# Coupled nonlinear Schrödinger equations for multifrequency wave packets in a dispersive nonlinear medium

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This paper examines the propagation of multifrequency wave packets in a dispersive nonlinear medium. When three-wave resonance is impossible, a system of coupled nonlinear Schrödinger equations is derived to describe the dynamics of the slowly varying amplitudes of the frequency components. The same system is shown to follow in the small-amplitude limit from the equations of Whitham's multiphase theory. 

The evolution of the envelope of a quasimonochromatic wave in a nonlinear medium is known to obey a nonlinear Schrödinger equation.[1] The propagation of the envelope of a multifrequency wave, however, cannot be described by a single nonlinear Schrödinger equation. This problem has been discussed by Oikawa and Yajima,[2,3] who concluded that the dynamics of each frequency component is still described by a nonlinear Schrödinger equation and that the effects of interaction manifest themselves only in small corrections to the phase of the carrier wave. In their classic experiments on the interaction of hydrodynamic solitons, Yuen and Lake[4] used wave packets with different carrier frequencies to ensure that their velocities were different, and explained the observed pattern by employing Oikawa and Yajima's theory.[2,3]

Note, however, that Oikawa and Yajima[2,3] arrived at their result by assuming corrections to the phase of the carrier in a form that ensured the decoupling of the evolution equations. From the standpoint of physics, their assumption was unjustified.

At the same time, several papers have lately appeared in which specific problems in the propagation of multifrequency wave packets have been solved by employing a system of coupled nonlinear Schrödinger equations (see, e.g., Refs. 5–7). Note that the difference between these two approaches is significant because the interaction of waves may change the nature of modulation instability considerably.[6,7] Hence, it is interesting to examine the problem without resorting to a specific model, thereby arriving at a result that is valid over a broad class of systems. To this end, this paper uses the Hamiltonian formalism, one of the most universal in the theory of nonlinear waves.[8]

Consider the simple situation in which only one type of wave with a dispersion relation $\omega(k)$ can propagate in the system. When no three-wave interaction processes are possible, the Hamiltonian of the system is[8]

$$H=H_0+H_{\text{int}}+\cdots, \tag{1}$$

$$H_0=\int \omega(k)a_k a_k^* dk, \tag{2}$$

$$H_{\text{int}}=\tfrac{1}{2}\int T_{k_1k_2k_3k_4}a_{k_1}^*a_{k_2}^*a_{k_3}a_{k_4} \times\delta(k_1+k_2-k_3-k_4)dk_1dk_2dk_3dk_4, \tag{3}$$

where $a_k$ is a Fourier component of the wave.

If we assume the wave field to be a single spectrally narrow wave packet with mean wave vector $k_0$, or

$$a_k=c(k_0+\kappa)\exp[-i\omega(k_0)t],$$

the appropriate equation of motion can be shown to be a nonlinear Schrödinger equation.[8]

Now let us take the wave field to be a two-frequency packet:

$$a_k=c_1(k_1+\kappa_1)\exp(-i\omega_1 t)+c_2(k_2+\kappa_2)\exp(-i\omega_2 t),$$

where $\omega_j=\omega(k_j)$, $j=1,2$. In terms of the $c_j$, the Hamiltonian (1) assumes the form

$$\tilde{H}=H-\omega_1\int|c_1|^2d\kappa-\omega_2\int|c_2|^2d\kappa. \tag{4}$$

The interaction Hamiltonian $H_{\text{int}}$ is

$$\begin{aligned}H_{\text{int}}=\tfrac{1}{2}\int&[T_{11}c_1^*(\kappa_1)c_1^*(\kappa_2)c_1(\kappa_3)c_1(\kappa_4)\\&+T_{12}c_1^*(\kappa_1)c_2^*(\kappa_2)c_1(\kappa_3)c_2(\kappa_4)\\&+T_{22}c_2^*(\kappa_1)c_2^*(\kappa_2)c_2(\kappa_3)c_2(\kappa_4)]\\&\times\delta(\kappa_1+\kappa_2-\kappa_3-\kappa_4)d\kappa_1d\kappa_2d\kappa_3d\kappa_4,\end{aligned}$$

where

$$T_{11}=T_{k_1k_1k_1k_1},\quad T_{22}=T_{k_2k_2k_2k_2},$$

$$T_{12}=T_{k_1k_2k_1k_2}+T_{k_1k_2k_2k_1}+T_{k_2k_1k_1k_2}+T_{k_2k_1k_2k_1}.$$

Taking the inverse Fourier transform

$$\psi_j=\int c_j\exp(-ikx)dk$$

and expanding the dispersion relation $\omega(k)$ in the neighborhood of each wave vector $k_j$,

