Solitary wave instability in the positive-dispersion media described by the two-dimensional Boussinesq equations

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The linear and nonlinear stages of transverse and longitudinal instabilities of plane solitons are investigated for the two-dimensional Boussinesq equations describing wave processes of moderate amplitude in media with weak positive-dispersion. The cause of the plane soliton transverse instability is found to be a resonance with "periodic solitons" resulting in decay of a plane solitary wave and in the formation of another one, periodically modulated along its front.

1. INTRODUCTION

Investigations performed over the past twenty years have shown that plane solitary waves are unstable with respect to long-wave transverse modulation of their fronts in weak positive-dispersion media.[1–5] This fact was first discovered by B. B. Kadomtsev and V. I. Petviashvili[1] within the framework of the approximate equation that is now known as the KP equation. Plane soliton instability of self-focusing type is encountered in wave processes of different physical origin: for example, for gravity-capillary waves on the surface of thin liquid films,[6,7] for oblique magnetoacoustic waves in a magnetized plasma,[8,9] for waves in a charged surface of a liquid dielectric,[10] etc. As is known now, in positive-dispersion media there exist not only unstable plane solitons but also stable two-dimensional (2D) solitons localized in all directions. Ablowitz and Segur[2,3] put forth a hypothesis that such 2D solitons may play the role of elementary particles that are formed as a result of breaking of plane nonlinear waves. Some numerical simulations on self-focusing instability of plane perturbations[11] confirm this idea.

However, another idea has prevailed for a long time,[12] according to which plane soliton instability is caused by the excitation of small wave oscillations at the soliton front. The spectrum of small perturbations in positive-dispersion media looks like a bowl and is decaying as compared with three-wave resonances. According to the hypothesis proposed by Zakharov,[12] the development of small perturbations due to the energy transferred from nonlinear waves and subsequently dispersed in space leads to breaking of the original wave and to formation of a quasi-stochastic wave field.

Recently, a new point of view on this problem was elaborated in a number of works[13–15] where new types of spatio-temporal resonance of plane and essentially 2D soliton structures were discovered for the KP equation with positive dispersion. According to the mechanism of self-focusing instability detected by Pelinovsky and Stepanyants,[15] an unstable plane soliton evolution under the action of a periodic perturbation results in its decay into a plane soliton of smaller amplitude and a chain of uniformly spaced 2D solitons parallel to the front of the original soliton. The parameters of the nonlinear structures appearing in the decay are determined by the nonlinear (soliton) resonance relations:

$$D(\omega_1-\omega_2,\mathbf{k}_1-\mathbf{k}_2)=0, \qquad (1.1)$$

where ω_i, $\mathbf{k}_i=(k_{xi},k_{yi})$ are the "frequency" (the reciprocal of the characteristic duration or period) and the "wave vector" (the components of which are proportional to the characteristic inverse scales along the x- and y-axes) of the soliton structures and $D(\omega_i,\mathbf{k}_i)=0$ is their nonlinear dispersion relation.

New structures are also unstable with respect to long-wave modulation of their fronts and, in turn, decay into plane solitons of still smaller amplitudes and longer chains of 2D solitons. Such a cascade may be infinitely long with gradual slowing down because the growth rate of the self-focusing instability decreases with increasing period of transverse modulation.

The KP model describes multidimensional wave processes whose characteristic scale is larger in the longitudinal direction than in the transverse direction.[1] As a consequence, the spatial variables do not enter the KP equation on equal terms. It is important to study the characteristic features of plane solitons in the framework of the basic equations of an isotropic medium. In the present paper this problem is considered for some modifications of the well known Boussinesq equation.[6–10] We show that the idea of soliton resonance has again a fundamental meaning for this problem.

2. BOUSSINESQ EQUATION AND MODIFICATIONS

There exist several variants of the Boussinesq equations containing different nonlinear and dispersive terms.[8] We restrict our consideration only to two, most typical models.

Vortex-free motion of long waves with small but finite amplitude on the surface of an inviscid incompressible liquid[6,7] or magnetoacoustic waves in a collisionless plasma[8] is described by the Boussinesq equation in the form

$$\phi_{tt}-c^2\Delta\phi+2\beta c\Delta^2\phi+\tfrac{2}{3}\alpha(2\nabla\phi\nabla\phi_t+\phi_t\Delta\phi)=0, \qquad (2.1)$$

where ϕ is the velocity potential $\nabla\phi=\mathbf{v}$ and $\Delta\equiv\nabla^2$ is the 2D Laplace operator.

Another well known Boussinesq equation is valid in the same approximation but is written for a different component of the wave field:[8–10]

$$H_{tt}-c^2\Delta H+2\beta c\Delta^2 H+\alpha c\Delta(H^2)=0. \qquad (2.2)$$

For water waves H denotes a displacement of the free surface and for plasma waves a perturbation of magnetic field intensity.

From a physical point of view, both models are correct for long waves of moderate amplitude (α and β small), and are equivalent over their range of validity, but they have different mathematical structures. Nevertheless, it will be shown below that the instability of moderate-amplitude solitons occurs in a similar manner in either of the two models.

