Nonlocal Josephson electrodynamics of layered structures

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A system of two-dimensional equations, which describe the distribution of the phase differences of the Cooper pairs and magnetic fields in tunnel junctions and form the foundation of the nonlocal Josephson electrodynamics of such structures, has been derived for structures consisting of alternating flat superconducting layers with weak coupling. A two-dimensional nonlocal generalization of the sine–Gordon equation has been obtained for thick and thin superconducting layers. The spectrum of generalized Swihart waves has been obtained. An approach to the construction of slightly nonlinear vortex structures has been formulated, and the spectrum of their excitations has been determined. The nonlinear one-dimensional picture of the vortices in a layered structure with thin superconducting layers has been considered. © *1995 American Institute of Physics.*

1. INTRODUCTION

Numerous results have recently been obtained in the field of nonlocal Josephson electrodynamics, which describes structures with a characteristic spatial scale smaller than the London penetration depth. We first note here the work on the description of Josephson junctions between the end surfaces of thin superconducting films.[1,2] Although the analysis of the equations appearing in this description was meager, recent studies[3,4] have shown that more detailed results of the theory of small-scale Josephson structures and junctions between thick superconductors can be used for the models described in Refs. 1 and 2. A general formulation of the electrodynamics of such tunnel junctions was given in Refs. 5–8. A theory of generalized Swihart waves and slightly nonlinear waves was formulated in Ref. 5. A solution describing a one-dimensional stationary kink corresponding to a motionless vortex similar in structure to an Abrikosov vortex was found in Ref. 6. Exact nonlinear solutions describing the establishment of a stationary kink with time, as well as vortex decay, were obtained in Ref. 9. The first traveling vortex structure, which was discovered in Refs. 7 and 8, corresponds to a 4π kink in a zero-current junction, in which dissipation is completely neglected. An expression describing a traveling 2π kink was found for a tunnel junction with a current under the conditions of strong dissipation in Ref. 10, and an expression describing a traveling periodic structure of vortices was derived in Ref. 11. This made it possible to obtain the current-voltage characteristic of a Josephson junction determined by small-scale vortices. Stationary periodic structures and relaxation to them were described in Ref. 12. Solutions for periodic traveling structures were obtained with the neglect of dissipation in the same work, and the foundation was laid for a nonlocal theory of the stability of stationary small-scale Josephson vortex structures. Of course, all these achievements far from exhaust the possibilities of nonlocal Josephson electrodynamics. At the same time, the need to describe additional objects that have yet to be reported is clear.

In this paper we formulate the main principles of the nonlocal electrodynamics of layered Josephson structures. The interest in such structures arose comparatively long ago in connection with the theory of layered compounds (see, for example, Refs. 13–16). On the other hand, models of high-T_c superconductors as layered Josephson structures are very popular.[16–19] The basic equations of the local electrodynamics of such structures were derived in Ref. 17. The solution of such equations in Ref. 20, which was obtained using perturbation theory in the limit of weak coupling between neighboring Josephson junctions, is noteworthy. However, local Josephson electrodynamics becomes inapplicable when the characteristic spatial scales of the Josephson vortices are smaller than the London depths. We point out some physical situations in which a nonlocal description is necessary. The simplest example refers to the linear theory of Swihart waves.[5] According to Ref. 5, the nonlocal effect results in slowing of the linear waves when their wavelength is shorter than the London depth λ. The latter occurs if the frequency of the Swihart wave $\omega \sim u/\lambda$, where the phase velocity u is large compared with the Josephson frequency ω_j, but is smaller than the limiting value determined by the width of the energy gap. Another situation pertains to the physics of the spectral properties of Josephson junctions in a strong magnetic field $\bar{H}$.[21] In this case the nonlocal theory is needed to describe vortex harmonics which vary on scales of the order of the magnetic length $L_H = \Phi_0/4\pi\lambda\bar{H}$, where Φ_0 is the magnetic flux quantum, but smaller than the London depth. Corresponding conditions are realized in superconductors with a large value for the Ginzburg–Landau parameter $\kappa = \lambda/\xi$ (ξ is the correlation length), when the magnetic field strength $\bar{H}$ exceeds $\Phi_0/4\pi\lambda^2$, but remains smaller than the lower critical value $H_{c1} = (\Phi_0/4\pi\lambda^2)\ln(\lambda/\xi)$. In addition, physical conditions under which, in contrast to the situation under ordinary Josephson electrodynamics, the London depth is greater than the Josephson length λ_j appear in superconductors with $\kappa \gg 1$.[6,12] This occurs in tunnel junctions with a critical current density j_c greater than $\Phi_0 c/16\pi^2\lambda^3$ (c is the speed of light), but smaller than the pair-breaking

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current $j_d=\Phi_0 c/12\pi^2\sqrt{3}\xi\lambda^2$ (Ref. 22). The application of nonlocal electrodynamics to the description of tunnel junctions with a large critical current made it possible to predict small scale Josephson vortices, which are structurally similar to Abrikosov vortices.[6,12] These situations, in which the nonlocal description must be used, pertain to the theory of a single Josephson junction. However, as will be shown below, the same situations are realized in the electrodynamics of layered Josephson structures if the thicknesses of the superconducting layers L are greater than the London depth. Due to the appearance of an additional small scale in layered structures with thin superconducting layers ($L<\lambda$), the nonlocal theory is also needed to describe the smaller spatial scales.

The principles of the nonlocal Josephson electrodynamics of layered structures are formulated in Sec. 2, where we derive the basic infinite system of coupled equations for the magnetic fields and for the phase differences of the Cooper pairs in all Josephson junctions.

New conditions, under which the influence of neighboring tunnel junctions on the Josephson vortices appearing in them can be neglected, are derived using nonlocal electrodynamics in Sec. 3. Two-dimensional integral equations describing small-scale vortices with a characteristic scale not exceeding the London depth are derived. In Sec. 4 nonlocal equations are obtained for simple vortex structures, such as, first, structures which periodically repeat the distribution of the phase differences of the Cooper pairs in periodically repeated tunnel junctions and, second, structures which alternately repeat the phase distributions in two neighboring tunnel junctions in a periodic system of superconducting layers and tunnel junctions. Section 5 includes a discussion of the electrodynamics of Josephson structures with thin superconducting junctions.