$$\omega(k)=\omega_j+(k-k_j)\omega_j'+\tfrac{1}{2}(k-k_j)^2\omega_j'',$$

where $\omega_j'=d\omega(k_j)/dk$, and $\omega_j''=d^2\omega(k_j)/dk^2$, we obtain from the equations of motion for the Hamiltonian (4) the coupled nonlinear Schrödinger equations

$$\begin{aligned}
&i\left(\frac{\partial\psi_1}{\partial t}+\omega_1'\frac{\partial\psi_1}{\partial x}\right)+\frac{\omega_1''}{2}\frac{\partial^2\psi_1}{\partial x^2}\\
&\qquad-(2T_{11}|\psi_1|^2+T_{12}|\psi_2|^2)\psi_1=0,\\
&i\left(\frac{\partial\psi_2}{\partial t}+\omega_2'\frac{\partial\psi_2}{\partial x}\right)+\frac{\omega_2''}{2}\frac{\partial^2\psi_2}{\partial x^2}\\
&\qquad-(2T_{22}|\psi_2|^2+T_{12}|\psi_1|^2)\psi_2=0.
\end{aligned}\tag{5}$$

This result can easily be generalized to an $n$-frequency wave

$$a_k=\sum_{j=1}^{n}c_j(k_j+\kappa_j)\exp(-i\omega_j t).$$

We assume that the dispersion relation is such that the conditions for four-wave resonance,

$$\omega_1+\omega_2=\omega_3+\omega_4,\quad k_1+k_2=k_3+k_4,\tag{6}$$

are satisfied only in the trivial case $k_1=k_3$ and $k_2=k_4$ or $k_1=k_4$ and $k_2=k_3$. Then a similar procedure leads to a system of $n$ coupled nonlinear Schrödinger equations of the form

$$\begin{aligned}
&i\left(\frac{\partial\psi_j}{\partial t}+\omega_j'\frac{\partial\psi_j}{\partial x}\right)+\frac{\omega_j''}{2}\frac{\partial^2\psi_j}{\partial x^2}\\
&\qquad-\left(2T_{jj}|\psi_j|^2+\sum_{j\neq 1}T_{ji}|\psi_i|^2\right)\psi_j=0,
\end{aligned}$$

where $T_{jj}$ and $T_{ji}$ are defined in the same way as $T_{11}$ and $T_{12}$.[1)]

As an example we take the nonlinear Klein–Gordon equation discussed by Oikawa and Yajima:[2,3]

$$\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial x^2}+y-\beta y^3=0.\tag{7}$$

The Hamiltonian of this system is

$$H=\tfrac{1}{2}\int\left[\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial y}{\partial x}\right)^2+y^2-\frac{\beta y^4}{2}\right]dx.$$

Taking Fourier transformations

$$q_k=\int y\exp(ikx)dx,\quad \rho_k=\int\left(\frac{\partial y}{\partial t}\right)\exp(ikx)dx$$

and passing to the variables $a_k$ and $a_k^*$ via the formulas[8]

$$q_k=\frac{1}{\sqrt{2\omega_k}}(a_k-a_k^*),\quad \rho_k=\sqrt{\frac{\omega_k}{2}}(a_k+a_k^*),$$

we reduce the Hamiltonian to the form (1), with $H_0$ and $H_{\text{int}}$ taking the same form as (2) and (3), respectively, and

$$T_{k_1k_2k_3k_4}=-\frac{3}{8}\frac{\beta}{\sqrt{\omega_1\omega_2\omega_3\omega_4}}.$$

Applying the procedure described above obviously leads to

$$\begin{aligned}
&i\left(\frac{\partial u_1}{\partial t}+\frac{k_1}{\omega_1}\frac{\partial u_1}{\partial x}\right)+\frac{1}{2\omega_1^3}\frac{\partial^2 u_1}{\partial x^2}\\
&\qquad+\frac{3\beta}{2\omega_1}(|u_1|^2+2|u_2|^2)u_1=0,\\
&i\left(\frac{\partial u_2}{\partial t}+\frac{k_2}{\omega_2}\frac{\partial u_2}{\partial x}\right)+\frac{1}{2\omega_2^3}\frac{\partial^2 u_2}{\partial x^2}\\
&\qquad+\frac{3\beta}{2\omega_2}(|u_2|^2+2|u_1|^2)u_2=0,
\end{aligned}$$

where $u_j=\psi_j\sqrt{2\omega_j}$.