In the one-dimensional case, both Boussinesq equations can be investigated using the inverse scattering transform, and possess multisoliton solutions.[16,17] However, the two-dimensional equations (2.1) and (2.2) are evidently not integrable, and have no N-soliton solutions. We analyze the linear and nonlinear stages of plane soliton instability describing these models by means of some approximate methods.

Solutions in the form of plane solitons can be found directly from the basic equations and are written as follows:

$$\phi_0=-\frac{6\beta c\mathbf{k}^2}{\alpha\omega}\tanh\frac{\mathbf{kr}-\omega t}{2}+\text{const} \qquad (2.3\text{a})$$

for Eq. (2.1) and

$$H_0=\frac{3\beta\mathbf{k}^2}{\alpha}\operatorname{sech}^{-2}\frac{\mathbf{kr}-\omega t}{2} \qquad (2.3\text{b})$$

for Eq. (2.2). Here the parameters ω and $\mathbf{k}$ are related by

$$D(\omega,\mathbf{k})=\omega^2-c^2\mathbf{k}^2+2\beta c\mathbf{k}^4=0. \qquad (2.4)$$

Both models are spatially isotropic, so that solitons can propagate in any direction in the xy-plane. We choose the direction of a plane soliton vector $\mathbf{k}=(k,0)$ coinciding with the x-axis. Further on, we consider the evolution of such a single-soliton solution with a weakly perturbed front. If the soliton modulation is smooth enough that the diffractive effects of perturbation diverging in the transverse direction have the same order of smallness as the dispersive effects, then both Boussinesq equations can be simplified and reduced to a completely integrable KP equation (see, for example,[9]):

$$\left(\pm\phi_t+c\phi_x\pm\frac{\alpha}{2}(\phi_x)^2-\beta\phi_{xxx}\right)_x=-\frac{c}{2}\phi_{yy}+O(\beta^2), \qquad (2.5)$$

where the plus and minus signs correspond to waves propagating to the right or left, respectively, and H and ϕ are related by $H=\mp\phi_x$.

However, if the characteristic scale of transverse perturbations is comparable to the longitudinal, the KP approximation is not correct and we have to use the basic Boussinesq equations.

3. TRANSVERSE AND LONGITUDINAL INSTABILITY OF A PLANE SOLITON

Here we investigate the principal characteristic features of plane soliton instability at $\beta>0$ by using an asymptotic approach to the solution of linear equations with variable coefficients.

First, we consider the model (2.2). We are looking for a localized linear mode $w(\xi)$, where $\xi=x-vt$, with the positive growth rate λ and the wave number of transverse modulation p in the problem linearized against the solution (2.3b), i.e., $H=H_0(\xi)+\varepsilon w(\xi)\exp(\lambda t+ipy)$:

$$\frac{d^2}{d\xi^2}\hat{\mathbf{L}}w\equiv\frac{d^2}{d\xi^2}\left(v^2-c^2+2\beta c\frac{d^2}{d\xi^2}+2\alpha cH_0\right)w$$
$$=-(\lambda^2+c^2p^2+2\beta cp^4)w+2\lambda v\frac{dw}{d\xi}+4\beta cp^2\frac{d^2w}{d\xi^2}$$
$$+2\alpha cp^2H_0w. \qquad (3.1)$$

If the modulation is small enough, the linear Eq. (3.1) can be solved by expanding the eigenfunction $w(\xi)$ and the eigenvalue λ in p (see Refs. 1–5):

$$w(\xi)=\sum_{n=0}^{\infty}p^nw^{(n)}(\xi),\quad \lambda=\sum_{n=1}^{\infty}p^n\lambda_n. \qquad (3.2)$$

This asymptotic expansion is based on the assumption that the presence of small transverse modulation of the soliton phase and velocity is the reason for its instability. In the absence of modulation ($p=0$), the soliton is stable ($\lambda=0$).

The first two corrections, determining unambiguously the entire series for the linear localized mode, have the form

$$w^{(0)}=\frac{dH_0}{d\xi},\quad w^{(1)}=-\lambda_1\frac{dH_0}{dv} \qquad (3.3)$$

and are, actually, the customary renormalization of the phase and velocity of a modulated soliton. The leading part of the eigenvalue λ_1 is contained in these formulas as an arbitrary parameter.

In the next order ($\sim p^2$), the linear inhomogeneous equation for $w^{(2)}$ has a bounded solution only if the inhomogeneous terms are orthogonal to the falling solution of the self-adjoint operator $\hat{\mathbf{L}}$. Then λ_1 is refined from the equation

$$\left(1-\frac{8\beta k^2}{3c}\right)\lambda_1^2=\frac{2\beta ck^2}{3}-\frac{16\beta^2k^4}{15}. \qquad (3.4)$$

This implies that the solitons of moderate amplitudes with $k\sim O(1)$ are unstable only for $\beta>0$. We shall call such an instability and the plane soliton resonance associated with it transverse instability (resonance).