The application of the general principles of nonlocal Josephson electrodynamics to specific problems is described in Secs. 6–8. More specifically, the spectrum of generalized Swihart waves, which differs qualitatively in the case of sufficiently short waves from the spectrum of ordinary Swihart waves, is found in Sec. 6. The problem of finding the exact stationary nonlinear one-dimensional vortex pattern in a periodic Josephson structure with thin superconducting layers is analyzed in Sec. 7. It is shown that the vortex states detected are qualitatively similar to the states appearing in local Josephson electrodynamics, which is based on the equation of a mathematical pendulum. It is established that the vortex states found cease to exist when the thickness of the superconducting layers exceeds a certain critical value (see below). Section 8 contains results based on the theory of weak vortex excitations in the tunnel junctions of a periodic Josephson system in a strong magnetic field. The novelty here is the dependence of the frequency of the excitations on the magnetic field due to the periodic arrangement of the layers, which specifies the discrete arrangement of the lines in the spectrum. In addition, the two-dimensional description made it possible to establish the conversion of the line spectrum into a spectrum of bands, which are broadened as a result of consideration of the dependence of the excitations on the coordinates parallel and perpendicular to the average magnetic field.

2. DERIVATION OF THE BASIC EQUATIONS

The basic equations of nonlocal Josephson electrodynamics for a layered structure of tunnel junctions that is not restricted in the direction of the x axis are obtained in this section.

The starting equations for phase differences of the Cooper pairs $\varphi_n(\rho,t)$ are the ordinary equations of Josephson electrodynamics:[22,23]

$$\frac{\partial}{\partial t}\varphi_n(\rho,t)=-\frac{2e}{\hbar}\int_{x_n-d_n}^{x_n+d_n}dxE_x(x,\rho,t)\simeq-\frac{4ed_n}{\hbar}E_x(x_n,\rho,t), \tag{2.1}$$

$$[\mathrm{curl}\,\mathbf{H}(x_n,\rho,t)]_x=\frac{4\pi}{c}\sigma_nE_x(x_n,\rho,t)+\frac{4\pi}{c}j_{nc}\times\sin\varphi_n(\rho,t)+\frac{\varepsilon_n}{c}\frac{\partial}{\partial t}E_x(x_n,\rho,t), \tag{2.2}$$

where ρ is a two-dimensional vector in the yz plane of the nth tunnel junction of thickness $2d_n$, x_n is the coordinate of the midplane of the nth nonsuperconducting layer, ε_n and σ_n are the dielectric constant and the conductivity of the material in the nth tunnel junction, and j_{nc} is the critical Josephson current density. It was assumed in writing Eq. (2.1) that the thickness of the junction is small compared with the distance over which the electric field varies. In turn, the variation of E_x and the magnetic field $\mathbf{H}=(0,H_y,H_z)$ across the junction is neglected in Eq. (2.2). At the same time, the components of the electric field which are tangential to the plane of the junction $\mathbf{E}_\perp$ undergo a jump. To determine it, we integrate the tangential components of the following equation across the junction:

$$\mathrm{curl}\,\mathbf{E}=-\frac{L}{c}\frac{\partial}{\partial t}\mathbf{H}. \tag{2.3}$$

Now, using nonstationary Josephson equation (2.1), we obtain

$$\mathbf{E}_\perp(x_n+d_n,\rho,t)-\mathbf{E}_\perp(x_n-d_n,\rho,t)=-\frac{\hbar}{2e}\frac{\partial^2}{\partial t\partial\rho}\varphi_n(\rho,t)+\frac{2d_n}{c}\left[\mathbf{e}_x\frac{\partial}{\partial t}\mathbf{H}(x_n,\rho,t)\right], \tag{2.4}$$

where $\mathbf{e}_x$ is a unit vector parallel to the x axis.

Then, in accordance with (2.1), Eq. (2.2) can be written in the form

$$[\text{curl } \mathbf{H}(x_n,\rho,t)]_x = \frac{4\pi}{c} j_{nc}\left\{ \sin \varphi_n(\rho,t) + \frac{\beta_n}{\omega_{jn}^2}\frac{\partial}{\partial t} \times \varphi_n(\rho,t) + \frac{1}{\omega_{jn}^2}\frac{\partial^2}{\partial t^2}\varphi_n(\rho,t)\right\}, \quad (2.5)$$

where $\beta_n = 4\pi\sigma_n/\varepsilon_n$ is a parameter which characterizes the dissipation and $\omega_{jn} = \sqrt{16\pi/e/d_n j_{nc}/\hbar\varepsilon_n}$ is the Josephson frequency. Therefore, the formulation of the equations of closed Josephson electrodynamics requires knowledge of the relationship of the tangential components of the magnetic field in the junctions to the phase difference of the wave functions in them. For this purpose we utilize the equations which describe the magnetic field $\mathbf{H}_n(x,\rho,t)$ and the electric field $\mathbf{E}_n(x,\rho,t)$ in the nth layer of a superconductor of thickness $2L_n$ with a London depth λ_n:

$$\lambda_n^2 \Delta \mathbf{H}_n - \mathbf{H}_n = 0, \quad (2.6)$$

$$\mathbf{E}_n = \frac{\lambda_n^2}{c} \text{curl}\left(\frac{\partial \mathbf{H}_n}{\partial t}\right). \quad (2.7)$$

Taking into account the continuity of the tangential components of the electric field $\mathbf{E}_\perp(x_n+d_n,\rho,t)$ on the boundaries of the junctions and assuming that the nth tunnel layer is bounded on one side by the nth superconducting and on the other side by the $(n+1)$-th superconducting layer, from Eqs. (2.4) and (2.7) we obtain

$$\lambda_{n+1}^2\left[\mathbf{e}_x \frac{\partial}{\partial x_n}\mathbf{H}_{n+1}(x_n+d_n,\rho,t)\right] - \lambda_n^2\left[\mathbf{e}_x \frac{\partial}{\partial x_n}\mathbf{H}_n(x_n-d_n,\rho,t)\right] = -\frac{\hbar c}{2e}\frac{\partial}{\partial \rho}\varphi_n(\rho,t) + 2d_n[\mathbf{e}_x \mathbf{H}(x_n,\rho,t)]. \quad (2.8)$$

The derivatives of the magnetic field in the superconductors appearing therein can be expressed in terms of the magnetic fields on the boundaries of tunnel junctions n and $n+1$. For this purpose we use the following solution of Eq. (2.6) in the nth superconductor:

$$\mathbf{H}_n(x,\rho,t) = \int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2}\exp-[i\mathbf{k}(\rho-\rho')] \times \text{csch}(2L_n\sqrt{k^2+\lambda_n^{-2}})\{\sinh[\sqrt{k^2+\lambda_n^{-2}} \times (x-x_{n-1}-d_{n-1})]\mathbf{H}_n(x_n-d_n,\rho',t) - \sinh[\sqrt{k^2+\lambda_n^{-2}}(x-x_n+d_n)] \times \mathbf{H}_n(x_{n-1}+d_{n-1},\rho',t)\}. \quad (2.9)$$

Substituting (2.9) into (2.8) and neglecting the small variation of the magnetic field in a thin tunnel junction, we obtain the following system of integral equations, which defines the relationship of the magnetic field to the phase differences of the wave functions of the Cooper pairs φ_n:

