Spectral problem for bound solitons (asymptotic behavior of ultrashort optical pulses in a dense resonant medium)

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The inverse scattering method is used to solve the spectral problem for the bound states of solitons and the asymptotic behavior of ultrashort optical pulses in a dense resonant medium is found. The time dependence of the roots of the scattering amplitude corresponding to soliton-breather bound states or 0π pulses is analyzed in the complex plane as a function of the time between the excited optical pulses and in terms of their "areas."

1. INTRODUCTION

As is well known, when high-power ultrashort optical pulses propagate in a dense resonant medium the phenomenon of self-induced transparency found by McCall and Hahn[1] can arise, in which solitary waves (solitons) develop. Although many theoretical results in this area were originally obtained using relatively simple methods, the greatest progress has resulted from the use of the inverse scattering method.[2–4]

Essentially the method consists of finding a pair of operators comprising the so-called Lax representation. This formalism was applied by Ablowitz *et al.*[3] and Lamb[4] to the Maxwell–Bloch equations describing propagation of ultrashort optical pulses with lengths less than the time for irreversible relaxation of the polarization in a resonant medium. The Lax operator is associated with a scattering problem in which the amplitude of the optical pulse introduced into the resonant medium, which varies slowly as a function of time, plays the role of the "potential," while the spatial coordinate is treated as a parameter. The spectral problem—the determination of the eigenvalue spectrum of the Lax operator—is one part of the inverse scattering method.

The eigenvalue spectrum of the Lax operator contains both discrete and continuum modes; since the form of the operator is in general non-Hermitian, the discrete eigenvalues correspond to complex numbers. These bound states determine the soliton part of the scattering problem, where each 2π pulse (soliton) is associated with a distinct eigenvalue lying on the imaginary axis of the complex plane. The continuum part of the spectrum corresponds to real values.[5]

In addition to the soliton or steady nonlinear solutions of the Maxwell–Bloch equations, an important role in the dynamics of ultrashort pulses is also played by solutions in the form of soliton waves, which have soliton properties but are not steady solutions.[4] These are the so-called "bound states" of solitons or breathers. In addition to breathers there are also "soliton-antisoliton" waves with envelope pulses having $\pm 2\pi$ areas and with various amplitudes. These soliton-antisoliton pairs and breathers both have total area $\theta=0$ and therefore belong to the class of 0π pulses.

To date questions involving the formation of solitons and 0π pulses when an individual ultrashort light pulse propagates and also when a resonant medium is excited by pairs of optical pulses with identical areas and phases shifted by π have been extensively studied.[4–8] In particular, it has been shown that a breather consists of a bound state of two solitons having the same group velocity but different phase velocities and is describable by the two-soliton solution of the Maxwell–Bloch equations. In terms of the spectral problem these states correspond to eigenvalues of the Lax operator lying in the complex plane.

Similar problems arise when several ultrashort light pulses pass through the resonant medium, e.g., in nonlinear optical phenomena involving three- or four-wave interactions and photon echo effects (see, e.g., Ref. 9).

The spectral problem associated with the propagation of two ultrashort optical pulses and the formation of photon echo signals was first treated by Zakharov and Manykin.[10] In particular, it was noted that for areas $\theta_1=\theta_2$ of the excited pulses and a particular value of the time interval between the pulses a new branch of the soliton solutions developed.[10] Subsequently this anomalous behavior of the asymptotic form of the solutions for large separation in a resonant medium was confirmed by the results of numerical studies of the Maxwell–Bloch equations.[11–13]

Hence it is of interest to generalize the results found in Ref. 10 (see also Refs. 14–16) and to include bound soliton states (i.e., breathers) in the treatment. Another motivation for studying them is that a number of articles have appeared to date in which self-induced transparency and photon echo effects were investigated in dense resonant medium. Specifically, these include optical fibers with res-

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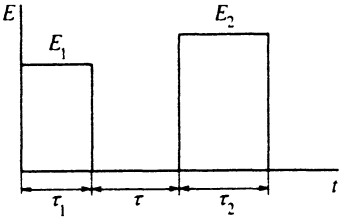

FIG. 1. Shape of the exciting optical pulses applied to the resonant medium.