The higher-order corrections in (3.4), which are important only for $k^2\sim O(\beta^{-1})$, lead to a new effect. When the expression in brackets before λ_1^2 in (3.4) vanishes, which occurs for high-amplitude solitons with $k^2=k_c^2=3c/8\beta$, λ can be finite even when $p=0$, i.e., in the $1D$ case. Such an instability and the related resonance of plane solitons of large amplitudes $k\geqslant k_c$ and small velocities $|v|\leqslant c/2$ are referred to as longitudinal ones. This phenomenon was first discovered by Spector *et al.*[18] and by Tajiri and Murakami.[19]

The last term in (3.4) diminishes the growth rate λ but its contribution is appreciable only for $k^2\geqslant 5c/8\beta$. However, a soliton with such a value of k no longer exists, because its velocity becomes imaginary: $v^2=-c^2/4$.

In analyzing the higher-order corrections, one should remember that all the Boussinesq equations obtained by asymptotic procedures[6–10] hold when the terms of order $O(\alpha^2,\beta^2,\alpha\beta)$ are negligible. Therefore the longitudinal instability may be physically meaningless and may be not observed in real systems.

When the orthogonality conditions are fulfilled, we can calculate the correction $w^{(2)}$. However, it is not localized in space and we need to set boundary conditions for $w^{(2)}$. The correction is not localized because of the radiation propagating from a perturbed soliton in the nonstationary problem. It is physically obvious that the growth of soliton modulation leads to formation of highly nonlinear 2D structures with larger amplitudes and smaller velocities than in the original soliton. The growing nonlinear perturbations propagate towards $\xi<0$ if $v>0$ and towards $\xi>0$ if $v<0$ in the reference frame ξ moving with the soliton velocity V. The lack of radiation in the opposite directions allows us to choose the boundary conditions for each correction $w^{(n)}$ in the form

$$w^{(n)}(\xi\rightarrow+\infty)=0 \quad \text{for } v>0, \tag{3.5a}$$

$$w^{(n)}(\xi\rightarrow-\infty)=0 \quad \text{for } v<0. \tag{3.5b}$$

Note that linear dispersive waves in positive-dispersion media propagate faster than soliton structures. Therefore, if small oscillations of a wave medium were excited as a result of soliton instability, as was supposed earlier,[12] we would choose the opposite boundary conditions for the linear discrete-spectrum mode. However, comparison with the known mode for the KP equation[12,15] reveals that this step would lead to an error.

A correct choice of the boundary conditions (3.5) gives an unambiguous correction $w^{(2)}$ in the form

$$w^{(2)}=a_1\frac{dH_0}{dv}+a_2\int_{\pm\infty}^{x}H_0dx+a_3\times\left(4xH_0+x^2\frac{dH_0}{dx}\right), \tag{3.6}$$

where

$$a_1=-\lambda_2-\frac{6c}{\beta|v|k^3}\left(\frac{v^2\lambda_1^2}{4c^2}-\frac{\beta^2k^4}{5}\right),$$

$$a_2=\frac{1}{\beta^2k^4}\left(\frac{v^2\lambda_1^2}{4c^2}-\frac{\beta^2k^4}{5}\right),\quad a_3=\frac{v^2\lambda_1^2}{8\beta^2c^2k^4},$$

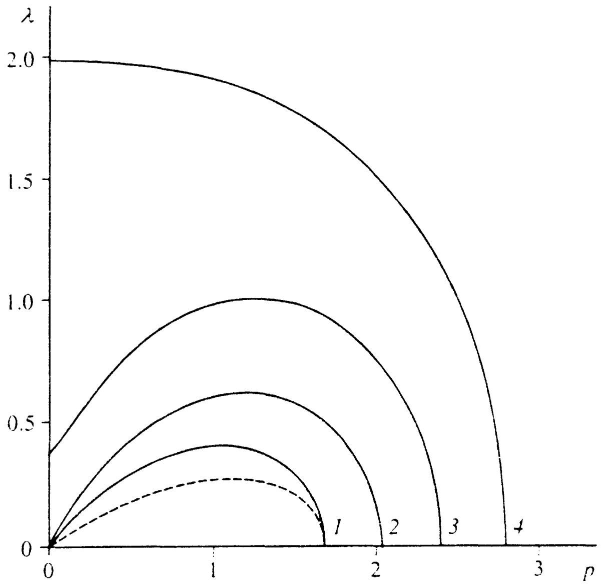

FIG. 1. The approximate dependence of the growth rate λ of the linear mode versus transverse wavenumber p (solid lines) for some values of the plane soliton amplitude in the linearized problem (3.1) for $c=1$, $\beta=0.01$, $k_c=6.124$. The numbers near the curves correspond to the following values of the parameter k: *1*—5.25; *2*—5.75; *3*—6.25; *4*—6.75. The dispersion dependence $\lambda(p)$, following from Eq. (3.9) is shown by the dashed line for $k=5.25$.

where the upper sign on the lower integration limit corresponds to $v>0$, and the lower one to $v<0$.

The correction λ_2 to the eigenvalue is calculated from the similar requirement of a bounded solution $w^{(3)}$ to order p^3 and is determined by the equation:

$$\lambda_2\left(1-\frac{8\beta k^2}{3c}\right)=-|v|\left(\frac{1}{\beta ck^3}\left(\frac{v^2\lambda_1^2}{4c^2}-\frac{\beta^2k^4}{5}\right)+\frac{v^2+\lambda_1^2}{2c^2k}\right). \tag{3.7}$$

Hence, the growth rate λ is restricted at large enough p (because $\lambda_2<0$), which specifies a finite size of the instability region on the p-axis.[5] The approximate dependence $\lambda(p)$ is shown in Fig. 1 as solid lines for some values of the soliton amplitude (velocity). The displacement of the curve from zero in the region of long-wave transverse modulation $\lambda(0)>0$ corresponds to the appearance of longitudinal instability at large enough soliton amplitudes.