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2}\exp[i\mathbf{k}(\rho-\rho')] \times\{-a_{n+1}(k)\mathbf{H}(x_{n+1},\rho',t) - a_n(k)\mathbf{H}(x_{n-1},\rho',t) + [b_n(k)+b_{n+1}(k)+2d_n]\mathbf{H}(x_n,\rho',t)\} = \frac{\hbar c}{2|e|}\left[\mathbf{e}_x \frac{\partial}{\partial \rho}\varphi_n(\rho,t)\right], \quad (2.10)$$

where we used the notation

$$a_n(k) = \lambda_n^2\sqrt{k^2+\lambda_n^{-2}}\,\text{csch}(2L_n\sqrt{k^2+\lambda_n^{-2}}), \quad (2.11)$$

$$b_n(k) = \lambda_n^2\sqrt{k^2+\lambda_n^{-2}}\,\text{coth}(2L_n\sqrt{k^2+\lambda_n^{-2}}). \quad (2.12)$$

The system of equations (2.5) and (2.10) comprises the foundation of the nonlocal Josephson electrodynamics of layered superconducting structures. We note that in the local limit, at which the variation of the magnetic field along the junction on the scale of the London depth is small, Eq. (2.10) corresponds to the result in Ref. 17 obtained for Josephson superlattices.

3. LIMIT FOR NEGLECTING THE COUPLING BETWEEN DIFFERENT TUNNEL JUNCTIONS

In this section we consider a system of junctions in which the influence of the different tunnel junctions on one another can be neglected. In nonlocal Josephson electrodynamics it corresponds to the model presented in Refs. 5, 7, and 8. In contrast to those papers, here we give a two-dimensional description, which is needed to devise the theory of two-dimensional small-scale vortex lattices. At the same time, we show how the condition for neglect of the coupling between the different junctions changes for small-scale vortex structures, i.e., how the concepts of thin and thick superconducting layers in a layered Josephson structure change.

The treatment of layered structures on the basis of local electrodynamics allows us to state that superconducting layers are thick when their thickness is much greater than the London depth:

$$L_n \gg \lambda_n. \quad (3.1)$$

In fact, in this case the quantity $a_n(k)$ in Eq. (2.10), i.e., the influence of one superconducting layer on another, can be neglected, and the theory can thus be reduced to the Josephson electrodynamics of one tunnel junction. The situation changes in our case of a nonlocal theory, since even when condition (3.1) is violated (in the usual terminology this means that we have thin superconducting layers), the influence of neighboring layers can be neglected for vortex structures which vary abruptly enough along the tunnel layer. In fact, according to (2.11) and (2.12), a sufficient condition for this is

$$2L_n\sqrt{k^2+\lambda_n^{-2}} \gg 1, \quad (3.2)$$

which signifies that the penetration depth of a magnetic field into the nth superconducting layer $(\sim 1/\sqrt{k^2+\lambda_n^{-2}})$ is small compared with its thickness L_n, where $1/k$ is the characteristic spatial scale of the variation of a vortex structure along the tunnel layer. For just this reason, we shall use (3.2), instead of the usual condition (3.1), as a criterion of the transition to the limit of thick superconducting layers. When inequality (3.2) is satisfied, Eq. (2.10) takes on the form

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2} \exp[ik(\rho-\rho')][\lambda_n^2\sqrt{k^2+\lambda_n^{-2}} + \lambda_{n+1}^2\sqrt{k^2+\lambda_{n+1}^{-2}}+2d_n]\mathbf{H}(x_n,\rho',t) = \frac{\hbar c}{2|e|}\left[\mathbf{e}_x \frac{\partial}{\partial\rho} \varphi_n(\rho,t)\right]. \quad (3.3)$$

The solution of this equation is

$$\mathbf{H}(\rho,t)=\frac{\hbar c}{2|e|}\int d\rho' Q(\rho-\rho')\left[\mathbf{e}_x \frac{\partial}{\partial\rho'} \varphi(\rho',t)\right], \quad (3.4)$$

where

$$Q(\rho)=\int \frac{d\mathbf{k}}{(2\pi)^2} \exp(i\mathbf{k}\rho)[2d+\lambda_+\sqrt{1+k^2\lambda_+^2} + \lambda_-\sqrt{1+k^2\lambda_-^2}]^{-1}. \quad (3.5)$$

Here we have omitted the subscripts n and $n+1$, replacing them, where necessary, by a plus or minus sign. Accordingly, λ_+ and λ_- are the London depths of the superconductors on opposite sides of a tunnel junction of thickness $2d$. Here Eq. (2.5) takes on the form

$$\sin\varphi(\rho,t)+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi(\rho,t)+\frac{1}{\omega_j^2}\frac{\partial^2}{\partial t^2}\varphi(\rho,t) = \lambda_0^3\Delta_\perp\int d\rho' Q(\rho-\rho')\varphi(\rho',t), \quad (3.6)$$

where $\Delta_\perp=\partial^2/\partial\rho^2$ and $\lambda_0^3=\hbar c^2/8\pi|e|j_c$. In this case the distribution of the magnetic field in the superconductors is described by the expression

$$\mathbf{H}_\pm(x,\rho,t)=\frac{\hbar c}{2|e|}\int d\rho' Q_\pm(\rho-\rho',\pm x-d) \times\left[\mathbf{e}_x \frac{\partial}{\partial\rho'} \varphi(\rho',t)\right] \quad (3.7)$$

with a kernel $Q_\pm$ of the form

$$Q_\pm(\rho,x)=\int \frac{d\mathbf{k}}{(2\pi)^2} \exp[i\mathbf{k}\rho - x\sqrt{k^2+\lambda_\pm^{-2}}] \times[2d+\lambda_-\sqrt{1+k^2\lambda_-^2}+\lambda_+\sqrt{1+k^2\lambda_+^2}]^{-1}. \quad (3.8)$$

Equation (3.6) together with relations (3.4), (3.5), (3.7), and (3.8) generalizes the results in Ref. 8 to the case of two-dimensional geometry. If the phase difference varies on scales much greater than the London depths, kernels (3.5) and (3.8) can be approximated using a delta function:

$$Q_\pm(\rho,x)=\frac{\exp(-x/\lambda_\pm)}{(2d+\lambda_++\lambda_-)}\,\delta(\rho),$$

$$Q(\rho)=Q_\pm(\rho,x=0), \quad (3.9)$$

In this case Eqs. (3.7) and (3.4) become

$$\mathbf{H}_\pm(x,\rho,t)=\frac{\hbar c\exp[-(\pm x-d)/\lambda_\pm]}{2|e|(2d+\lambda_++\lambda_-)}\left[\mathbf{e}_x\frac{\partial}{\partial\rho}\varphi(\rho,t)\right],$$

$$\mathbf{H}(\rho,t)=\mathbf{H}_\pm(\pm d,\rho,t), \quad (3.10)$$

and Eq. (3.6) becomes the two-dimensional sine–Gordon equation

$$\sin\varphi+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi+\frac{1}{\omega_j^2}\frac{\partial^2}{\partial t^2}\varphi=\lambda_j^2\frac{\partial^2}{\partial\rho^2}\varphi, \quad (3.11)$$

where $\varphi=\varphi(\rho,t)$ and $\lambda_j^2=\lambda_0^3/(2d+\lambda_++\lambda_-)$.