onant Er^{3+} ion doping[17–19] and optically dense $LaF_3:Pr^{3+}$ crystals.[20,21]

2. BASIC EQUATIONS

As noted above, the inverse scattering method enables one to find the asymptotic behavior of the solutions for an ultrashort optical pulse propagating in a resonant medium analytically.[3] Here the search for the soliton part of the solution in the problem where square-wave pulses in time are incident reduces to finding the zeroes of a complex transcendental equation for the scattering coefficient a on the imaginary axis of the complex argument ζ (the eigenvalues of the Lax evolution operator $\hat{L}$).[10]

It is of interest to find the total number of zeros, including the complex values, and the corresponding soliton solutions, and also to study the way these roots change as the time τ between the two exciting pulses incident on the medium increases (Fig. 1).

Consider the case when there is no phase difference between the exciting optical pulses, so that the amplitudes E_1 and E_2 can be regarded as real quantities. As is well known,[7] when this condition holds the roots of the equation

$$a(\zeta)=0$$

are either on the imaginary axis or are in pairs symmetric with respect to the imaginary axis, so that their imaginary parts are the same and their real parts differ in sign.

In this case the expression for the scattering coefficient is given in Ref. 10. Employing the notation

$$\Omega_i=\sqrt{E_1E_2}\tau_i \quad i=1,2; \quad \tau_0=\tau\sqrt{E_1E_2}; \quad z=2\zeta/\sqrt{E_1E_2}; \quad r=E_1/E_2, \tag{1}$$

we can write this expression in the form

$$a(z)=\exp[iz(\Omega_1+\Omega_2)/2]A(z), \tag{2}$$

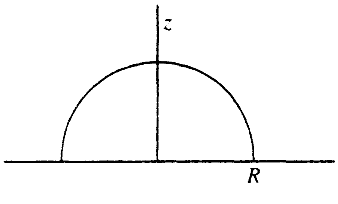

FIG. 2. Contour to be traced in the complex region.

where

$$A(z)=\cos(\Omega_1/2)\sqrt{z^2+r}\cos(\Omega_2/2)\sqrt{z^2+r^{-1}} - \frac{\sin(\Omega_1/2)\sqrt{z^2+r}\sin(\Omega_2/2)\sqrt{z^2+r^{-1}}}{\sqrt{(z^2+r)(z^2+r^{-1})}} \times[z^2+\exp(iz\tau_0)]-iz \times\left(\frac{\sin(\Omega_1/2)\sqrt{z^2+r}\cos(\Omega_2/2)\sqrt{z^2+r^{-1}}}{\sqrt{(z^2+r)}} + \frac{\cos(\Omega_1/2)\sqrt{z^2+r}\sin(\Omega_2/2)\sqrt{z^2+r^{-1}}}{\sqrt{(z^2+r^{-1})}}\right). \tag{3}$$

The equation

$$A(z)=0 \tag{4}$$

has roots whose number and location in the complex z plane depend on the four parameters Ω_1, Ω_2, τ_0, and r. Here we consider the important (and interesting from an experimental standpoint) case for which $E_1=E_2$ holds, so that $r=1$.

In what follows we will show that the number of zeros of Eq. (4) can increase without bound as the parameter τ_0 increases.

3. EIGENVALUES OF THE LAX OPERATOR

In order to find the number of zeros of Eq. (4) it is convenient to use the principle of the argument given by Lavrent'ev and Shabat,[22] according to which the argument of an entire transcendental function changes by an amount equal to the product of the number of zeros n by 2π when a closed contour is traversed. In the present instance the entire transcendental function is $a(z)$, and in what follows we will analyze the increment $\Delta \mathrm{Arg}\, A(z)$. We choose the contour in the form of a segment of the real axis and a semicircle located in the upper half-plane whose points are given by $z=R\exp(i\varphi)$ (Fig. 2).