In principle, we are able to calculate successively all the terms in the series (3.2) for $k<k_c$. In order to allow for the existence of instability in the 1D problem at $k\geqslant k_c$, it is necessary to modify the expansions (3.2) and include the leading part $\lambda_0\neq 0$ into the series in terms of λ. If we keep in all relations only the lowest term with respect to the parameter β, then the series (3.2) correspond exactly to the expansion of the linear localized mode and its growth rate, which is known for the KP equation (2.5):[12,15]

$$w(\xi;k,k')=\frac{d^2}{d\xi^2}\left(\frac{\exp(k'\xi/2)}{\cosh(k\xi/2)}\right), \tag{3.8}$$

where

$$\lambda = -\frac{\beta k'}{2}(k-k'^2), \tag{3.9a}$$

$$p^2 = \frac{3\beta}{8c}(k^2-k'^2)^2, \tag{3.9b}$$

and the condition of mode localization implies that $|k'| \leqslant k$.

The dispersion dependence $\lambda(p)$ following from Eqs. (3.9) is shown in Fig. 1 by the dashed curve. As is seen, the curves are qualitatively similar at small amplitudes of the plane soliton. However, there is no longitudinal soliton instability in the framework of the KP equation.

By repeating similar calculations for the other Boussinesq equation described by (2.1) and writing

$$\phi = \phi_0(\xi) + \varepsilon\psi(\xi)\exp(\lambda t + ipy),$$

where $\phi_0(\xi)$ is expressed by (2.3a), we can readily find the first terms (3.3) of the series (3.2) for the linear mode $w(\xi) = \psi_\xi$ and the dispersion relation for the eigenvalue λ_1

$$\left(1 + \frac{8\beta^2 k^4}{5v^2}\right)\lambda_1^2 = \frac{2\beta c k^2}{3} - \frac{8\beta^2 k^4}{15}. \tag{3.10}$$

It follows from this relation that except the transverse instability of a plane soliton, other instability types including the longitudinal one are absent. Besides, the higher-order corrections in β do not interfere with the transverse instability throughout the region of soliton existence, and similarly for the formulas (3.4) and (3.7).

4. TRANSVERSE SOLITON RESONANCE

In the previous section the problem of plane soliton instability was considered for the basic models, without allowance for their range of applicability. We formally kept terms to the $O(\beta^2)$ and higher orders which are usually supposed to be small values. Results of our calculations reveal that the influence of these terms on the growth rates of the transverse soliton instability and on distortions of the linear mode is actually negligible. Therefore, the problem can be adequately investigated to first order in $O(\beta)$. In this approximation, the basic equation (2.1) can be reduced by replacing the dependent variable by a bilinear form which has partial explicit solutions.

For this purpose we use the approach proposed by Yajima *et al.*[20] and rewrite Eq. (2.1) in the equivalent form neglecting terms of order $O(\alpha^2, \alpha\beta, \beta^2)$:

$$\phi_{tt} - c^2\Delta\phi + \frac{2\beta}{3c}\left(2\Delta\phi_{tt} + \frac{\phi_{tttt}}{c^2}\right) + \frac{2\alpha}{3}\left(2\nabla\phi\nabla\phi_t + \frac{\phi_t\phi_{tt}}{c^2}\right) = 0. \tag{4.1}$$

It is not difficult to find (see Ref. 20) that the substitution $\phi = 12\beta/\alpha c(\partial/\partial)t \ln f$ transforms (4.1) to a bilinear equation for $f(\mathbf{r},t)$:

$$\left(D_t^2 - c^2 D_\mathbf{r}^2 + \frac{4\beta}{3c} D_t^2 D_\mathbf{r}^2 + \frac{2\beta}{3c^3} D_t^4\right) ff + \frac{8\beta}{3c^3} \times (D_t^2 - c^2 D_\mathbf{r}^2) f_t f_t = O(\beta^2), \tag{4.2}$$

where we use the Hirota differential operator:

$$(D_t^n) fg = (\partial_t - \partial_{t'})^n f(t) g(t') |_{t=t'}$$

and

$$D_\mathbf{r}^2 = D_x^2 + D_y^2.$$

For the variable f the single-soliton solution (2.3a) of Eq. (4.1) has the form

$$f_1 = 1 + \exp(\eta_i), \quad \eta_i = \mathbf{k}_i\mathbf{r} - \omega_i t + \eta_{i0}, \tag{4.3}$$

where η_{i0} is a phase constant and the parameters $\omega_i, \mathbf{k}_i$ satisfy the relation

$$\tilde{D}(\omega_i, \mathbf{k}_i) = \omega_i^2 - c^2\mathbf{k}_i^2 + \frac{4\beta}{3c}\omega_i^2\mathbf{k}_i^2 + \frac{2\beta}{3c^3}\omega_i^4 = 0, \tag{4.4}$$

which coincides with (2.4) when terms $O(\beta^2)$ are neglected.