In the opposite limit of abrupt variation of the phase difference on scales of the London depths, kernels (3.8) and (3.5) may be represented in the form ($\lambda_\pm \gg d$)

$$Q_\pm(\rho,x)=\frac{1}{2\pi(\lambda_+^2+\lambda_-^2)\sqrt{\rho^2+x^2}},$$

$$Q(\rho)=Q_\pm(\rho,x=0). \quad (3.12)$$

Accordingly, Eqs. (3.6) and (3.7) take on the forms

$$\sin\varphi(\rho,t)+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi(\rho,t)+\frac{1}{\omega_j^2}\frac{\partial^2}{\partial t^2}\varphi(\rho,t) = \frac{\lambda_0^3}{2\pi(\lambda_-^2+\lambda_+^2)}\Delta_\perp \fint \frac{d\rho'}{|\rho-\rho'|}\varphi(\rho',t), \quad (3.13)$$

$$\mathbf{H}_\pm(x,\rho,t)=\frac{\hbar c}{4\pi|e|(\lambda_-^2+\lambda_+^2)}\int\frac{d\rho'}{\sqrt{(\rho-\rho')^2+(\pm x-d)^2}} \times\left[\mathbf{e}_x\frac{\partial}{\partial\rho'}\varphi(\rho',t)\right]. \quad (3.14)$$

Equation (3.13) is the two-dimensional analog of the sine–Hilbert equation.[8]

In the special case of identical superconductors ($\lambda_+=\lambda_-=\lambda$) and with neglect of the thickness of the tunnel junction in comparison with the penetration depth of a magnetic field into the superconducting layers, expression (3.8) takes on the form

$$Q_\pm(\rho,x)=\frac{1}{4\pi\lambda^2\sqrt{\rho^2+x^2}}\exp\left[-\frac{1}{\lambda}\sqrt{\rho^2+x^2}\right]. \quad (3.15)$$

For scales exceeding the London depth, result (3.9) with $\lambda_+=\lambda_-=\lambda\gg d$ follows from (3.15). In the opposite limit of small scales, we obtain expression (3.12) from (3.15).

4. SIMPLEST VORTEX STRUCTURES

We now analyze the system of equations (2.5) and (2.10) in the case of periodic layered Josephson structures. For simplicity, we restrict ourselves to a treatment of structures consisting of identical superconducting layers ($\lambda_n=\lambda, L_n=L$) and identical tunnel junctions ($\sigma_n=\sigma$, $\varepsilon_n=\varepsilon$, $d_n=d$, $j_{nc}=j_c$). Then, in view of the fact that

$$a_n(k)=a(k),\quad b_n(k)=b(k),\quad n=0,\pm1,\pm2,...,$$

the system of integral equations (2.10) takes on the form

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2} \exp[i\mathbf{k}(\rho-\rho')]\{-a(k)$$
$$\times[\mathbf{H}(x_{n+1},\rho',t)+\mathbf{H}(x_{n-1},\rho',t)]$$
$$+2[d+b(k)]\mathbf{H}(x_n,\rho',t)\}$$
$$=\frac{\hbar c}{2|e|}\left[\mathbf{e}_x \frac{\partial}{\partial\rho} \varphi_n(\rho,t)\right]. \quad (4.1)$$

Let us consider a vortex structure in which the phase difference and the magnetic fields in all the junctions are identical:

$$\mathbf{H}(x_n,\rho,t)=\mathbf{H}(\rho,t), \quad \varphi_n(\rho,t)=\varphi(\rho,t),$$
$$n=0,\pm 1,\pm 2,\ldots .$$

In this case from Eq. (4.1) we find that the magnetic field $\mathbf{H}(\rho,t)$ is described by relation (3.4) with the kernel

$$Q(\rho)=\frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^2} \exp(i\mathbf{k}\rho)$$
$$\times[d+\lambda\sqrt{1+k^2\lambda^2} \tanh(L\sqrt{k^2+\lambda^{-2}})]^{-1}. \quad (4.2)$$

For its part, the distribution of the phase difference is described by nonlocal equation (3.6) with replacement of the kernel Q by (4.2). Under conditions for the applicability of local electrodynamics, under which the characteristic scale of the variation of the phase difference greatly exceeds the London depth, Q is approximated by a delta function:

$$Q(\rho)=\frac{1}{2}\,\delta(\rho)\left[d+\lambda \tanh\frac{L}{\lambda}\right]^{-1}. \quad (4.3)$$

This leads to the following relation between the magnetic field and the phase difference:

$$\mathbf{H}(\rho,t)=\frac{\hbar c}{4|e|}\left[\mathbf{e}_x \frac{\partial}{\partial\rho} \varphi(\rho,t)\right]\left[d+\lambda \tanh\frac{L}{\lambda}\right]^{-1}. \quad (4.4)$$

The equation for $\varphi(\rho,t)$ becomes a local equation of the form (3.11), where the role of the Josephson length λ_j is played by the quantity $\{\lambda_0^3/2[d+\lambda \tanh(L/\lambda)]\}^{1/2}$.

We proceed to an examination of another simple example of a nonlinear electrodynamic structure which is realized in layered compounds. More specifically, we assume that the distributions of the magnetic fields and the phase differences in neighboring tunnel junctions differ in sign:

$$\varphi_n=-\varphi_{n\pm 1}=\varphi, \quad \mathbf{H}_n=-\mathbf{H}_{n\pm 1}=\mathbf{H}.$$

In this case it follows from Eqs. (4.1) that such a nonlinear vortex structure is described by Eq. (3.4) and by Eq. (3.6) with the kernel

$$Q(\rho)=\frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^2} \exp(i\mathbf{k}\rho)$$
$$\times[d+\lambda\sqrt{1+k^2\lambda^2} \coth(L\sqrt{k^2+\lambda^{-2}})]^{-1}. \quad (4.5)$$

In the limit corresponding to the local theory, in which the characteristic scale of the variation of the phase difference significantly exceeds the London depth λ, Eqs. (3.4) and (4.5) give

$$\mathbf{H}(\rho,t)=\frac{\hbar c}{4|e|}\left[\mathbf{e}_x \frac{\partial}{\partial\rho} \varphi(\rho,t)\right]\left[d+\lambda \coth\frac{L}{\lambda}\right]^{-1}. \quad (4.6)$$