If the radius of the semicircle satisfies $R\gg 1$, then the function $A(z)$ at these points is given by an exponential

$$A(z)\to\exp[-i(\Omega_1+\Omega_2)z/2],$$

so that

$$\mathrm{tg}\ \mathrm{Arg}\, A(z)=-\mathrm{tg}[(\Omega_1+\Omega_2)R\cos\varphi/2].$$

We choose R so that $(\Omega_1+\Omega_2)R/2=2\pi N$, where N is a whole number much greater than unity. Then we find that the argument (Arg) of the function $A(z)$ changes from $-2\pi N$ to $+2\pi N$ as we move over the semicircle. Hence it follows that to determine the number of zeros we must find $\Delta\ \mathrm{Arg}\, A(z)$ only for movement along the real axis. Moreover, in this case when E_1 and E_2 are real, it suffices to trace the increase of the argument only on a positive segment of the real axis.

If $\Delta\ \mathrm{Arg}\, A(x)$ becomes equal to πN_1 as we move from zero to $x=R$, then the number of zeros n of Eq. (4) is found to be

$$n=2N-N_1. \tag{5}$$

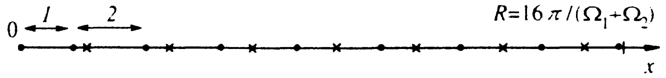

FIG. 3. Disposition of the zeros of the imaginary (points) and real (crosses) parts of the function $A(x)$ when the inequality $\pi<\Omega_1+\Omega_2<2\pi$ holds; the limits of the interval correspond to $N=4$; the separation in time under the applied pulses is equal to zero.

Now let us consider the case of incident pulses whose separation in time vanishes, $\tau_0=0$. It is evident that the two successive pulses are identical to a single pulse with a total area beneath the curve equal to $\Omega_1+\Omega_2$. At points x of the real axis the argument of the scattering coefficient $A(x)$ is such that we have

$$\text{tg Arg}\,A(x)=-\frac{x\sin(\Omega_1+\Omega_2)\sqrt{1+x^2/2}}{\sqrt{1+x^2}\cos(\Omega_1+\Omega_2)\sqrt{1+x^2/2}}. \qquad (6)$$

As x increases the right-hand side of this expression goes over to $-\text{tg}(\Omega_1+\Omega_2)x/2$. This suggests that from the relative position of the zeros of the numerator and denominator, which constitute the imaginary and real parts of the function $A(x)$, we can easily find $\Delta\,\text{Arg}\,A(x)$ associated with motion along the specified segment of the real axis. In fact, each zero of the functions $\text{Im}\,A(x)$ and $\text{Re}\,A(x)$ corresponds to a complex vector directed parallel to either the real or the imaginary axis of the z plane and representing the complex number $A(x)$. The total angle through which this sector turns in the clockwise direction as x goes from zero to R will be equal to the change in the argument of $A(x)$, i.e., πN_1.

We now list the specific results for the different areas of the excited optical pulses.

Case 1: $(\Omega_1+\Omega_2)/2<\pi/2$.

In this case the numerator of (6), like the undeformed $\sin(\Omega_1+\Omega_2)x/2$, has $2N+1$ zeros in the interval $(0,R)$. In each interval bounded by two successive roots of the numerator there is one root of the denominator. In each such interval the complex vector $A(x)$ turns through an angle π in the clockwise direction. This implies $N_1=2N$, and hence n from Eq. (5) vanishes, so that the scattering coefficient $a(z)$ has no zeros anywhere in the complex plane.

Case 2: $\pi/2<(\Omega_1+\Omega_2)/2<\pi$.

The number of intervals of the x axis bounded by neighboring zeros of the numerator of (6) remains unchanged. But now the first zero of the denominator of (6) is located in the second interval (Fig. 3). The change in the argument of the function $A(x)$ in the first interval, denoted in Fig. 3 by the interval *1*, now is equal to zero. Accordingly, N_1 (the number of intervals in which a phase equal to π "accumulates") decreases by unity. This implies that there is one root in the complex plane.

Case 3: $\pi<(\Omega_1+\Omega_2)/2<3\pi/2$.

The number of zeros of both the imaginary and the real parts of the function $A(x)$ is smaller by one than in case 1, as can be seen from expression (6). In each interval bounded by adjacent zeros *1*, *2* of the imaginary part, etc. (Fig. 3), there is one root of the imaginary part.

The complex vector $A(x)$ contains $2N-1$ half-rotations as x varies from 0 to R. Altogether $A(z)$ has one root in the z plane.

Case 4: $3/2\pi<(\Omega_1+\Omega_2)/2<2\pi$.