If the parameters k_{xi}, k_{yi} are supposed to be complex values, the solution (4.3) is complex too. In this case, the corresponding function $\phi(x,y)$ is localized in the direction of $\mathrm{Re}\,\mathbf{k}_i$ and periodic in the direction of $\mathrm{Im}\,\mathbf{k}_i$. We shall call such periodic and localized complex solution a periodic soliton.

A two-soliton solution in conventional form[20,21]

$$f_2 = 1 + \exp(\eta_1) + \exp(\eta_2) + A_{12}\exp(\eta_1 + \eta_2) \tag{4.5}$$

can also be obtained for Eq. (4.2) with the function A_{12} $(\mathbf{k}_1, \mathbf{k}_2)$

$$A_{12} = -\frac{\tilde{D}(\omega_1 - \omega_2, \mathbf{k}_1 - \mathbf{k}_2)}{\tilde{D}(\omega_1 + \omega_2, \mathbf{k}_1 + \mathbf{k}_2)}\left(1 + \frac{8\beta}{3c^3}\omega_1\omega_2\right). \tag{4.6}$$

A detailed investigation of Eqs. (4.5) and (4.6), described below, allows us to analyze the development of transverse instability of plane solitons and the formation of 2D modulated waves in media with positive dispersion, $\beta > 0$.

4.1. Interaction of Plane Solitons

To analyze the solution (4.5), (4.6) describing the interaction of two plane solitons, it is convenient to introduce the angle between their propagation directions $\theta = \widehat{\mathbf{k}_1\mathbf{k}_2}$. Then function $A_{12}(\mathbf{k}_1, \mathbf{k}_2)$ can be rewritten in the form

$$A_{12} = \frac{1 - \cos\theta + \frac{5\beta}{3c}(|\mathbf{k}_1| - |\mathbf{k}_2|)^2\left(1 + \frac{4}{5}\cos\theta\right)}{1 - \cos\theta + \frac{5\beta}{3c}(|\mathbf{k}_1| + |\mathbf{k}_2|)^2\left(1 + \frac{4}{5}\cos\theta\right)} \times \left(1 + \frac{8\beta}{3c}\left|\mathbf{k}_1\right|\left|\mathbf{k}_2\right|\right). \tag{4.7}$$

Obviously, at $\beta > 0$ the function A_{12} is positive over the entire range of real values of θ. Therefore, the plane soliton interaction is nonresonant and leads only to appearance of a spatio-temporal phase shift $\delta = \ln A_{12}$ of the soliton fronts relative to the narrow region of their interaction. For large enough angles $\theta \sim O(1)$, this shift is small and is equal to

$$\delta = -\frac{2\beta}{c}|\mathbf{k}_1||\mathbf{k}_2|\frac{1 + 2\cos\theta}{1 - \cos\theta} + O(\beta^2). \tag{4.8}$$

It differs from the soliton shift in negative-dispersion media only by its sign. It follows from (4.8) that at $\theta=2\pi/3$, $4\pi/3$ the phase shift vanishes, i.e., two solitons do not interact in the model (2.1) at small β, and the field $\phi(\mathbf{r},t)$ is a linear superposition of the fields of two solitons. For negative-dispersion media this effect was first detected by Benney and Luke.[22,23]

However, at small angles $\theta\sim O(\beta^{1/2})$, for which the solitons interact strongly, the difference in the dispersion is important. Whereas the spatial resonance of plane solitons and the formation of soliton triplets intersecting at an angle to one another are possible at $\beta<0$ [23], for $\beta>0$ the field does not strongly depend upon θ, and A_{12} can be rewritten as

$$A_{12}\approx\frac{\theta^2+\frac{6\beta}{c}(|\mathbf{k}_1|-|\mathbf{k}_2|)^2}{\theta^2+\frac{6\beta}{c}(|\mathbf{k}_1|+|\mathbf{k}_2|)^2}. \tag{4.9}$$

4.2. Resonance of Periodic Solitons

As follows from (1.1) and (4.6), soliton resonance occurs when A_{12} tends to zero or infinity. For media with positive dispersion, this is only possible by virtue of analytic continuation of the real soliton parameters into the complex plane. The conditions $A_{12}=0$ or $A_{12}=\infty$ determine the relationship between the parameters of resonant solitons:

$$\frac{k_{y2}}{k_{x2}}=\frac{k_{y1}}{k_{x1}}+\sqrt{-\frac{6\beta}{c}}\,\frac{|\mathbf{k}_1|^2}{k_{x1}^2}\begin{cases}(|\mathbf{k}_1|+|\mathbf{k}_2|)+O(\beta^{3/2}), & \text{(4.10a)}\\ (|\mathbf{k}_1|-|\mathbf{k}_2|)+O(\beta^{3/2}). & \text{(4.10b)}\end{cases}$$

To understand the physical meaning of this resonance, we suppose one of the solitons to be real, with $\mathbf{k}_1$ $(k_1,0)$ and $A_{12}=0$. Then the solution (4.5), which can be rewritten in the form

$$f_2=1+\exp(\eta_1)+\exp(\eta_2), \tag{4.11}$$

remains a complex function and formally describes the decay of a real plane soliton with parameters ω_1, $\mathbf{k}_1$ into two complex ones with parameters ω_2, $\mathbf{k}_2(k_2,ip)$ and $\omega_3=\omega_1-\omega_2$, $\mathbf{k}_3=\mathbf{k}_1-\mathbf{k}_2$. However, in the initial stage of the decay, this solution is a plane soliton with small periodic perturbation which is nothing but the linear localized mode (3.8) with parameters $k=k_1$, $k'=2k_2-k_1$, and $\lambda=k_2(v_1-v_2)$. The resonance condition (4.10b) leads to Eq. (3.9b), allowing us to find a dispersion relation $\lambda(p)$ for this mode by eliminating k'.

