Propagation of small perturbations in a relativistic gas: The relation between the kinetic and hydrodynamic descriptions

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(Submitted 21 October 1993)
Zh. Eksp. Teor. Fiz. **106**, 161–171 (July 1994)

A derivation of the hydrodynamic equations from the linearized relativistic kinetic equation is given, which differs from the method of expansion in a small parameter. No *a priori* assumptions are made regarding the smallness of the dissipative terms, spatial gradients, or collision integral. The resulting hydrodynamic equations are exact in the sense that any arbitrary solution of these equations can in principle be used to reproduce an exact solution of the linearized kinetic equation. The hydrodynamic model is nonlocal. Despite the presence of dissipation, the signal propagation velocity is finite. The present derivation is general, independent of the specific form of the collision integral.

1. INTRODUCTION

In the usual description of dissipative processes in a relativistic gas the Eckart[1] and Landau–Lifshitz[2] models are used. De Groot *et al.*[3] showed that both these models can be derived from the relativistic kinetic equation using the Chapman–Enskog method as special cases of a more general class of models which differ in the definition of the local four-velocity of the medium. All these models share a certain property of a Navier–Stokes–Fourier nonrelativistic, viscous, thermally conducting gas,[4] namely that a signal propagates with infinite velocity in the comoving inertial coordinate system.

The unsatisfactory nature of such models in a systematic relativistic theory has been widely discussed in recent years in the literature.[5–7] Dinariev[8] showed that in order to avoid the paradox of superluminal signal velocities, it suffices to include the dispersion of the transport coefficients, i.e., to use a model with a memory. Such models have long been used in nonrelativistic mechanics.[9,10] Nonlocal space–time behavior in the constitutive relations are also predicted by statistical mechanics and kinetic theory.[11–13]

Thus, there exist relativistic hydrodynamic models with dissipation and a finite signal velocity. Here the use of the Chapman–Enskog method in order to pass from the relativistic kinetic equation to the hydrodynamic description[3] is inadequate, since the original equation yields a finite signal velocity (see Sec. 1 below), while the resulting Eckart or Landau–Lifshitz equations yield an infinite velocity.

In the present work a derivation of the hydrodynamic equations from the linearized kinetic equation is proposed which differs from the methods of Chapman–Enskog, Grad, and Hilbert. No *a priori* assumptions are made regarding the smallness of the dissipative terms, spatial gradients, or collision integral. The resulting hydrodynamic equations are exact in the sense that any arbitrary solution of these equations can in principle be used to recover an exact solution of the linearized kinetic equation. The hydrodynamic model is found to be nonlocal. This model was considered on purely phenomenological grounds previously by Dinariev,[14] where restrictions on the corresponding kernels were derived from thermodynamics and time reversibility on the microscopic level (analogous to the Onsager relations). In the present work these limitations are derived as a direct consequence of the kinetic equation. All conclusions in the present work are general in nature, independent of the specific form of the collision integral.

In the present work a system of units is employed in which the speed of light in vacuum c, the Planck constant $\hbar$, and the Boltzmann constant k are equal to unity. Greek subscripts run through the values 0, 1, 2, 3, corresponding to the coordinates x^α of an inertial system. The Minkowski metric is used, $(g_{\alpha\beta})=\mathrm{diag}(1,-1,-1,-1)$. The roman subscripts i, j, k run through the values 1, 2, 3, corresponding to the spatial coordinates x^i. The roman subscripts A, B run through the values 0, 1, 2, 3, 4. Summation is implied over repeated indices unless otherwise specified.

In Sec. 1 we discuss the general properties of the relativistic kinetic equation. In Sec. 2 we derive the hydrodynamic equations from the linearized kinetic equation and study their properties. In Sec. 3 we summarize the results of this work.