The number of intervals distinguished by roots of the numerator of expression (6) remains the same as in the previous example, but the first root of the denominator disappears from the first interval. The increase in the argument of $A(x)$, which is equal to π, is "accumulated" over a number of intervals which is smaller by two than in case 1 when x varies from zero to π, so that $A(z)$ has two roots.

Case 5: $2\pi<(\Omega_1+\Omega_2)/2<5\pi2$.

Under these conditions another root of the imaginary part disappears. The number of intervals has decreased by two relative to case 1. In each of the other intervals there is one root of the real part. As in the previous case the number of roots is equal to two.

Now we can easily sum up the results obtained. When the condition

$$\pi+2\pi(n-1)<\Omega_1+\Omega_2<3\pi+2\pi(n-1) \qquad (7)$$

holds the scattering coefficient $a(z)$ corresponding to two pulses with identical amplitudes and zero separation in time and having a total area $\Omega_1+\Omega_2$ in the pulses has n roots in the complex plane ($n=1, 2,...$). These results also holds for a single pulse with area equal to $\Omega_1+\Omega_2$.

We note now that for $z=iy$ Eq. (4) becomes real. It is easy to tabulate the number of roots of this equation for $\tau_0=0$. For this it suffices to find the number of intersections of the two functions $f_1(y)$ and $f_2(y)$:

$$f_1(y)=y/\sqrt{1-y^2},\quad f_2(y)=-\text{ctg}(\Omega_1+\Omega_2)\sqrt{1-y^2/2}$$

on the interval (0, 1). Outside this interval the equation $f_1=f_2$ obviously has no solutions. The results can be expressed by analogy with Eq. (7): if the total area of the pulses satisfies Eq. (7), then Eq. (4) allows n roots on the imaginary axis.

From this a simple and useful conclusion follows: if two pulses are separated by a vanishing or small time interval, then all roots of the scattering coefficient lie on the imaginary axis, and the complex roots can appear only when this time interval increases.

Let us now consider some specific examples of applied pulses separated in time. We choose $\Omega_1+\Omega_2$ so as to satisfy the conditions

$$\pi<\Omega_1<3\pi,\quad \pi<2\Omega_2<3\pi.$$

This example is of interest because the function $A(z)$ corresponding to each pulse has a single root, but the scattering coefficient of the total applied pulse also has one root.

The disposition of the roots of the imaginary and real parts of the scattering coefficient for $\Omega_1+\Omega_2$ and $\tau_0=2$ is shown in Fig. 4.

It follows from the material given above that the number of roots of the imaginary and real parts for $\tau_0=0$ is smaller by one than the number of zeros of the imaginary and real parts of the undeformed function

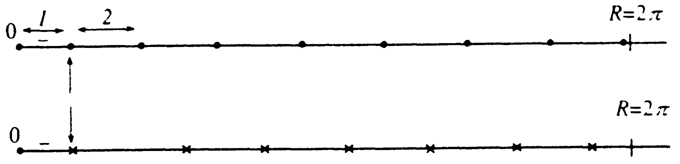

FIG. 4. Disposition of the zeros of the imaginary (points) and real (crosses) parts of the function $A(x)$ for $\Omega_1=\Omega_2=4$ and $\tau_0=2.0$ on the segment of the real axis corresponding to $N=4$; the zeros, marked with arrows, change their relative position as τ_0 increases, so that for $\tau_0=2.1$ the scattering amplitude $A(x)$ has a root on the real x axis [the signs of both parts of $A(x)$ are explicitly shown on the first intervals].

$\exp[-(\Omega_1+\Omega_2)x/2]$. But for $\tau_0=2$ the imaginary part has a double root, indicated in Fig. 4 by an arrow. Nevertheless, the number of roots of the complex scattering coefficient remains equal to unity. In fact, the first root of the real part is located beyond the second root of the imaginary part, so that in the first interval of the x axis, distinguished in Fig. 4 by the label *1*, we have $\Delta\ \mathrm{Arg}\, A(x)=0$.