It is obvious from Eq. (3.9b) that the wave number of transverse modulation at which resonant soliton instability can be observed is a small quantity of order $O(\beta^{1/2})$. In other words, the instability of a plane soliton with moderate amplitude is caused by long transverse perturbations when the dispersion and diffraction of the wave field have the same order of smallness. In this case, the KP equation is again a universal model for description of transverse soliton instability in weakly nonlinear dispersive media.

Note that the possibility of periodic soliton resonance in the Boussinesq equation at $\beta>0$ was first pointed out by Miles.[24] However, he did not associate this phenomenon with plane soliton instability. Moreover, he cautioned that such an instability makes the solutions thus derived physically meaningless.

4.3. Solitary waves with periodically modulated fronts

Using complex-valued periodic solutions, one can construct real solutions describing waves that are periodic in one coordinate (the y-axis, for instance) and localized in the other (the x-axis). Such solutions are described by Eq. (4.5) at $\omega_1=\omega_2=\omega$ and $\mathbf{k}_1(k,ip)=\bar{\mathbf{k}}_2$, where k and p are real, and the overbar denotes complex conjugation. This formula can be rewritten in a more convenient form:[24]

$$f_2=\cosh(kx-\omega t)+\mu\cos(py), \tag{4.12}$$

where

$$\mu^2=\frac{1}{A_{12}}=1-\frac{2\beta}{c}\,\frac{3k^4-2k^2p^2-p^4}{p^2}+O(\beta^2). \tag{4.13}$$

This solution is regular in the x,y-plane at $0\leqslant\mu^2\leqslant1$, which is met only if $\beta>0$ and $p_c\leqslant p\leqslant k$, where $p_c^2=(6\beta/c)k^4$. The meaning of the upper bound is obvious from the dispersion relation $\omega(k,p)$ coinciding with the dispersion relation (4.4) of the periodic solitons with parameters $\mathbf{k}(k,ip)$: the wave has vanishing velocity at $p=k+O(\beta)$ and does not exist in the form (4.12).

On the other hand, for $p=p_c$, two complex-conjugate periodic solitons merge and the solution (4.12) transforms into an ordinary plane soliton with parameters $\omega_s=2\omega_1$ and $\mathbf{k}_s(2k,0)$. It is physically clear that for $p<p_c$ the diffractive effects are negligible compared with the dispersive ones. Therefore, plane solitons are the only possible stationary solitary waves in this case. Solitary waves with transverse periodic structure described by Eq. (4.12) can be formed solely by virtue of the balance between diffraction and dispersion for $p^2\sim O(\beta)$.

Note that the critical point $p=p_c$ coincides with the cutoff of the instability region for a plane soliton with parameters ω_s, $\mathbf{k}_s$.

4.4. Two-Dimensional Soliton

At small $p\sim O(\beta^{1/2})$, the solution (4.12) describes amplitude-modulated quasiplane solitons, the modulation depth growing with increasing $(p-p_c)$. At large enough $p\sim O(k)$, the modulation depth is so great that the solution looks like a periodic chain of 2D solitons whose fields are essentially nonoverlapping. If the parameter k characterizing the longitudinal localization of the wave tends to zero consistently with the transverse wavenumber when $p=k\rho$, where $6\beta k^2/c\leqslant\rho^2\leqslant1$, then in this limit one can find the solution describing an individual 2D soliton propagating with velocity $v = \pm c\sqrt{1-\rho^2}$, localized in all directions, and falling off as a power law (see, e.g., [25]):

$$f_2=(x-vt)^2+\rho^2y^2+\frac{2\beta}{c\rho^2}(3-2\rho^2-\rho^4). \tag{4.14}$$

The solutions (4.12) and (4.14) generalize the known soliton solutions to the classical KP equation (2.5)[15,25] for the Boussinesq equation (2.1). In positive-dispersion media, both plane solitons and 2D solitons can be regarded as special (critical, in a certain sense) forms of the family of solitary waves with periodically modulated fronts.