In the same limit, from (3.6) we obtain the sine–Gordon equation for the phase difference

$$\sin\varphi+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi+\frac{1}{\omega_j^2}\frac{\partial^2}{\partial t^2}\varphi=\lambda_{j\,\mathrm{eff}}^2\Delta_\perp\varphi, \quad (4.7)$$

where the effective Josephson length depends on the thickness of the superconducting layers:

$$\lambda_{j\,\mathrm{eff}}=\lambda_j\sqrt{\frac{d+\lambda}{d+\lambda \coth(L/\lambda)}}. \quad (4.8)$$

Let us consider the example of a more complicated nonlinear vortex structure, in which the alternating distributions of the phase difference

$$\varphi_{2n}(\rho,t)=\varphi_1(\rho,t), \quad \varphi_{2n+1}(\rho,t)=\varphi_2(\rho,t)$$

and the alternating distributions of the magnetic field

$$\mathbf{H}(x_{2n},\rho,t)=\mathbf{H}_1(\rho,t), \quad \mathbf{H}(x_{2n+1},\rho,t)=\mathbf{H}_2(\rho,t)$$

are realized in the tunnel junctions. In this case from the system of equations (4.1) we find that the distribution of the magnetic field strength in each tunnel junction is determined by the values of the phase differences in both the junction in question and in the two neighboring junctions:

$$\mathbf{H}_\alpha(\rho,t)=\frac{\hbar c}{2|e|}\sum_{\beta=1,2}\int d\rho' Q_{\alpha\beta}(\rho-\rho')\varphi_\beta(\rho',t),$$
$$\alpha=1,2, \quad (4.9)$$

where

$$Q_{11}(\rho)=Q_{22}(\rho)=\frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^2} \exp(i\mathbf{k}\rho)[d+b(k)]$$
$$\times\{[d+a(k)]^2-b^2(k)\}^{-1}, \quad (4.10)$$

$$Q_{12}(\rho)=Q_{21}(\rho)=\frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^2} \exp(i\mathbf{k}\rho)a(k)$$
$$\times\{[d+a(k)]^2-b^2(k)\}^{-1}. \quad (4.11)$$

According to (2.5), the distribution of the phase difference in the junctions is described here by a system of two coupled equations

$$\sin\varphi_\alpha(\rho,t)+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi_\alpha(\rho,t)+\frac{1}{\omega_j^2}\frac{\partial^2}{\partial t^2}\varphi_\alpha(\rho,t)$$
$$=\sum_{\beta=1,2}\lambda_0^3\Delta_\perp\int d\rho' Q_{\alpha\beta}(\rho-\rho')\varphi_\beta(\rho',t). \quad (4.12)$$

In superconducting layers whose thickness is greater than the penetration depth of a magnetic field in them, the kernels $Q_{\alpha\beta}(\alpha\neq\beta)$ are exponentially small. This makes it possible to describe the distribution of the phase difference in neighboring junctions using perturbation theory (see, for example, Ref. 20).

Finally, we dwell on the case in which the distribution of the phase differences and the magnetic fields in neighboring junctions differ slightly from one another. Here the system of difference equations (4.1) may be written in the form

$$\int d\rho' \int \frac{d\mathbf{k}}{(2\pi)^2} \exp[i\mathbf{k}(\rho-\rho')] \bigg\{ -2a(k)(d+L)^2 \times \frac{\partial^2}{\partial x^2} \mathbf{H}(x,\rho,t) + [d+b(k)-a(k)]\mathbf{H}(x,\rho,t) \bigg\} = \frac{\hbar c}{4|e|} \left[\mathbf{e}_x \frac{\partial}{\partial \rho} \varphi(x,\rho,t) \right], \tag{4.13}$$

where the coordinate x replaces the discrete values of the midpoints of the junctions x_n.

5. JOSEPHSON STRUCTURES WITH THIN SUPERCONDUCTING LAYERS

We move on to an investigation of vortex formations in Josephson structures consisting of thin superconducting layers under conditions in which the reverse of inequality (3.2) is satisfied, i.e., the thickness of the layers is much smaller than the penetration depth of a magnetic field into the superconductor.

We begin the analysis with an examination of symmetric vortex formations ($\varphi_n = \varphi$, $\mathbf{H}_n = \mathbf{H}$). In this case, from (3.4) and (4.2) we have

$$\Delta_\perp \mathbf{H}(\rho,t) - \lambda_{\text{eff}}^{-2} \mathbf{H}(\rho,t) = -\frac{\hbar c}{4|e|\lambda^2 L} \left[\mathbf{e}_x \frac{\partial}{\partial \rho} \varphi(\rho,t) \right], \tag{5.1}$$

where $\lambda_{\text{eff}}^2 = \lambda^2 L/(L+d)$. This relation must be supplemented by Eq. (2.5). The system of equations (5.1) and (2.5) can be rewritten by introducing the vector potential $\mathbf{A} = (A(\rho,t),0,0)$, which is defined by the relation $\mathbf{H} = \text{curl}\, \mathbf{A}$, in the following form

$$\sin \varphi + \frac{\beta}{\omega_j^2} \frac{\partial}{\partial t} \varphi + \frac{1}{\omega_j^2} \frac{\partial^2}{\partial t^2} \varphi = -\frac{\lambda_0^3}{2\lambda^2 L} \left[\varphi + \frac{4\pi(d+L)}{\phi_0} A \right], \tag{5.2}$$

$$\Delta_\perp A - \lambda_{\text{eff}}^{-2} A = \frac{\phi_0}{4\pi\lambda^2 L} \varphi, \tag{5.3}$$

where $\Phi_0 = \pi\hbar c |e|^{-1}$ is the magnetic flux quantum and $A = A(\rho,t)$. Eliminating the vector potential from (5.2) with the aid of the solution of (5.3)

$$A(\rho,t) = -\frac{\Phi_0}{2\lambda^2 L} \int \frac{d\rho'}{(2\pi)^2} K_0\left(\frac{|\rho-\rho'|}{\lambda_{\text{eff}}} \right) \varphi(\rho',t), \tag{5.4}$$

we obtain the following nonlocal equation:

$$\sin \varphi + \frac{\beta}{\omega_j^2} \frac{\partial \varphi}{\partial t} + \frac{1}{\omega_j^2} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\lambda_0^3}{4\pi\lambda^2 L} \Delta_\perp \int d\rho' K_0 \times \left(\frac{|\rho-\rho'|}{\lambda_{\text{eff}}} \right) \varphi(\rho',t), \tag{5.5}$$

where $K_0(x)$ is the modified Bessel function of the second kind. We stress that the investigation of specific situations may be simpler under certain conditions when Eqs. (5.2) and (5.3) are used. This is especially clear in the stationary case, in which Eqs. (5.2) and (5.3) reduce to a single nonlinear differential equation.