2. PROPERTIES OF THE RELATIVISTIC KINETIC EQUATIONS

In the formulation of relativistic kinetic theory we will generally follow the approach and notation of de Groot *et al.*[3]

Consider a relativistic gas of structureless particles with mass m, whose state is described by the distribution function $f=f(x^\alpha,p^\beta)$. Here the four-momentum p^α belongs to the mass surface

$$g_{\alpha\beta}p^\alpha p^\beta=m^2,\quad p^0>0. \tag{1.1}$$

On the mass surface the Lorentz-invariant measure $d\mu_m=(p^0)^{-1}dp^1dp^2dp^3$ is prescribed. The distribution function satisfies the dynamical equation

$$p^\alpha\partial_\alpha f=\mathrm{St}[t]. \tag{1.2}$$

 1063-7761/94/070088-06$10.00

Here St[t] is in general a nonlinear operator in function space on the mass surface (1.1), which satisfies a number of conditions:

$$\int \mathrm{St}[t](x^\alpha,p^\beta)d\mu_m=0 \tag{1.3}$$

(conservation of particle number in collisions),

$$\int p^\sigma \mathrm{St}[t](x^\alpha,p^\beta)d\mu_m=0 \tag{1.4}$$

(conservation of particle energy–momentum in collisions), and

$$\mathrm{St}[f_e]=0, \tag{1.5}$$

where f_e is the equilibrium distribution

$$f_e(p^\beta)=(2\pi)^{-3}\exp((\mu-p_\alpha U^\alpha)/T),$$
$$U_\alpha U^\alpha=1. \tag{1.6}$$

From the distribution function we can compute the four-vector flux of the gas particles

$$j^\alpha(x^\beta)=\int p^\alpha f(x^\beta,p^\delta)d\mu_m$$

and the energy–momentum tensor of the gas

$$T^{\alpha\beta}(x^\gamma)=\int p^\alpha p^\beta f(x^\gamma,p^\delta)d\mu_m.$$

From Eqs. (1.2), (1.3), and (1.4) we find the particle and energy–momentum conservation laws

$$\partial_\alpha j^\alpha=0, \tag{1.7}$$

$$\partial_\beta T^{\alpha\beta}=0. \tag{1.8}$$

Now we consider the evolution of linear perturbations about the equilibrium distribution

$$f_0(p^\beta)=(2\pi)^{-3}\exp((\overset{0}{\mu}-p^0)/\overset{0}{T}),$$

where we assume that the perturbations are produced by weak external forces and particle and energy sources. We use the standard representation of the linearized distribution function[3] $f=f_0(1+\varphi)$.

When we take the sources into account, Eq. (1.2) can be transformed to

$$p^\alpha\partial_\alpha\varphi-L\varphi=S, \tag{1.9}$$

where $S=S(x^\alpha,p^\beta)$ is the source function and L is a linear operator defined by a derivative of the collision integral: $L\varphi=f_0^{-1}D\mathrm{St}[f_0](f_0\varphi)$. In terms of their dependence on the arguments p^α the functions φ belong to a Hilbert space H_m with the scalar product

$$(\varphi_1,\varphi_2)=\int f_0\varphi_1^*\varphi_2 d\mu_m.$$

In the space H_m we define the operation I corresponding to sign reversal of the spatial components of the momentum: $(I\varphi)(p^i)=\varphi(-p^i)$.

Without specifying a particular form of the collision integral we can assert that the operator L satisfies a number of conditions:

$$L^+=ILI \tag{1.10}$$

(resulting from reversibility at the microscopic level),

$$(L\varphi)^*=L\varphi^* \tag{1.11}$$

(reality), and

$$L\leqslant 0 \tag{1.12}$$

(the presence of dissipation).

In a large number of cases the operator L commutes with the operator I, so in place of (1.10) we have the stronger condition $L^+=L$. Furthermore, from (1.3), (1.4), (1.10) or by differentiating (1.5) with respect to the three parameters we derive the relations

$$Le^A=0, \tag{1.13}$$

where $e^4=1$, $e^\alpha=p^\alpha$.