4.5. Development of Plane Soliton Instability

Here we analyze the nonlinear stage of plane soliton instability under the action of periodic perturbations. It follows from (4.11) that the solution corresponding to this process is real if two growing complex-conjugate modes (3.8) are given in the vicinity of the soliton. This is possible in the framework of resonant processes described by a three-soliton solution:[14,15]

$$f_3=1+\exp(\eta_1)+\exp(\eta_2)+\exp(\eta_3)$$
$$+\frac{1}{\mu^2}\exp(\eta_2+\eta_3), \qquad (4.15)$$

where the variable η_1 corresponds to a phase of a plane soliton with parameters ω_1, $\mathbf{k}_1(k_1,0)$, the complex-conjugate variables η_2 and η_3 correspond to the phases of the periodic solitons with parameters ω_2, $\mathbf{k}_2(k_2,ip)$ and ω_2, $\bar{\mathbf{k}}_2(k_2,-ip)$, and μ^2 is defined by the formula (4.13) with k_2 substituted for k. The direct substitution of (4.15) into Eq. (4.2) reveals that f_3 is its solution if the relations

$$\tilde{D}(\omega_1-\omega_2,\mathbf{k}_1-\mathbf{k}_2)=\tilde{D}(\omega_1-\omega_2,\mathbf{k}_1-\bar{\mathbf{k}}_2)=0, \qquad (4.16)$$

$$\tilde{D}(\omega_2+\omega_2-\omega_1,\mathbf{k}_2+\bar{\mathbf{k}}_2-\mathbf{k}_1)=0 \qquad (4.17)$$

between the soliton parameters are met.

Equations (4.16) determine the resonant conditions (4.10b) between the plane soliton and each of the periodic solitons. One condition follows from the other by virtue of symmetry. The additional condition (4.17), which overdetermines the system of algebraic equations with respect to the soliton parameters, indicates the existence of yet another (fourth-order) resonance of two plane and two periodic solitons. Moreover, this resonance is described by the solution (4.15). Neglecting terms that are $O(\beta^2)$, Eq. (4.17) holds identically if Eqs. (4.16) are fulfilled.

Analysis of (4.15) reveals that the development of plane soliton instability leads to the formation, at $t\to+\infty$, of a solitary wave with a periodically modulated front determined by the periodic solitons with parameters ω_2, $\mathbf{k}_2$ and ω_2, $\bar{\mathbf{k}}_2$, and a plane soliton with parameters $\omega_4=2\omega_2-\omega_1$, $\mathbf{k}_4(2k_2-k_1,0)$.[15]

Thus, while a linear mode growing against the background of the plane soliton can be constructed in the framework of a complex solution describing a triple resonance of a plane and two periodic solitons, the development of a real transverse instability is largely determined by the characteristic features of the fourth resonance (4.17). Therefore, instead of yet another modulated wave with parameters $\omega_3=\omega_1-\omega_2$, $\mathbf{k}_3=\mathbf{k}_1-\mathbf{k}_2$, a plane soliton with parameters ω_4, $\mathbf{k}_4$ is formed.

To conclude this section, we note that in the framework of the three-soliton solution we can find a linear mode growing against the background of the wave with a periodically modulated front (4.12). To this end, one should analyze the resonance of one of the periodic solitons forming the wave and an individual periodic soliton which occurs when only one condition of (4.16) is met. This mode was constructed for the KP equation by Burtsev[26] in the framework of the $\bar{\partial}$-problem.

A complete description of the instability of a wave with a periodically modulated front can be given in the framework of a four-soliton solution. For long-period waves with $O(\beta^{1/2})$, transverse wavenumbers this process is a decay of the original wave into two new similar waves.[13,15] However, since according to Eq. (4.10) resonance of periodic solitons and the waves considered is also possible for moderate transverse wavenumbers, we suppose that similar processes of transverse instability and decay are inherent in the entire family of solutions (4.12). Thus, for the general Boussinesq models describing wave processes in positive-dispersion media, we also conclude that an arbitrary nonlinear perturbation unlocalized in one direction is unstable with respect to long-wave front modulations, and decays into other waves with greater separations between 2D solitons.[15]

5. LONGITUDINAL SOLITON RESONANCE

Transverse instability and resonance of plane solitons are inherent in different Boussinesq models and weakly depend on the types of nonlinear and dispersive terms. On the other hand, longitudinal instability is directly related to the nonlinearity, as follows from Sec. 3, and is observed only for solitons with large amplitudes.

We consider the phenomenon of longitudinal instability by analyzing Eq. (2.2) in the 1D case. For this purpose we reduce this equation to bilinear form by replacing $H=(12\beta/\alpha)(\partial^2/\partial x^2)\ln f$:

$$(D_t^2-c^2D_x^2+2\beta cD_x^4)ff=0. \qquad (5.1)$$

The N-soliton solutions of this equation were found by Hirota.[17] For $N=1$ and $N=2$, they have the same functional structure as the solutions (4.3) and (4.5) but with a dispersion relation of the form

$$D(\omega_i,\mathbf{k}_i)=\omega_i^2-c^2k_i^2+2\beta ck_i^4=0 \qquad (5.2)$$

and the function A_{12}:

$$A_{12}=\frac{(v_1-v_2)^2-6\beta c(k_1-k_2)^2}{(v_1-v_2)^2-6\beta c(k_1+k_2)^2}, \qquad (5.3)$$

where $v_i=\omega_i/k_i$.

Analysis of two-soliton interaction, including the resonant one, was carried out for Eq. (5.1) by Tajiri and Nishitani.[19] The linear instability of a soliton in this model was investigated by Spector *et al.*.[18] There exists a close relationship between the two phenomena, the role of which has not been elucidated yet.