If the phase difference depends only on the single coordinate z, from (5.5) we have

$$\sin \varphi + \frac{\beta}{\omega_j^2} \frac{\partial \varphi}{\partial t} + \frac{1}{\omega_j^2} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\lambda_0^3}{4\lambda \sqrt{L(L+d)}} \frac{\partial^2}{\partial z^2} \int_{-\infty}^{\infty} dz' \times \exp\left[-\frac{|z-z'|}{\lambda_{\text{eff}}} \right] \varphi(z',t). \tag{5.6}$$

For nearly symmetric structures, in which the distribution of the phase difference φ_n and the magnetic fields upon the transition from layer to layer is a slowly varying function of n, from (4.13) we obtain

$$\left(\frac{d+L}{L} \right)^2 \frac{\partial^2}{\partial x^2} \mathbf{H}(x,\rho,t) + \Delta_\perp \mathbf{H}(x,\rho,t) - \lambda_{\text{eff}}^{-2} \mathbf{H}(x,\rho,t) = -\frac{\hbar c}{4|e|\lambda^2 L} \left[\mathbf{e}_x \frac{\partial}{\partial \rho} \varphi(x,\rho,t) \right], \tag{5.7}$$

where the continuous coordinate x replaces the discrete coordinate of the midplane of the tunnel junction. Equation (5.7) together with (2.5) describes three-dimensional vortex structures under conditions under which the reverse of inequality (3.2) is satisfied. We note that Eq. (5.7) coincides with the equation obtained in Ref. 24, which was devoted to an investigation of vortex structures in layered Josephson compounds (see also Refs. 13 and 14). For antisymmetric vortex distributions ($\varphi_n = -\varphi_{n+1} = \varphi$, $\mathbf{H}_n = -\mathbf{H}_{n+1} = \mathbf{H}$), from (3.4) and (4.5) we obtain a local relation between the magnetic field and the gradient of the phase difference, which is defined by Eq. (4.6) in the limit $L \ll \lambda$. Accordingly, the distribution of the phase difference is described by the sine–Gordon equation with the Josephson length

$$\lambda_{j\text{eff}} = \lambda_j \sqrt{(d+\lambda)L/(dL+\lambda^2)},$$

which is smaller than the usual Josephson length λ_j.

6. SWIHART WAVES

We now consider the limit of the linear theory corresponding to small perturbations of the phase difference which permit the use of the approximation $\sin \varphi_n \simeq \varphi_n$. In view of the linear nature of the problem, it is convenient to write the expressions for the magnetic fields and the phase differences in the form $\exp(-i\omega t + i\mathbf{k}\rho)$. The substitution of such expressions into the linearized equation (2.5) together with the relations (3.4) and (4.2) gives a dispersion relation between the frequency $\omega(\mathbf{k})$ of the linear modes of the multilayer structure and the wave vector $\mathbf{k}$:

$$\omega^2 + i\beta\omega = \omega_j^2 \{1 + \tfrac{1}{2} k^2 \lambda_0^3 \times [d + \lambda \sqrt{1+k^2\lambda^2} \tanh(L\sqrt{k^2+\lambda^{-2}})]^{-1}\}. \tag{6.1}$$

In the case of small ohmic losses in the junctions, in which $|\omega| \gg \beta$, the decay constant γ equals $\mathrm{Im}\,\omega = \beta/2$. The real frequency $\omega(\mathbf{k})$ is specified by the square root of the right-hand side of (6.1):

$$\omega(\mathbf{k}) = \omega_j\{1 + \tfrac{1}{2}k^2\lambda_0^3[d + \lambda\sqrt{1 + k^2\lambda^2} \times \tanh(L\sqrt{k^2 + \lambda^{-2}})]^{-1}\}^{1/2}. \tag{6.2}$$

For superconducting layers which are so thick that $L \gg \lambda$ or $kL \gg 1$, the spectrum of generalized Swihart waves

$$\omega^2(\mathbf{k}) = \omega_j^2\{1 + \tfrac{1}{2}k^2\lambda_0^3[d + \lambda\sqrt{1 + k^2\lambda^2}]^{-1}\} \tag{6.3}$$

coincides with the result obtained in Ref. 5.

In the limit of thin superconducting layers ($L \ll \lambda, kL \ll 1$), Eq. (6.2) gives

$$\omega^2(\mathbf{k}) = \omega_j^2 + \frac{d}{\varepsilon}\frac{k^2c^2}{d + L + Lk^2\lambda^2}. \tag{6.4}$$

If it turns out here that $k^2\lambda^2$ is greater than unity and d/L, we have

$$\omega^2(\mathbf{k}) = \omega_j^2 + \frac{dc^2}{\varepsilon L\lambda^2}. \tag{6.5}$$

This frequency does not depend on the wave vector and greatly exceeds the Josephson frequency, if $\lambda_0^3 \gg 2\lambda^2 L$. In the latter limit we have

$$\omega^2(\mathbf{k}) = \frac{dc^2}{\varepsilon L\lambda^2}. \tag{6.6}$$

Since this equation does not contain an influence on the part of the critical current, it corresponds to a regime under which the Josephson effect is negligible.

We move on to a treatment of the antisymmetric mode. Here kernel (4.5) must be used instead of (4.2) to obtain result (6.1). Accordingly, we obtain

$$\omega^2 + i\beta\omega = \omega_j^2\{1 + \tfrac{1}{2}k^2\lambda_0^3 \times [d + \lambda\sqrt{1 + k^2\lambda^2}\coth(L\sqrt{k^2 + \lambda^{-2}})]^{-1}\}. \tag{6.7}$$

In the case of small losses, the decay of this mode, like that of the symmetric mode, is specified by $\gamma = \beta/2$, and the real part of the frequency $\omega(\mathbf{k})$ is given by the square root of the right-hand side of (6.7). For sufficiently thick superconducting layers ($L\sqrt{k^2 + \lambda^{-2}} \gg 1$) the frequencies of the symmetric and antisymmetric modes are similar, and relation (6.3) is valid for them. In the limit of thin superconducting layers, from (6.7) we obtain

$$\omega^2(\mathbf{k}) = \omega_j^2 + \frac{d}{\varepsilon}k^2c^2\left(d + \frac{\lambda^2}{L}\right)^{-1}. \tag{6.8}$$

Frequency (6.8) is significantly smaller than the frequency of the symmetric mode.

