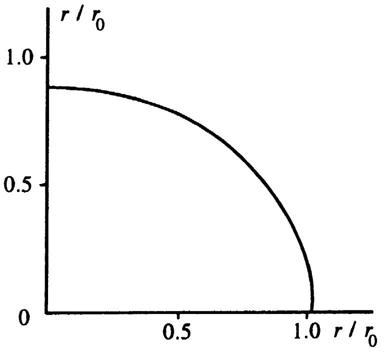

FIG. 2. Shape of the sonic surface for $\Gamma=4/3$, $\varepsilon^2(\mathcal{M}^2/r_0^2)=0.1$.

$$b+4d-2q=\frac{2}{\sin\theta}\frac{\partial f}{\partial\theta}\bigg|_{r=r_0}-\frac{\mathcal{M}^2}{r_0^2}\left(1+\cos^2\theta+2\frac{\mathcal{M}}{r_0}\sin^2\theta\right). \quad (78)$$

On the right-hand side of this equation, due to the term $(1/\sin\theta)(\partial f/\partial\theta)$, we have besides $g_2(r_0)$ the zeroth harmonic g_0:

$$\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\bigg|_{r=r_0}=g_0+g_2(r_0)(3\cos^2\theta-1). \quad (79)$$

It can easily be shown that the left-hand side of Eq. (78) is a linear combination of Eqs. (69)–(72). This is as it should be, since the differential equation (19), which leads to Eq. (69), can be derived directly from (16). For the five equations (69)–(72) and (78) to be compatible, the determinant of the generalized matrix must vanish, implying that

$$\frac{\mathcal{M}^2}{r_0^2}\left(3\cos^2\theta-1+6\frac{\mathcal{M}}{r_0}\sin^2\theta\right)+2g_0+2g_2(r_0)\times(3\cos^2\theta-1)=0, \quad (80)$$

so that

$$g_0=-2\mathcal{M}^3/r_0^3, \quad (81)$$

and the expression for $g_2(r_0)$ is the same as in (59). We see that the expression for g_0 also contains an additional factor $\mathcal{M}/r_0\ll 1$.

We can thus write the complete solution of Eq. (52) that describes axisymmetric accretion onto a slowly rotating black hole:

$$\Phi(r,\theta)=\Phi_0[1-\cos\theta+\varepsilon^2g_0(1-\cos\theta)+\varepsilon^2g_2(r)\sin^2\theta\cos\theta], \quad (82)$$

where Φ_0 is given by (38), and the radial functions g_0 and $g_2(r)$ are given by Eqs. (62) and (81). The solution is completely determined by the two thermodynamic functions s_∞ and μ_∞, which are specified at infinity.

To conclude this discussion, we note that for the physically reasonable condition $c_\infty^2\ll 1$, the radius r_0 of the sonic surface is significantly greater than $2\mathcal{M}$, the radius of the black hole. Rotation effects under these circumstances are exceedingly weak. Likewise, the shape of the sonic surface will be only slightly distorted, so that the effects considered in this section are scarcely of practical interest under realistic conditions. In our opinion, therefore, the fundamental result here is more of a mathematical nature. In fact, it was shown above that the well-known earlier spherically symmetric accretion regime is stable against small perturbations associated with black hole rotation.

5. ACCRETION ONTO A SLOWLY MOVING BLACK HOLE

We now examine the problem of accretion onto a Schwarzschild black hole moving at velocity $v_\infty \ll c_\infty$ relative to the ambient medium. We take as the small parameter here

$$\varepsilon_1 = v_\infty / c_\infty . \tag{83}$$

We transform immediately to the rest frame of the black hole, in which the velocity of the medium is v_∞. As $r \to \infty$, the stream function Φ should display the asymptotic behavior (47) and (48), i.e., it should be completely specified by the first harmonic $g_1(r)\sin^2\theta$. Making use of the definition (38) of Φ_0, that behavior can conveniently be rewritten in the form

$$\Phi = \varepsilon_1 K(\Gamma)\Phi_0 \frac{r^2}{r_0^2} \sin^2 \theta, \tag{84}$$

where

$$K(\Gamma) = \frac{1}{2}\frac{M_0^2}{M_\infty^2}\frac{c_\infty}{c_0} = \begin{cases} \dfrac{1}{2}\left(\dfrac{5-3\Gamma}{2}\right)^{(\Gamma+1)/2(\Gamma-1)}, & \Gamma \neq 5/3, \\ \dfrac{9}{8} c_\infty^2, & \Gamma = 5/3 \end{cases} \tag{85}$$

is a numerical coefficient (for which values are listed in Table II). Clearly the asymptotic function (84) is formally linear in the small parameter (83). On the other hand, the Bernoulli integral $E = \mu_\infty \gamma_\infty$ for $\varepsilon_1 \neq 0$ differs from the value $E^{(0)} = \mu_\infty$ for a black hole at rest by a quantity of order ε_1^2:

$$E = E^{(0)}\left(1 + \frac{1}{2}\varepsilon_1^2 c_\infty^2\right). \tag{86}$$

As in the preceding section, we seek a solution of the equilibrium equation (12) in the form

$$\Phi(r,\theta) = \Phi_0\left[1 - \cos\theta + \varepsilon_1 \sum_{n=0}^{\infty} g_n(r) Q_n(\theta)\right]. \tag{87}$$