For an arbitrary function g of the space–time coordinates we will denote by g_F the Fourier transform

$$g_\mathrm{F}(k_\alpha)=\int \exp(-ik_\alpha x^\alpha)g(x^\alpha)dx^\alpha.$$

Applying the Fourier transformation to Eq. (1.9) we find the linear operator equation

$$G\varphi_\mathrm{F}=S_\mathrm{F},$$
$$G=p^\alpha ik_\alpha-L. \tag{1.14}$$

Equation (1.14) allows the velocity of signal propagation from a source in the relativistic kinetic theory to be investigated. For this purpose we consider Eq. (1.14) for complex quantities k_α, where the imaginary parts of the wave four-vector $K_\alpha=\mathrm{Im}\,k_\alpha$ satisfy the conditions

$$M=K_\alpha K^\alpha>0,\quad K^0<0. \tag{1.15}$$

From (1.1), (1.12), and (1.15) it follows that

$$G+G^+=-2K_\alpha p^\alpha-(L+L^+)\geqslant 2mM^{1/2}.$$

This inequality implies that the operator G is invertible in the complex tube (1.15) and says that the inverse operator G^{-1} is an analytic function of the wave vector components. It follows from Ref. 15 that the latter implies that infinitesimal perturbations cannot exceed the speed of light.

Thus, the kinetic equation (1.2) is consistent with the causality principle of special relativity. Since the application to Eq. (1.2) of analogs of the Chapman–Enskog method[3] yields models with superluminal signals, this method is evidently inadequate in rigorous relativistic theory. In the following section we obtain equations for the hydrodynamic variables which follow exactly from Eqs. (1.9) in the sense that any arbitrary solution of the hydrodynamic equations can be used to recover a solution of Eq. (1.9).

3. DERIVATION OF THE LINEAR HYDRODYNAMIC EQUATIONS FOR A RELATIVISTIC GAS

It is convenient to introduce some new notation in what follows. Let H_h be a subspace of the Hilbert space H_m spanned by the set of vectors e^A, and let H_α be the orthogonal complement to H_h: $H_m=H_h\oplus H_\alpha$. We define P_h: $H_m\to H_h$, and P_α: $H_m\to H_\alpha$ to be the corresponding projection operators and I_h: $H_h\to H_m$, and I_α: $H_\alpha\to H_m$ to be the corresponding injections. In the space H_h we define the metric tensor $\eta^{AB}=(e^A,e^B)$ (see Appendix), by means of which we can raise and lower the upper-case roman subscripts. In particular, a set of vectors e_A is defined for which $(e_A,e^B)=\delta_A^B$ holds. Note that (1.10) and (1.13) imply $LH_h=0$, $LH_\alpha\subseteq H_\alpha$.

We assume that the source term in (1.9) as a function of the four-momenta takes values in the space H_h. From the hydrodynamic standpoint this source term creates particle sources $(1,S)=\nu$, energy sources $(p^0,S)=F^0$, and external forces $(p^i,S)=F^i$. The hydrodynamic equations (1.7) and (1.8) are transformed into equations with the sources

$$\partial_\alpha j^\alpha=\nu, \tag{2.1}$$

$$\partial_\beta T^{\alpha\beta}=F^\alpha. \tag{2.2}$$

It is easy to find a formal solution of Eq. (1.14). Specifically, we introduce the notation

$$\begin{gathered} h=P_h\varphi=e^A(e_A,\varphi),\quad a=P_\alpha\varphi=\varphi-h,\\ G_{hh}=P_hGI_h,\quad G_{\alpha h}=P_\alpha GI_h,\\ G_{h\alpha}=P_hGI_\alpha,\quad G_{\alpha\alpha}=P_\alpha GI_\alpha. \end{gathered} \tag{2.3}$$

The "hydrodynamic" part of the distribution function is found in an obvious fashion from the system of linear equations

$$(G_{hh}-G_{h\alpha}G_{\alpha\alpha}^{-1}G_{\alpha h})h_{\mathrm{F}}=S_{\mathrm{F}},$$

and the "nonhydrodynamic" part is found from

$$a_{\mathrm{F}}=-G_{\alpha\alpha}^{-1}G_{\alpha h}h_{\mathrm{F}}. \tag{2.4}$$

We follow Eckart[1] in defining the material four-vector u^α. In this approach the four-velocity and the particle density n are determined from the relation