7. VORTEX STATES IN A PERIODIC JOSEPHSON STRUCTURE WITH THIN SUPERCONDUCTING LAYERS

In this section we discuss several consequences of Eq. (5.6) for a periodic structure in which the thickness of the superconducting layers is smaller than the London depth. This has been greatly simplified owing to the mathematical analysis of such an equation in Ref. 25. Following that paper, we introduce the function

$$a(z,t) = \int_{-\infty}^{\infty}\frac{dz'}{2\lambda_{\mathrm{eff}}}\exp\left(-\frac{|z - z'|}{\lambda_{\mathrm{eff}}}\right)\varphi(z',t). \tag{7.1}$$

Then the integral equation (5.6) reduces to the system of differential equations

$$\sin\varphi + \frac{\beta}{\omega_j^2}\frac{\partial\varphi}{\partial t} + \frac{1}{\omega_j^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{1}{\mu}(a - \varphi), \tag{7.2}$$

$$\lambda_{\mathrm{eff}}^2\frac{\partial^2 a}{\partial z^2} - a + \varphi = 0, \tag{7.3}$$

where $\mu = 2\lambda^2 L/\lambda_0^3$, $a = a(z,t)$, and $\varphi = \varphi(z,t)$.

In the stationary case, the solution of this system of equations can be written in quadratures:

$$\pm\sqrt{2\mu}\,\frac{z - z_0}{\lambda_{\mathrm{eff}}} = \int_{\varphi_0}^{\varphi}\frac{d\phi(1 + \mu\cos\phi)}{\sqrt{C_0 - \cos\phi + (\mu/2)\sin^2\phi}}, \tag{7.4}$$

where $\varphi_0 = \varphi(z = z_0)$, and C_0 is an integration constant.

The last equation reveals that solutions with a bounded derivative appear only when (compare Ref. 25)

$$\mu = 2\lambda^2 L/\lambda_0^3 < 1. \tag{7.5}$$

The value $\mu = 1$ corresponds to a bifurcation point, at which there is restructuring of the phase portrait of the differential equation

$$\lambda_{\mathrm{eff}}^2\frac{d^2}{dz^2}(\varphi + \mu\sin\varphi) = \mu\sin\varphi. \tag{7.6}$$

Assuming that the thickness of the superconducting layers is sufficiently small, i.e., that $L < L_c = \lambda_0^3/2\lambda^2$, we can state in accordance with Eq. (7.4) that values of the integration constant in the range $-1 < C_0 < 1$ correspond to a vortex chain described by a solution $\varphi(z)$ which oscillates near a special point like a "center" in the phase plane and has a periodically vanishing derivative $d\varphi/dz$. Similar solutions of the Peierls equation are found to be unstable when they are analyzed with the sine–Hilbert equation.[12] This also applies to similar solutions of the sine–Gordon equation. The values $C_0 > 1$ correspond to periodic (rotational) solutions, which correspond to the presence of a magnetic field in the case of Josephson structures. Such states are usually stable. Finally, the value of the integration constant $C_0 = 1$ corresponds to the separatrix in the phase plane of Eq. (7.6). In this case the solution can be written in the form[25]

$$\frac{\sqrt{1 + \mu\cos^2\psi} + \sqrt{1 + \mu}\cos\psi}{\sqrt{1 + \mu\cos^2\psi} - \sqrt{1 + \mu}\cos\psi} \times \left(\frac{\sqrt{\mu^{-1} + \cos^2\psi} - \cos\psi}{\sqrt{\mu^{-1} + \cos^2\psi} + \cos\psi}\right)^{2/\sqrt{1 + \mu^{-1}}} = \exp\left[\pm\frac{2(z - z_0)}{\sqrt{1 + \mu^{-1}}\lambda_{\mathrm{eff}}}\right], \tag{7.7}$$

where $\psi=\varphi/2$. This solution defines a 2π kink when the sign is positive or an antikink when the sign is negative.

Let us dwell on the solution describing a nonlinear periodic vortex chain with a sufficiently strong magnetic field. Assuming that $C_0 \gg 1$, we have

$$\pm\sqrt{2\mu C_0}\,\frac{z}{\lambda_{\rm eff}}=\varphi+\mu\sin\varphi. \tag{7.8}$$

According to (3.4), the magnetic field $\bar{H}$ averaged over the oscillation period of the phase difference of the Cooper pairs is specified by the constant C_0 in the following manner:

$$\bar{H}=\frac{\Phi_0}{4\pi}\frac{\sqrt{2\mu C_0}}{\lambda\sqrt{L(d+L)}}\left\langle\frac{1}{1+\mu\cos\varphi}\right\rangle. \tag{7.9}$$

This equation makes it possible to find the lower bound for the thickness of the superconducting layers, if it is borne in mind that $\bar{H}$ does not exceed the lower critical field.

8. SPECTRAL PROPERTIES OF LAYERED JOSEPHSON STRUCTURES IN A STRONG MAGNETIC FIELD

Following Ref. 21, we consider the frequency spectrum of layered Josephson structures in a strong magnetic field $\mathbf{H}=(0,\bar{H},0)$, in which the distribution of the phase difference is described by Eq. (3.6) with the kernel Q in (4.2). When the layered structure has a strong mean magnetic field [compare (7.9)]

$$\bar{H}=\frac{\Phi_0}{4\pi L_H[d+\lambda\tanh(L/\lambda)]}\gg\frac{\Phi_0}{4\pi\lambda_j}\times\left[(d+\lambda)\left(d+\lambda\tanh\frac{L}{\lambda}\right)\right]^{-1/2}, \tag{8.1}$$

the solution of Eq. (3.6) for the ground state of the system may be represented in the form

$$\varphi_0(z)=-\frac{z}{L_H}+\frac{L_H^2}{\lambda_j^2(d+\lambda)}\left[d+\lambda\sqrt{1+\frac{\lambda^2}{L_H^2}}\times\tanh\left(\frac{L}{\lambda}\sqrt{1+\frac{\lambda^2}{L_H^2}}\right)\right]\sin\frac{z}{L_H}, \tag{8.2}$$

where L_H is a scale characterizing the mean magnetic field, and the coefficient of $\sin(z/L_H)$ is assumed to be much smaller than unity. In this case the first term in Eq. (8.2) corresponds to the mean magnetic field, and the second term describes its periodic variation in space, which is weak under the conditions of inequality (8.1):

$$H_y=\bar{H}\left[1-\frac{L_H^2[d+\lambda\tanh(L/\lambda)]}{\lambda_j^2(d+\lambda)}\cos\frac{z}{L_H}\right]. \tag{8.3}$$