We note straightaway that formally the expansion (87) holds only at small distances $r \ll r_m$ from the black hole:

$$r_m = r_0/\sqrt{\varepsilon_1}, \tag{88}$$

since at large distances the perturbed term (84) becomes larger than the term $\Phi_0(1-\cos\theta)$, which describes spherically symmetric accretion. Nevertheless, we will show that the expansion (87) correctly details the behavior of the stream function throughout all space, because the term proportional to ε_1 corresponds exactly to the asymptotic behavior far from the black hole.

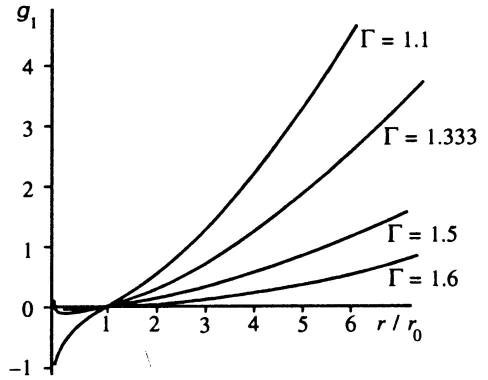

FIG. 3. The behavior of $g_1(r)$ for various values of Γ.

Substituting the expansion (87) into the general equilibrium equation and retaining only terms proportional to ε_1, we obtain for the radial function $g_1(r)$

$$-D\frac{d^2 g_1}{dr^2} + N_r\frac{dg_1}{dr} + 2\frac{\mu^2}{E^2 - \alpha^2\mu^2}\frac{c_s^2}{1-c_s^2}\frac{g_1}{r^2} = 0. \tag{89}$$

Since $D(r_0) = N_r(r_0) = 0$, the regularity condition at the sonic point $r = r_0$ is

$$g_1(r_0) = 0. \tag{90}$$

The boundary condition at infinity, according to (84) and (85), becomes

$$\lim_{r\to\infty} g_1(r) = K(\Gamma)\, r^2/r_0^2. \tag{91}$$

Furthermore, the "radial constant" g_0, as in the case of accretion onto a rotating black hole, cannot be determined from the equilibrium equation. Finally, by virtue of the regularity condition $g_n(r_0) = 0$ at the sonic point and the limit (60) at infinity, the remaining radial functions $g_2(r)$, $g_3(r)$,..., all turn out to vanish:

$$g_n(r) = 0, \quad n = 2,3,\dots .$$

In Fig. 3 we have plotted $g_1(r)$, obtained by numerically integrating Eq. (89) with boundary conditions (90) and (91) for various polytropic indices Γ. We see that $g_1(r)$ approaches its asymptotic behavior $g_1(r) \propto r^2$ [Eq. (91)] fairly rapidly for $r > r_0$. At the same time, the derivative $g_1'(r_0)$ is conveniently given by

$$g_1'(r_0) = \frac{k_1(\Gamma)}{r_0}. \tag{92}$$

Table II also lists values of $k_1(\Gamma)$. Note that in contrast to the situation for accretion onto a rotating black hole, the expression for $g_1'(r_0)$ contains no additional small parameters.

On the other hand, for $r \ll r_0$, the asymptotic form of $g_1(r)$ is

TABLE II.

Γ	1.01	1.1	1.2	1.333	1.5	1.6
$K(\Gamma)$	0.49	0.090	0.070	0.044	0.016	0.003
$k_1(\Gamma)$	3.0	0.56	0.46	0.31	0.12	0.023
$K_{in}(\Gamma)$	−0.74	−0.090	−0.026	0.025	0.0081	0.0002

$$g_1(r)=K_{in}(\Gamma)\left(\frac{r}{r_0}\right)^{-1/2}.$$

Values of $K_{in}(\Gamma)$ are listed in Table II. We see that when

$$\varepsilon_1>|K_{in}|^{-1}\sqrt{\frac{\mathcal{M}}{r_0}},$$

the perturbed term $\Phi_0\varepsilon_1 g_1(r)\sin^2\theta$ at distances $2\mathcal{M}<r<r_0\varepsilon_1^2K_{in}^2$ becomes greater than $\Phi_0(1-\cos\theta)$, which corresponds to spherically symmetric accretion. This means that the linear approximation (89) breaks down over that radial interval, and it is necessary to solve the full nonlinear equation (12). Incidentally, the linear approximation is also inapplicable in the hyperbolic region, which has no effect on the behavior of the solution at $r>r_0$.