Indeed, the resonance of two solitons (when $A_{12}=0$) occurs only at $k_1^2\geqslant 3c/8\beta$, $|v_1|\leqslant c/2$, which corresponds to the range of longitudinal instability of a soliton with parameters V_1, k_1 (see Sec. 3).

The resonant solution described by the function (4.11) remains real for any time t. At the initial stage, it is a single soliton solution with parameters $v_1 \equiv v$, $k_1 \equiv k$ that is perturbed by the mode $w(x-vt;k,k')$ in the form (3.8) with $p=0$, $\lambda=-\beta ck'(k^2-k'^2)/2v$ and $k'^2=3v^2/2\beta c$. This mode grows on the soliton background according to a linear theory. As the instability develops further, the original soliton decays into two new solitons. One of them moves in the same direction and has the parameters

$$v_2=+\frac{3k}{4}-\frac{v}{2}, \quad k_2=\frac{k-k'}{2}, \tag{5.4a}$$

while the other moves in the opposite direction and has the parameters

$$v_3=-\frac{3k}{4}-\frac{v}{2}, \quad k_3=\frac{k+k'}{2}. \tag{5.4b}$$

Thus, longitudinal soliton instability is caused by resonant excitation of the soliton in the oncoming wave that is also described by the two-wave Boussinesq equation. Obviously, when the minus sign is chosen before the last term in the solution (4.11) (which can be done by replacing $\eta_{20} \to i\pi+\tilde{\eta}_{20}$), the solitons appearing in the decay have $\operatorname{cosech}^2 x$-type profiles and are singular. Such an appearance of singularities in the evolution of the bounded single-soliton perturbation can be regarded as a collapse of the unstable soliton.[18] Note that construction of singular solutions is impossible and there is no soliton collapse for the transverse instability of solitons with moderate amplitudes.

Omitting detailed investigation of the wave collapse due to the development of longitudinal instability, we shall show that this phenomenon is sensitive to weak changes of the dispersion relation (5.2). Namely, we choose it in the form

$$D(\omega,k)=\omega^2-c^2k^2+2\beta\omega^2k^2/c=0. \tag{5.5}$$

Such a relation is physically more appropriate for the description of waves in weakly dispersive media (see, e.g., Ref. 6) because it approximates more accurately real dispersion curves and allows for the existence of solitons with arbitrary amplitudes.

In this case, a resonant triplet of plane solitons (5.4) is formed if the condition (1.1) is fulfilled. This condition determines the dependence $k'(k)$ where we must set $|k'| \leqslant k$. Omitting lengthy calculations, we present an equation for k' in the final form:

$$\frac{2\beta}{c}k'^2\left(1+\frac{\beta}{4c}(k'^2-k^2)\right)^2=3\left(1+\frac{\beta}{6c}(3k^2+k'^2)\right)\times\left(1+\frac{\beta}{2c}(k'+k)^2\right)\left(1+\frac{\beta}{2c}(k'-k)^2\right). \tag{5.6}$$

It is easy to see that for $k^2 \sim O(1)$, $k'^2 \sim O(\beta^{-1})$, which means that the condition is not met, i.e., $|k'|>k$. We assume that there exists $k=k_c$ such that $k'(k_c)=\pm k_c$. Then Eq. (5.6) is simplified and transforms to the biquadratic equation

$$\left(\frac{2\beta}{c}k_c^2\right)^2+3\left(\frac{2\beta}{c}k_c^2\right)+3=0, \tag{5.7}$$

which has no real roots k_c. Thus, the conditions of longitudinal soliton resonance are not fulfilled for the dispersion relation (5.5).

6. CONCLUSION

The results presented in this paper imply that the transverse instability of solitary waves is a general phenomenon in isotropic media with weak positive dispersion and is adequately described by the KP approximation. As to the longitudinal instability and the soliton collapse, this problem may be solved correctly only in the framework of the basic equations describing strongly nonlinear waves with large amplitudes.

As was to be expected, the self-focusing of small perturbations leads to the formation of multidimensional solitary waves. For a sufficiently long modulation period ($p \to 0$), such a wave is a long-period chain of 2D solitons formed from the original plane soliton, which results in a decrease of amplitude. For a short enough modulation period ($p \to p_c$), a solitary wave modulated along the front is formed and emits a plane soliton with small amplitude. We suppose that the existence of the critical wave number p_c as well as the scenario of the formation of 2D soliton chains from a plane soliton are general for isotropic and anisotropic media with a decaying dispersion law.[27]

The mechanism of solitary wave instability that is determined by the characteristic features of decaying configurations of resonantly interacting solitons enables us to regard these processes as a transformation of the wave field from one soliton to another soliton state that is not accompanied by radiation of quasilinear dispersive waves of small amplitude. The resonant conditions in the form (1.1) for the waves which are periodic in one direction and localized in the other give a reliable criterion of plane soliton instability to multidimensional perturbations. Unfortunately, it is not easy to establish the form of the nonlinear dispersion relation in the general case. Therefore, the instability criterion based on a decaying spectrum of linear perturbations is more convenient. At present, there is no rigorous proof of such a criterion, although the examples of weakly dispersive media considered here confirm its validity. A search for the relationship between the two criteria mentioned above is a timely problem for further investigation.

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