$$j^\alpha=nu^\alpha$$

according to

$$n=j^\alpha u_\alpha, \tag{2.5}$$

$$u^\alpha=n^{-1}j^\alpha. \tag{2.6}$$

Then the energy–momentum tensor can be represented in the form

$$T^{\alpha\beta}=\varepsilon u^\alpha u^\beta+q^\alpha u^\beta+u^\alpha q^\beta+\pi^{\alpha\beta}, \tag{2.7}$$

where the thermal flux vector q^α and stress tensor $\pi^{\alpha\beta}$ are subject to the restrictions

$$q^\alpha u_\alpha=0\quad \pi^{\alpha\beta}u_\alpha=0. \tag{2.8}$$

Relation (2.7) together with (2.8) can be regarded as the definition of the internal energy ε, as well as q^α and $\pi^{\alpha\beta}$:

$$\varepsilon=T^{\alpha\beta}u_\alpha u_\beta, \tag{2.9}$$

$$q^\alpha=T^{\alpha\beta}u_\beta-\varepsilon u^\alpha, \tag{2.10}$$

$$\pi^{\alpha\beta}=T^{\alpha\beta}-(\varepsilon u^\alpha u^\beta+q^\alpha u^\beta+u^\alpha q^\beta). \tag{2.11}$$

The stress tensor is used to define the viscous stress tensor

$$\tau^{\alpha\beta}=p\Delta^{\alpha\beta}-\pi^{\alpha\beta}, \tag{2.12}$$

where we have written $\Delta^{\alpha\beta}=u^\alpha u^\beta-g^{\alpha\beta}$, and p is the hydrostatic pressure given, e.g., by a function $p=p(n,\varepsilon)$, which can be evaluated for the class of equilibrium distributions (1.6) (see Appendix).

To obtain a closed hydrodynamic model we must express the energy–momentum tensor in terms of the hydrodynamic variables, which we take to be the perturbed density, internal energy, and velocity. This has already been done for the particle flux four-vector.

Let us consider the physical interpretation of the h^A components found by expanding h in the e_A basis. For this we assume that any physical quantity A in the problem can be expanded in powers of $A=\overset{0}{A}+\overset{1}{A}$, where $\overset{0}{A}$ is the equilibrium value of the variable and $\overset{1}{A}$ is the linear perturbation. In particular,

$$\overset{0}{u}{}^\alpha=\delta_0^\alpha. \tag{2.13}$$

Assume that φ is a solution of Eq. (1.13) and relations (2.3) hold. It is easy to show that

$$h^A=(e^A,h)=(e^A,\varphi).$$

Hence from Eqs. (2.5) and (2.6) we find using (2.13) that

$$h^0=\overset{1}{n},\quad h^i=\overset{0}{n}\overset{1}{u}{}^i,\quad \overset{1}{u}{}^0=0. \tag{2.14}$$

The physical interpretation of the quantity h^4 presents certain difficulties. Specifically, we find using (1.1), (2.7), (2.9), and (2.12) that

$$\begin{aligned} h^4&=m^{-2}g_{\alpha\beta}(p^\alpha p^\beta,\varphi)\\ &=m^{-2}g_{\alpha\beta}\overset{1}{T}{}^{\alpha\beta}=m^{-2}(\overset{1}{\varepsilon}-3\overset{1}{p}-g_{\alpha\beta}\overset{1}{\tau}{}^{\alpha\beta}). \end{aligned} \tag{2.15}$$

Consequently, in order to develop a physical interpretation of the quantity h^4 (i.e., to express it in terms of the hydrodynamic variables $\overset{1}{n}$, $\overset{1}{\varepsilon}$, $\overset{1}{u}{}^i$) we must make the concept of the viscous stress tensor more precise. This will be done below.

To construct a closed hydrodynamic model we must find expressions for the internal energy $\overset{1}{\varepsilon}$, the heat flow vector $\overset{1}{q}{}^i$, and the viscous stress tensor $\overset{1}{\tau}{}^{\alpha\beta}$ in terms of the variables h^A.