An alternative approach in describing multifrequency wave packets is Whitham's multiphase theory.[10,11] We now show that the system of equations (8) follows from this theory in the small-amplitude limit, just as the nonlinear Schrödinger equations follow from the equations of the single-phase theory.[1] For Eq. (7), the dynamics of the slowly varying amplitudes, frequencies, and wave numbers of two interacting waves obeys the following system of equations:[11]

$$\omega_1^2-k_1^2=1-\beta(\tfrac{3}{4}A_1^2+\tfrac{3}{2}A_2^2)+\frac{\varepsilon^2}{A_1}\left(\frac{\partial^2 A_1}{\partial T^2}-\frac{\partial^2 A_1}{\partial X^2}\right),\tag{9a}$$

$$\omega_2^2-k_2^2=1-\beta(\tfrac{3}{4}A_2^2+\tfrac{3}{2}A_1^2)+\frac{\varepsilon^2}{A_2}\left(\frac{\partial^2 A_2}{\partial T^2}-\frac{\partial^2 A_2}{\partial X^2}\right),\tag{9b}$$

$$\frac{\partial}{\partial T}(\omega_1A_1^2)+\frac{\partial}{\partial X}(k_1A_1^2)=0,\tag{10a}$$

$$\frac{\partial}{\partial T}(\omega_2A_2^2)+\frac{\partial}{\partial X}(k_2A_2^2)=0,\tag{10b}$$

$$\frac{\partial k_1}{\partial T}+\frac{\partial\omega_1}{\partial X}=0,\ \frac{\partial k_2}{\partial T}+\frac{\partial\omega_2}{\partial X}=0,\tag{11}$$

where the $A_j$ are the amplitudes of the wave packets, $T$ and $X$ are the "stretched" time and coordinate, and $\varepsilon$ is a small parameter. In Eqs. (9) we allowed for terms of order $\varepsilon^2$, which play an important role in the small-amplitude limit.[1]

The transition to two almost monochromatic packets with small amplitudes is accomplished via the substitutions

$$k_j\to k_{0j}-\varepsilon\frac{\partial\varphi_j}{\partial X},\ \omega_j\to\omega_{0j}-\varepsilon\frac{\partial\varphi_j}{\partial T},$$

$$A_j\to\varepsilon A_j,\ \frac{\partial}{\partial T}\to\frac{\partial}{\partial T}+\varepsilon\frac{\partial}{\partial\tau},$$

with $\omega_{0j}^2-k_{0j}^2=1$. Then, isolating terms of order $\varepsilon$ in Eqs. (9) and of order $\varepsilon^2$ in Eqs. (10), we see that $A_j$ and $\varphi_j$ can depend on $X$ and $T$ only in the combination $\xi_j=X-\omega_{0j}'T$, where $\omega_{0j}'=d\omega_{0j}/dk_{0j}=k_{0j}/\omega_{0j}$. The terms of order $\varepsilon^2$ in (9) and of order $\varepsilon^3$ in (10) yield

$$\frac{\partial\varphi_1}{\partial\tau}=\frac{1-\omega_{01}'^2}{2\omega_{01}}\left[\frac{1}{A_1}\frac{\partial^2 A_1}{\partial\xi_1^2}-\left(\frac{\partial\varphi_1}{\partial\xi_1}\right)^2\right]+\frac{3\beta}{8\omega_{01}}(A_1^2+2A_2^2),$$

$$\frac{\partial A_1}{\partial\tau}=\frac{\omega_{01}'^2-1}{2\omega_{01}}\left(2\frac{\partial A_1}{\partial\xi_1}\frac{\partial\varphi_1}{\partial\xi_1}+A_1\frac{\partial^2\varphi_1}{\partial\xi_1^2}\right)$$

and a pair of similar equations for $A_2$ and $\varphi_2$, which after the substitution

$$u_j=\tfrac{1}{2}A_j\exp(i\varphi_j)$$

leads to the system of equations (8).

Thus, we have shown that for a broad class of systems characterized by a Hamiltonian of the form (1)–(3), a meaningful description of propagation of weakly nonlinear multifrequency waves requires the use of a system of coupled nonlinear Schrödinger equations. The proposed approach simplifies the derivation of these equations, since although in many cases the use of traditional methods based on asymptotic expansions presents no difficulties in principle, it leads to extremely cumbersome calculations, especially when the number of interacting waves is large.

[1)]Otherwise, if for instance conditions (6) are met for frequencies $\omega_i$, $i=1,2,3,4$, the equation for $\psi_1$ acquires terms proportional to $\psi_2^*\psi_3\psi_4$, etc. (see, e.g., Ref. 9). But we deliberately restrict ourselves to the simplest case, which requires no specific resonance conditions.


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