We emphasize again that this becomes obvious if for each zero x_k of either the real or the imaginary part we draw a vector corresponding to $A(x_k)$. Then, true, we need the signs of the corresponding parts of the function $A(x_k)$. In the course of an actual computational procedure it suffices to know the signs in the first intervals. In Fig. 4 the signs of the imaginary and real parts are indicated by the symbol "$-$." as x varies along the computational axis the angle through which this vector rotates is the change in the argument of the function $A(x)$.

Thus we can perform a check which is readily implemented in the form of an algorithm. If the first interval bounded by two adjacent zeros is the imaginary part of $A(x)$ has an even number of zeros of the real part, then the change $\Delta\ \mathrm{Arg}\, A(x)$ in this interval vanishes. If the number of roots is odd, then $\Delta\ \mathrm{Arg}\, A(x)$ is equal to $\pm\pi$. The plus sign results if $\mathrm{Im}\, A(x) \mathrm{Re}\, A(x_1)<0$ holds. Here x_1 represents the start of the first interval, and an arbitrary number x located beyond x_1, but before the first root x_2 of the real part, is inside this interval. We can take $x=(x_1+x_2)/2$. The change in the argument will be $-\pi$ if the opposite inequality holds.

When τ_0 varies in the interval (2.0–2.1), the roots indicated in Fig. 4 by the arrows change places. In the first range of roots of the imaginary part, indicated in Fig. 4 by the label *1*, there is now a single root of the real part, while the second interval is found to be empty. In accordance with the above discussion the function $A(x)$ in the first interval acquires the phase π; in all the others except the second, where $\Delta\ \mathrm{Arg}\, A(x)=0$ holds, the phase change is equal to $-\pi$. From this we conclude that for $\tau_0>\tau_0^{(1)}=2.1$ the scattering coefficient $A(z)$ now has three roots. New roots appear near the real axis. In this example the value x_0 at which the real and imaginary parts of $A(x)$ vanish simultaneously is given by $x_0=0.495\pm0.001$. As τ_0 increases these roots should move out into the complex planes and either go to zero or move to the imaginary axis. It is interesting to note that as τ_0 increases without bound the number of branches occurring near the real axis for $x_0=0.495$ becomes arbitrarily large. In fact, τ_0 enters in $A(x)$ through the power in the exponent. Hence the zeros display periodicity depending on τ_0. Thus, if $\tau_0^{(2)}$ satisfies the condition

$$x_0\tau_0^{(1)}+2\pi=x_0\tau_0^{(2)},$$

i.e., for $\tau_0^{(2)}=14.7$, a second branch of the complex roots comes out of the point at x_0.

The computational procedure implemented using this technique confirms the presence of five roots of the functions $A(z)$, beginning with $\tau_0=14.7$, seven roots beginning at 27.4, etc. Each time the appearance of a new branch is associated with a change in the relative positions of the roots of the imaginary and real parts near the point $x_0=0.495$.

Knowing the approximate location of the roots in the complex plane we can easily trace their evolution as the parameter τ_0 increases. Figure 5 displays two branches of the roots, starting at $\tau_1^{(1)}=2.1$ and $\tau_0^{(2)}=14.7$. From Fig. 5 we see that at $\tau_0=16.0$ a multiple root of second order arises on the imaginary axis. As τ_0 increases further this root divides into two roots moving apart along the imaginary axis. Their subsequent evolution together with the original root is traced in Fig. 6.

According to Fig. 6 one of the newly arisen roots on the imaginary axis at $\tau_0=16.0$ approaches zero, while the

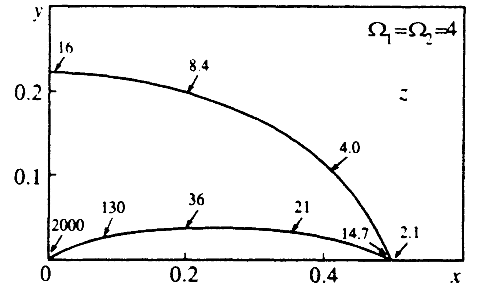

FIG. 5. Two branches of the complex roots of the scattering amplitude $A(x)$: the numbers next to the arrows label the time interval τ_0.