From (2.9), (2.13), and (2.4),

$$\begin{gathered} \overset{1}{\varepsilon}=\overset{1}{T}{}^{00}=(p^0p^0,\varphi),\\ \overset{1}{\varepsilon}_{\mathrm{F}}=(p^0p^0,(1-G_{\alpha\alpha}^{-1}G_{\alpha h})h_{\mathrm{F}}). \end{gathered} \tag{2.16}$$

Using relations (2.10) and (2.4) we find

$$\begin{gathered} \overset{1}{q}{}^0=0,\\ \overset{1}{q}{}^i=\overset{1}{T}{}^{i0}-\overset{0}{h}\overset{1}{u}{}^i=((p^ip^0-p^i\overset{0}{\kappa}\overset{0}{n}{}^{-1}),\varphi),\\ \overset{1}{q}{}^i_{\mathrm{F}}=((p^ip^0-p^i\overset{0}{\kappa}\overset{0}{n}{}^{-1}),(1-G_{\alpha\alpha}^{-1}G_{\alpha h})h_{\mathrm{F}}), \end{gathered} \tag{2.17}$$

where $\overset{0}{\kappa}=\overset{0}{\varepsilon}+\overset{0}{p}$ is the equilibrium enthalpy. Note that we have

$$((p^ip^0-p^i\overset{0}{h}\overset{0}{n}{}^{-1}),h)=0,\quad h\in H_h$$

since a relation of this type is satisfied for the class of equilibrium distributions (1.6). By virtue of this we find from (2.17) the expression

$$\overset{1}{q}{}^i_{\mathrm{F}}=-(p^ip^0,G^{-1}_{\alpha\alpha}G_{\alpha h}h_{\mathrm{F}}). \tag{2.18}$$

Here we have written

$$P_1=\left(\frac{\partial p}{\partial n}\right)_{\varepsilon}(\overset{0}{n},\overset{0}{\varepsilon}),\quad P_2=\left(\frac{\partial p}{\partial \varepsilon}\right)_{n}(\overset{0}{n},\overset{0}{\varepsilon}).$$

Using relations (2.4), (2.11), and (2.12) we find

$$\begin{aligned}
&\overset{1}{\tau}{}^{ij}=\overset{0}{T}{}^{ij}-\overset{1}{T}{}^{ij}+\overset{1}{p}\delta^{ij}=((-p^ip^j),\varphi)+\overset{1}{p}\delta^{ij},\\
&\overset{1}{p}=P_1\overset{1}{n}+P_2\overset{1}{\varepsilon},\\
&\overset{1}{\tau}{}^{ij}_{\mathrm{F}}=((-p^ip^j),(1-G^{-1}_{\alpha\alpha}G_{\alpha h})h_{\mathrm{F}})+\overset{1}{p}_{\mathrm{F}}\delta^{ij},\\
&\overset{1}{p}_{\mathrm{F}}=P_1\overset{1}{n}_{\mathrm{F}}+P_2\overset{1}{\varepsilon}_{\mathrm{F}}.
\end{aligned} \tag{2.19}$$

We introduce the auxiliary variables

$$Z^{\alpha\beta A}=(p^\alpha p^\beta,e^A),$$

$$A^{\alpha\beta A}=A^{\alpha\beta A}(k_\gamma)=(p^\alpha p^\beta,G^{-1}_{\alpha\alpha}G_{\alpha h}e^A).$$

Then Eqs. (2.16) and (2.19) can be rewritten in the compact form

$$\overset{1}{\varepsilon}_{\mathrm{F}}=(Z^{00A}-A^{00A})h_{A\mathrm{F}}, \tag{2.20}$$

$$\begin{aligned}
&\overset{1}{\tau}{}^{ij}_{\mathrm{F}}=(-Z^{ijA}+A^{ijA})h_{A\mathrm{F}}+\overset{1}{p}_{\mathrm{F}}\delta^{ij},\\
&\overset{1}{p}_{\mathrm{F}}=P_1\overset{1}{n}_{\mathrm{F}}+P_2\overset{1}{\varepsilon}_{\mathrm{F}}.
\end{aligned} \tag{2.21}$$