Let us now determine g_0, which dictates the behavior of the accretion rate. To do so, we again write out the expressions for the radius $r_*(\theta)$ of the sonic surface and the thermodynamic functions $M_*^2(\theta) = M^2(r_*,\theta)$, $\mu_*(\theta) = \mu(r_*,\theta)$, and $c_*^2(\theta)=c_s^2(r_*,\theta)$:

$$r_*(\theta)=r_0[1+\varepsilon_1 d(\theta)],\quad c_*^2(\theta)=c_0^2[1+\varepsilon_1 b(\theta)],$$

$$M_*^2(\theta)=M_0^2[1+\varepsilon_1 q(\theta)],\quad \mu_*(\theta)=\mu_0[1+\varepsilon_1 p(\theta)].$$

In the process, two of the five equations relating the dimensionless functions $d(\theta)$, $b(\theta)$, $p(\theta)$, and $q(\theta)$ are given by Eqs. (69) and (70), as before. Equations (71), (72), and (78), on the other hand, may now be written in the form

$$\frac{c_0^2}{1-c_0^2}b+\frac{4c_0^2}{1-c_0^2}d+2p=0, \tag{93}$$

$$\frac{2}{1-c_0^2}b+\frac{2}{\alpha_0^2}d=\frac{r_0}{\sin\theta}\frac{\partial^2 f}{\partial r\partial\theta}\bigg|_{r=r_0}, \tag{94}$$

$$-b-4d+2q=-\frac{2}{\sin\theta}\frac{\partial f}{\partial\theta}\bigg|_{r=r_0}. \tag{95}$$

Here we have taken advantage of Eq. (86), according to which the Bernoulli integral remains the same, to order $\sim\varepsilon_1$, as in the case of spherically symmetric accretion. Inserting the equation

$$\frac{1}{\sin\theta}\cdot\frac{\partial f}{\partial\theta}\bigg|_{r=r_0}=g_0+2g_1(r_0)\cos\theta$$

into (95), we obtain from the matching requirement on Eqs. (69), (70), and (93)–(95) that

$$g_0=0. \tag{96}$$

Thus, to first order in ε_1, the accretion rate onto a moving black hole does not vary. To the same accuracy, then, the solution of the equilibrium equation (12) can be written out as

$$\Phi(r,\theta)=\Phi_0[1-\cos\theta+\varepsilon_1 g_1(r)\sin^2\theta. \tag{97}$$

In particular, at $r\gg r_0$, we have

$$\Phi(r,\theta)=\Phi_0\left[1-\cos\theta+\varepsilon_1K(\Gamma)\frac{r^2}{r_0^2}\sin^2\theta\right]. \tag{98}$$

In point of fact, for $\varepsilon_1\ll(\mathcal{M}/r_0)^{1/2}$, the analytic expression (98) accurately describes the flow of matter throughout all space.

The solution of Eqs. (69), (70), (93), and (94) is now finally given by

$$d(\theta)=\frac{2(\Gamma+1)}{D_1}k_1(\Gamma)\cos\theta,$$

$$b(\theta)=-\frac{8(\Gamma-1)}{D_1}k_1(\Gamma)\cos\theta, \tag{99}$$

$$p(\theta)=-\frac{4c_0^2}{D_1}k_1(\Gamma)\cos\theta,$$

$$q(\theta)=\frac{8}{D_1}k_1(\Gamma)\cos\theta,$$

where D_1 is given by (77). Accordingly, the shape of the sonic surface is given by

$$r_*(\theta)=r_0\left[1+2\varepsilon_1\frac{(\Gamma+1)}{D_1}k_1(\Gamma)\cos\theta\right]. \tag{100}$$

We now show that our solution (97) correctly describes the accretion of matter throughout all space, including the region $r>r_m$, where the perturbation term $\Phi_0\varepsilon_1 g_1(r)\sin^2\theta$ becomes greater than the zeroth-order term $\Phi_0(1-\cos\theta)$. To do so, we note that both the stream function $\Phi_0(1-\cos\theta)$ corresponding to spherically symmetric accretion and the asymptotic function $\varepsilon_1\Phi_0K(r^2/r_0^2)\sin^2\theta$ corresponding to a uniform flux are asymptotic solutions of the linear part of Eq. (12), which is identical to (42) at $r\gg r_0$. We can show that in the asymptotic region $r>r_m$, the contribution of nonlinear terms to (12) is of order ε_1^2, so that they need only be taken into account in the next order of the expansion of the solution in powers of ε_1.