Let us consider weak periodic perturbations of the ground state of the system parallel to the mean magnetic field:

$$\varphi(\rho,t)=\varphi_0(z)+\varphi_1(z,t)\cos(k_y y). \tag{8.4}$$

In the linear approximation for determining the small correction φ_1, from Eq. (3.6) we have

$$\varphi_1(z,t)\cos\varphi_0(z)+\frac{\beta}{\omega_j^2}\frac{\partial}{\partial t}\varphi_1(z,t)+\frac{1}{\omega_j^2}\frac{\partial^2}{\partial t^2}\varphi_1(z,t)$$
$$=-\lambda_0^3\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}\frac{dk}{4\pi}\exp[ik(z-z')](k_y^2+k^2)\times\left[d+\lambda\sqrt{1+k_y^2\lambda^2+k^2\lambda^2}\times\tanh\left(\frac{L}{\lambda}\sqrt{1+k_y^2\lambda^2+k^2\lambda^2}\right)\right]^{-1}\varphi_1(z',t). \tag{8.5}$$

Neglecting the weak spatial modulation of the magnetic field $\bar{H}$, we rewrite linear equation (8.5) in the form

$$\varphi_1(z,t)=\sum_{m=-\infty}^{\infty}\varphi_m(t)\exp\left(i\frac{m}{L_H}z+ik_z z\right), \tag{8.6}$$

where k_z varies within the one-dimensional Brillouin zone over the range $-\pi/L_H<k_z<\pi/L_H$. Then in the case of weak coupling of the harmonics, which is realized when

$$\omega_m^2/\omega_j^2\gg 1, \tag{8.7}$$

we find

$$\varphi_m(t)=\varphi_m\exp\left(-\frac{\beta}{2}t\right)\sin\left(t\sqrt{\omega_m^2-\frac{\beta^2}{4}}\right)$$

and the frequency spectrum of the excitations:

$$\omega_m=c_S\left\{\left[k_y^2+\left(k_z+\frac{m}{L_H}\right)^2\right](d+\lambda)\times\left[d+\lambda\sqrt{1+k_y^2\lambda^2+\left(k_z+\frac{m}{L_H}\right)^2\lambda^2}\times\tanh\left(\frac{L}{\lambda}\sqrt{1+k_y^2\lambda^2+\left(k_z+\frac{m}{L_H}\right)^2\lambda^2}\right)\right]^{-1}\right\}^{1/2}, \tag{8.8}$$

where $c_x=\omega_j\lambda_j$ is the velocity of the Swihart wave.[22]

According to (8.8), only excitations at the frequencies ω_m, which cause weak modulation of the magnetic field with the periods $2\pi L_H/m$, appear in the linear approximation with respect to the amplitude $\varphi_1(z,t)$. Consideration of nonlinear corrections in both the equation for the ground state and the equation for $\varphi_1(z,t)$ creates a possibility for the appearance of excitations with frequencies which are multiples of ω_m.[21]

Result (8.8) is distinguished qualitatively from the result obtained in the theory of the spectral properties of a single Josephson junction[21] by the fact that the previously obtained discrete spectrum of excitations is replaced by a set of spectral bands owing to the consideration of the spatial modulation of the phase difference parallel and perpendicular to the constant magnetic field. In addition, the width of each spectral band is determined by the wave numbers k_y and k_z. Such inhomogeneous broadening of the spectral lines results in overlap of the spectral bands, if $k_y\gg|m|/L_H$. The frequencies ω_m (8.8) of such short-wavelength perturbations are almost independent of the magnetic field.

When $k_z L_H$ and $k_y L_H \ll |m|$, the spectrum of excitations is represented by a set of narrow overlapping spectral bands. In this case, in the limit of comparatively thick superconducting layers, where $L\sqrt{1+(m\lambda/L_H)^2} \gg \lambda$, from (8.8) we find

$$\omega_m = mc_S \frac{4\pi\bar{H}(d+\lambda)}{\Phi_0} \left\{ (d+\lambda) \times \left[d+\lambda\sqrt{1+(d+\lambda)^2\left(\frac{4\pi m\lambda}{\Phi_0}\bar{H}\right)^2} \right]^{-1} \right\}^{1/2}. \quad (8.9)$$

Expression (8.9) repeats the result in Ref. 21 and leads to two asymptotic dependences of the frequencies on $\bar{H}$: $\omega_m \propto \bar{H}$ and $\omega_m \propto \sqrt{\bar{H}}$.

The influence of the neighboring Josephson junctions on the spectral composition of the excitations becomes significant when

$$L[1+k_y^2\lambda^2+(k_z+m/L_H)^2\lambda^2]^{1/2} \ll \lambda.$$

Expression (8.8) then takes on the form

$$\omega_m = c_s \left\{ (d+\lambda)\left[k_y^2 + \left(k_2+(d+L)\frac{4\pi m}{\Phi_0}\bar{H}\right)^2\right] \times \left[d+L+L\lambda^2 k_y^2 + L\lambda^2 \times \left(k_z+(d+L)\frac{4\pi m}{\Phi_0}\bar{H}\right)^2\right]^{-1} \right\}^{1/2}. \quad (8.10)$$

A dependence of ω_m on $\bar{H}$ in the form (8.10) can appear only when k_z, $k_y \ll (d+L)4\pi|m|\bar{H}/\Phi_0$.

For such k_y and k_z in weak magnetic fields we have $\omega_m \propto \bar{H}$, and in stronger fields the ω_m do not depend on the magnetic field strength. When d, k_y, and k_z are small, $\omega_m = \text{const}$ if

$$|m|\bar{H} \gg \frac{\Phi_0}{4\pi\lambda L}, \quad (8.11)$$

where $L \ll \lambda$. Since in the theory presented here the field $\bar{H}$ should be smaller than the lower critical field[22] H_{c1}, (8.11) can hold only when $L \gg \lambda/\ln(\lambda/\xi)$. For its part, the smallness of k_y and k_z signifies that the scales of the perturbations parallel and perpendicular to the magnetic field should surpass the correlation length.

9. CONCLUSIONS

A system of coupled equations for the phase difference of the Cooper pairs and the magnetic fields of the different junctions in a layered structure of tunnel junctions between superconducting layers has been derived. The conditions under which the different junctions do not influence one another have been revealed. They are possible, in particular, for superconducting layers whose thickness is smaller than the London depth. A comparatively simple integral equation describing small-scale two-dimensional Josephson vortices has been derived. For a periodic Josephson structure with strong coupling between the different junctions, integrodifferential equations describing simple vortex structures have been derived for the phase difference of the Copper pairs. The conditions for applicability of a description of Josephson vortices based on a system of two differential equations have been determined. The spectrum of generalized Swihart waves in a periodic multilayer structure has been obtained. An exact one-dimensional stationary description of the phase difference of the Cooper pairs in a Josephson structure with thin superconducting layers has been considered. Finally, a theory of the spectrum of excitations in a periodic structure of Josephson junctions in a magnetic field has been devised.

Thus, the basic principles of the nonlocal electrodynamics of small-scale vortex formations in multilayer Josephson structures have been formulated, and some comparatively simple solutions describing vortices have been obtained.

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