Note that

$$Z^{ijA}h_{A\mathrm{F}}=(P_1\overset{1}{n}_{\mathrm{F}}+P_2Z^{00A}h_{A\mathrm{F}})\delta^{ij}$$

holds identically, since both sides of the equation contain perturbations of the pressure field in the class of equilibrium distributions. Hence after substituting (2.20) into (2.21) and using (2.18) we find the constitutive relations of the hydrodynamic model

$$\overset{1}{q}{}^i_{\mathrm{F}}=-A^{i0A}h_{A\mathrm{F}},\quad \overset{1}{\tau}{}^{ij}_{\mathrm{F}}=(A^{ijA}-P_2A^{00A}\delta^{ij})h_{A\mathrm{F}}. \tag{2.22}$$

Let us study the properties of the coefficients $A^{\alpha\beta A}$ which follow from the concepts of symmetry and the restrictions (1.10)–(1.13) on the operator L. We note first that since $G_{\alpha h}e^4=P_\alpha ik_\alpha p^\alpha=0$, holds we have

$$A^{\alpha\beta 4}=0.$$

Thus, the expressions for the dissipative fluxes do not depend on the function h_4.

From the invariance under the three-dimensional orthogonal group $SO(3)$ we find the representation

$$A^{000}=X_0,\quad A^{00i}=X_1ik^i,\quad A^{0i0}=X_2ik^i,$$

$$A^{0ij}=X_3ik^iik^j+X_4\delta^{ij},\quad A^{ij0}=X_5ik^iik^j+X_6\delta^{ij} \tag{2.23}$$

$$A^{ijk}=X_7ik^iik^jik^k+X_8ik^k\delta^{ij}+X_9(ik^i\delta^{jk}+ik^j\delta^{ik}),$$

where X_α, $a=0,1,...9$ are functions of the components k_α, which are invariant under the action of the group $SO(3)$. From (1.11) it follows that

$$(X_\alpha(k_\alpha))^*=X_\alpha(-k_\alpha),$$

which enables us to interpret the functions X_α as Fourier transforms of some real kernels $Y_\alpha=Y_\alpha(x^\alpha)$: $X_\alpha=Y_{\alpha\mathrm{F}}$. As follows from the results of Sec. 1, the functions X_α are analytic in the tube (1.15). From the theory of Ref. 15 this implies that the functions Y_α vanish outside the cone $g_{\alpha\beta}x^\alpha x^\beta\geqslant 0$, $x^0\geqslant 0$ (causality).

By virtue of conditions (1.1) we have

$$g_{\beta\gamma}A^{\beta\gamma\alpha}=0 \tag{2.24}$$

from which two identities follow:

$$X_0-k_ik^iX_5-3X_6=0, \tag{2.25}$$

$$X_1-k_ik^iX_7-3X_8-2X_9=0. \tag{2.26}$$

Returning now to expression (2.15) we see that the quantity h^4 can be expressed in terms of the hydrodynamic variables in a nonlocal fashion by means of the kernels Y_α. In more detail, we have using relations (2.23) and (2.24)

$$h^4_{\mathrm{F}}=m^{-2}(\overset{1}{\varepsilon}_{\mathrm{F}}-3\overset{1}{p}_{\mathrm{F}}-(3P_2-1)(X_0h_{0\mathrm{F}}+ik^iX_1h_{i\mathrm{F}})). \tag{2.27}$$

Recall that

$$h_0=\eta_{00}\overset{1}{n}+\eta_{04}h^4, \tag{2.28}$$

$$h_i=\eta_{iA}h^A=\overset{0}{T}{}^{-1}\overset{1}{u}{}^i. \tag{2.29}$$

Substituting expressions (2.28) and (2.29) in (2.27) we finally obtain the representation of h^4_{F} in terms of hydrodynamic variables:

$$\begin{aligned}
h^4_{\mathrm{F}}=m^{-2}(\overset{1}{\varepsilon}_{\mathrm{F}}-3\overset{1}{p}_{\mathrm{F}}-(3P_2-1)(X_0\eta_{00}\overset{1}{n}_{\mathrm{F}}\\
-X_1\overset{0}{T}{}^{-1}ik_i\overset{1}{u}{}^i_{\mathrm{F}}))D^{-1},
\end{aligned}$$