Indeed, at distances $r\gg r_0$, the value of D, which appears in the denominator of the nonlinear terms in (12), is dominated by the small quantity $(E^2-\alpha^2\mu^2)/\mu^2$ in the denominator of the second term. Making use of (16) to find the difference $E^2-\alpha^2\mu^2$, we have

$$D^{-1}\simeq\frac{M^4\Phi_0^2}{r^4\mu^2c_s^2}\cdot\frac{\Phi^2}{\Phi_0^2}.$$

As a result, we can write the second and third terms of (12), to order of magnitude, as

$$A_2+A_3=\varepsilon_1^2\left(K\frac{M^4c_0^2}{M_0^4c_s^2}\right)\cdot\left(\frac{\Phi^2r_m^4}{\Phi_0^2r^4}\right)\cdot\left(\frac{\Phi}{r^2}\right),$$

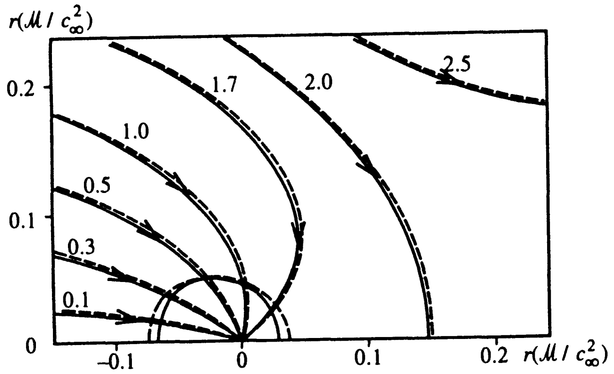

FIG. 4. Flow structure and shape of the sonic surface for $\Gamma=4/3$, $\varepsilon_1=0.6$. Labels on the curves denote the value of Φ/Φ_0, and dashed curves indicate streamlines and the sonic surface obtained numerically.[14]

where, by virtue of (85), $K^2(M^4/M_0^4)(c_0^2/c_s^2)\sim 1$ for all Γ. Therefore, both at distances $r\sim r_m$ (where $\Phi\sim\Phi_0$) and at $r\gg r_m$ where $\Phi\sim\Phi_0(r/r_m)^2$, we have

$$\frac{A_2+A_3}{A_1}\sim\varepsilon_1^2.$$

Here $A_1\sim\Phi/r^2$ corresponds, to order of magnitude, to the term in Eq. (42), and consequently to the linear term in Eq. (12). At distances $r>r_m$, the second and third terms of (12) are thus at most of order ε_1^2.

A similar analysis shows that to order of magnitude, the nonlinear part of the fourth term in (12) is

$$A_4^{\text{nonlin}}\sim\varepsilon_1^2 A_1,$$

so that the last term is also of order ε_1^2.

Figure 4 shows the structure of the flow (97) and the shape of the sonic surface (100) for $\Gamma=4/3$, $\varepsilon_1=0.6$. The dashed curves are streamlines and the sonic surface obtained numerically by Hunt.[14] Clearly, the solution (97) is essentially identical to the numerical result, despite the fact that ε_1 is close to unity in the case considered here. Here, as for a slowly rotating black hole, there are no discontinuities whatever in the flow. This is consistent with numerical results that predict shock formation only for $\varepsilon_1\geqslant 1$.[4,15–18] Lastly, we point out that the boundary condition $g_1(r_0)=0$ implies a flow that is all but spherically symmetric near the sonic surface, which is also consistent with numerical results.[16]

6. CONCLUSION

To summarize, the presence of a small parameter in the problem of accretion onto a black hole makes it possible to carry out a systematic analysis of the equilibrium equation (12), and to determine the shape of the sonic surface and the behavior of the accretion rate.

We have shown that neither slow rotation nor slow linear motion of the black hole alters the essential character of the accretion. As before, the flow can extend out to supersonic velocities and be collisionless; in that sense, the familiar spherically symmetric solution is stable. On the other hand, a systematic analysis of the stability of the resulting flow naturally requires that one consider equations that are both more general and time dependent. In particular, the possibility that the flow may be shocked at supersonic velocities cannot be ruled out, i.e., a shock wave may exist near the gravitating center.[21,22]

We note that the foregoing approach can be adopted to analyze flows in many other astrophysical objects: generalizing Eq. (12) to the $L\neq 0$ case enables one to study hydrodynamic accretion of matter with nonvanishing angular momentum, as well as mass ejection from rotating stars. Lastly, the full magnetohydrodynamic Grad–Shafronov equation,[11,23] which has been widely discussed in the context of active galactic nuclei in quasars,[24–28] young stellar objects,[28,29] and stellar (solar) wind and radio pulsars,[30,31] can also be investigated in similar fashion (see Ref. 32, for example). These problems, however, lie outside the scope of the present paper.

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