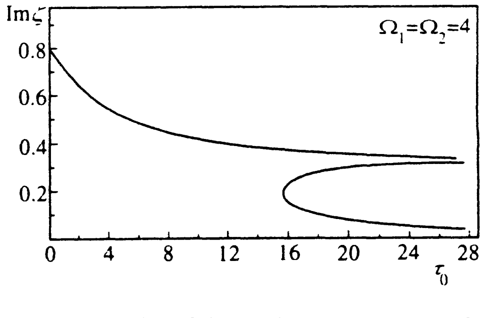

FIG. 6. Evolution of the purely imaginary roots of the scattering coefficient as a function of the time τ_0 between successive pulses.

other merges with the original root on the imaginary axis. The explanation for this is that in the limit $\tau_0 \to \infty$ the scattering coefficient should have roots corresponding to the product of the scattering coefficients for two independent but identical ($\Omega_1 = \Omega_2 = 4$) imposed pulses. But both of these scattering coefficients have only one root by virtue of the inequality $\pi < \Omega_1 < 3\pi$.

For the same reason the second branch of the complex roots arising for $\tau_0 > 14.7$ (Fig. 5) does not go to the imaginary axis, but approaches zero. This assertion holds for all the other roots as well.

On the basis of the above discussion we arrive at a simple conclusion. If the conditions

$$\pi + (n_i - 1)2\pi < \Omega_i < 3\pi + (n_i - 1)2\pi, \quad i = 1,2,$$

$$\pi + (n-1)2\pi < \Omega_1 + \Omega_2 < 3\pi + (n-1)2\pi,$$

$$n + 1 = n_1 + n_2$$

hold then one branch goes to the imaginary axis, where a multiple root arises for some value of τ_0. As τ_0 increases further the degeneracy is lifted and one of the newly arisen roots approaches zero while the others go to their limiting values, corresponding to the roots of each imposed pulse taken separately. As τ_0 increases new branches of the complex roots arise periodically; these, however, do not go to the imaginary axis, but approach zero in the limit of large τ_0.

If $n = n_1 + n_2$ holds, then all the branches of the roots that occur in the complex plane approach zero as τ_0 increases; as τ_0 increases, n roots of the imaginary axis reach values corresponding to the individual applied pulses.

If $n > n_1 + n_2$ holds, then as τ_0 increases all $n - (n_1 + n_2)$ values of the roots of the imaginary axis approach zero, just like all the complex branches of the roots that arise at the real axis.

Note that $n_1 + n_2$ differ from n only by unity. Thus, for $\Omega_1 = \Omega_2 = 10$ we have $n_1 = n_2 = 2$ and $n = 3$. In accordance with the above results for small τ_0, three solitons will be observed. Then for some value of τ_0 two conjugate roots arise, differing by the sign of the real part (they correspond to a breather). As τ_0 further increases, another breather develops and the first one turns into a soliton. Then this degenerate soliton decays into two, one of which disappears as τ_0 increases.

New breathers arise periodically, are transformed into degenerate solitons, and disappear in the limit of large τ_0.

4. CONCLUSION

Thus, a method based on the argument principle in complex variable theory enables us to find the number of zeros of the scattering amplitude in the inverse problem associated with the Maxwell–Bloch system of equations. We have shown that if the area of an excited pulse satisfies the inequality $\pi + 2\pi(n-1) < \Omega < 3\pi + 2\pi(n-1)$, then n solitons arise as an ultrashort pulse propagates in the resonant medium. When the medium is excited by two closely spaced optical pulses separated by an interval τ, assuming there is no phase difference between them, n solitons develop in the medium when the inequality

$$\pi + (n-1)2\pi < \Omega_1 + \Omega_2 < 3\pi + (n-1)2\pi$$

holds. When the numbers n_1 and n_2 determined from these inequalities for the areas Ω_1 and Ω_2 of the pulses add up to a total which exceeds n by unity, then for some value of τ a multiple soliton develops which decays into two solitons as τ increases further. One of these disappears in the limit of large τ, while the other corresponds to the excitation of the medium by the individual applied pulses. If $n_1 + n_2 \leqslant n$ holds, then no further solitons arise.

In every instance breathers develop periodically as τ increases. If $n_1 + n_2 = n + 1$ holds, then the first branch of the complex roots corresponding to such breathers goes to the imaginary axis, which gives rise to an additional soliton pair. The other branches go to zero very slowly as τ increases.

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