$$D=D(k_\alpha)=1+m^{-2}(3P_2-1)X_0\overset{0}{\eta}_{04}. \tag{2.30}$$

Next we define the temperature by means of the function $T=T(n,\varepsilon)$, found in the class of equilibrium states [e.g., from Eq. (A1)]. Now, substituting expression (2.30) in (2.28) and using Eq. (A2) we find

$$h_{0\mathrm{F}}=m^{-2}(-\overset{0}{\Delta}\overset{0}{F}{}'\overset{1}{T}_{\mathrm{F}}+(3P_2-1)X_1\overset{0}{T}{}^{-1}ik_iu^i_{\mathrm{F}})D^{-1}.$$

Thus, we have shown that the dissipative fluxes (2.22) can be determined from the temperature and velocity fields.

Let us continue the study of the functions X_α. Consider $B^{\alpha\beta}=-ik_\gamma A^{\alpha\gamma\beta}$. As can easily be seen, we can write

$$B^{\alpha\beta}=(G_{\alpha h}e^\alpha,G^{-1}_{\alpha\alpha}G_{\alpha h}e^\beta),$$

from which, using (1.10), we find the Onsager symmetry relations:

$$B^{\alpha\beta}(k_0,k_i)=\varepsilon_\alpha\varepsilon_\beta B^{\beta\alpha}(k_0,-k_i), \tag{2.31}$$

where $\varepsilon_0=1$, $\varepsilon_i=-1$.

Substituting Eq. (2.23) in (2.31) for the case in which $\alpha=0$, $\beta=1$, we find a single nontrivial relation:

$$-ik_0X_2+k_jk^jX_5+X_6+ik_0X_1-k_jk^jX_3-X_4=0. \tag{2.32}$$

Now let us find the restrictions that follow from the dissipation condition (1.12). We can write (no summation over α is assumed here!)

$$\begin{aligned}\mathrm{Re}\,B^{\alpha\alpha}&=\tfrac{1}{2}\,(G_{\alpha h}e^\alpha,(G_{\alpha\alpha}^{-1}+G_{\alpha\alpha}^{+-1})G_{\alpha h}e^\alpha)\\&=-\tfrac{1}{2}\,(G_{\alpha\alpha}^{-1}G_{\alpha h}e^\alpha,(L+L^+)G_{\alpha\alpha}^{-1}G_{\alpha h}e^\alpha).\end{aligned}$$

This implies the inequality

$$\mathrm{Re}\,B^{\alpha\alpha}\geqslant 0.$$

Substituting expressions (2.23) in this inequality for the cases $\alpha=0$, i, we find the inequalities

$$\begin{aligned}&\mathrm{Re}(-ik_0X_0+k_ik^iX_2)\geqslant 0,\\&\mathrm{Re}(ik_0X_4)\leqslant 0,\\&\mathrm{Re}(-(k_kk^k)k^ik^iX_7-k^ik^i(X_8+X_9)-(k_kk^k)X_9\\&\qquad+ik_0k^ik^iX_3-ik_0X_4)\geqslant 0,\end{aligned} \tag{2.33}$$

where no summation over i is assumed in the last inequality.

Since the dissipative fluxes must vanish for the assumed motion of a uniform medium, we must have the relations

$$X_0|_{k_\alpha=0}=0,\quad X_4|_{k_\alpha=0}=0,\quad X_6|_{k_\alpha=0}=0.$$

This completes the derivation of the hydrodynamic model.

The hydrodynamic variables $\overset{1}{n}$, $\overset{1}{\varepsilon}$, $\overset{1}{u}{}^i$ satisfy Eqs. (2.1) and (2.2). The dissipative fluxes are defined by expressions (2.22) using the set of scalar functions (2.23). These functions are related by the identities (2.25), (2.26), (2.32) and satisfy the inequality (2.33) implied by the dissipation of the system.

4. CONCLUSIONS

In Sec. 2 of the present work we have derived the defining relations and evolution equations for the hydrodynamic model that follows from the linearized relativistic kinetic equation. An arbitrary solution of the hydrodynamic equations together with relations (2.14) and (2.30) and (2.4) can be used to recover the solution of the original kinetic equation.

Since the original kinetic equation does not admit of superluminal velocities, the hydrodynamic model also has this property. This and the fact that the model is an exact consequence of a general kinetic equation are responsible for the superiority of this model to the classical Eckart and Landau–Lifshitz models. On the other hand, the appropriateness of the nonlocal model depends sensitively on how well the structure functions X_α have been chosen. In order that no superluminal signals occur it is necessary and sufficient that the determinant of the linear system of Eqs. (2.1), (2.2) be nonzero in the complex tube (1.15) (Ref. 14). On the other hand, it is sufficient that the functions X_2, X_8, X_9, $X_1=3X_8+2X_9$ be nonzero and frequency-dependent [see Eq. (2.26) of Ref. 8].

To conclude this work it is appropriate to discuss the relation between this method of transforming to the hydrodynamic limit and the classical methods (those of Hilbert, Chapman–Enskog, and Grad). We emphasize that the hydrodynamic equations obtained in the present work are exact and are related to the power-series expansion in some small parameter that characterizes the interaction of the gas particles. Hence we can apply any of the classical expansion methods using a small parameter to the exact nonlocal hydrodynamic equations and obtain approximate equations. Thus, in the Chapman–Enskog method it is necessary to substitute $L\to\varepsilon^{-1}L$ and look for the hydrodynamic variables in the form of series

$$\begin{aligned}g&=g(k_0,k_i)\\&=g_0(k_0,\varepsilon k_i)+\varepsilon g_1(k_0,\varepsilon k_i)+\varepsilon^2 g_2(k_0,\varepsilon k_i)+\dots,\end{aligned}$$

which gives rise to a polynomial in k_α for any finite approximation with respect to ε. This is what is responsible for the nonlocal behavior of the model. It is obvious that this approach yields an incorrect description of the propagation of short waves, which, in particular, leads to an infinite signal velocity in the second approximation.

APPENDIX

In this appendix we give some of the formulas used in the text of the paper. Using the equilibrium distribution (1.6) we can calculate the thermodynamic functions in the usual way:[3]

$$\begin{aligned}&n=4\pi m^2T(2\pi)^{-3}K_2(z)\exp(\mu/T),\quad p=nT,\\&\varepsilon=n(F(T)+3T),\quad F(T)=m\frac{K_1(z)}{K_2(z)},\end{aligned} \tag{A1}$$

where $z=m/T$ and K_α are modified Bessel functions of the second kind. From these equations we can find the function $p=p(n,\varepsilon)$ by eliminating the temperature T.

Using the definition of the thermodynamic variables in an equilibrium state at rest we can readily calculate the components of the metric η^{AB}:

$$\begin{aligned}&\eta^{00}=\varepsilon,\quad \eta^{04}=\eta^{40}=n,\quad \eta^{44}=m^{-2}(\varepsilon-3p),\\&\eta^{ij}=\delta^{ij}p,\quad \eta^{i0}=\eta^{i4}=\eta^{0i}=\eta^{4i}=0.\end{aligned}$$

Inverting the matrix η^{AB} we find the covariant components of the metric:

$$\begin{aligned}&\eta_{00}=m^{-2}(F-3T)\Delta,\quad \eta_{04}=\eta_{40}=-\Delta,\quad \eta_{44}=F\Delta,\\&\eta_{ij}=\delta^{ij}p^{-1},\quad \eta_{i0}=\eta_{i4}=\eta_{0i}=\eta_{4i}=0,\end{aligned}$$

where

$$\Delta=n^{-1}(m^{-2}(F+3T)F-1)^{-1}.$$

Using these expressions and Eq. (A1) we can readily demonstrate the relation

$$\eta_{04}m^{-2}(\overset{1}{\varepsilon}-3\overset{1}{p})+\eta_{00}\overset{1}{n}=m^{-2}\Delta(-F')\overset{1}{T}. \tag{A2}$$

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Translated by David L. Book