Chiral field theory of membrane shape fluctuations

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Fluctuations in the shape of membranes of arbitrary topology are analyzed within the framework of the classical theory of surfaces. It is assumed that the relaxation of the membrane to the equilibrium shape is described by a Langevin equation for a set of variables which characterize the softest corresponding mode of the system. It is convenient to take these variables to be a moveable Frenet frame of reference. The classical Gauss-Weingarten equations, which determine the variation of this frame of reference from point to point, are formulated as equations for the currents in a chiral field theory. The energy of the membrane can also be expressed in terms of chiral currents. As a result, static correlation functions of fluctuations in the membrane shape can be calculated in closed form. An effective action is constructed for the chiral field in the standard way from the Langevin equation. This action can be used to find dynamic correlation functions. It is shown that this action is supersymmetric. A fluctuation—dissipation theorem which relates the binary correlation function with a response function is derived in general form. An iterative procedure is proposed for finding arbitrary correlation functions in the form of a perturbation-theory series in the curvature of the membrane.

1. LANGEVIN EQUATION

Recent interest in the physics of complex liquids has led to the present boom in this field. The number of publications in physics journals devoted to various aspects of this problem continues to grow. Reviews[1,2] and monographs[3–5] have been published. The interest stems from the biological importance of complex liquids (e.g., blood) and also the very interesting fundamental physics of these systems (see Refs. 6 and 7, where an analogy was drawn between the physics of membranes and $2D$ gravitation and also a vortex lattice in a superconductor). A more natural analogy, however, is that with the theory of a chiral field, as we pointed out in a recent letter.[8] We take a detailed look at this analogy in the present paper.

Generally speaking, there are a huge number of diverse complex liquids, which exhibit different specific properties. A common feature of all these systems is that the molecules which make them up are capable of self-aggregation into membranes. These membranes are entities of molecular thickness with significant sizes in the two other dimensions. From the standpoint of macroscopic scales, a membrane can thus be pictured as a $2D$ entity immersed in a $3D$ space or (in geometric terms) as a surface.

An important circumstance for the entire physics of membranes is the fact that the surface tension of membranes is extremely small or in fact zero. If a membrane is in equilibrium with a solution of the molecules making up the membrane, zero surface tension is simply one of the equilibrium conditions. One way or the other, we will thus be ignoring the surface tension. The energy of the membrane (which, we recall, is a surface, i.e., its energy is purely a surface energy) is determined by the higher-order terms of an expansion in the curvature. The general form of this expansion was first proposed by Helfrich.[9] The corresponding expression, the "Helfrich energy," is

$$E_H=\int d^2r\left(\tfrac{1}{2}\kappa K_H^2+\bar{\varkappa}K_G\right). \tag{1}$$

Here κ and $\bar{\varkappa}$ are elastic moduli, which depend on the temperature, the composition, and other characteristics of the membrane and the external conditions; K_H is the average curvature; and K_G is the Gaussian curvature of the surface. The quantities K_H^2 and K_G are the lowest-order geometric invariants of the surface:

$$K_H=\frac{1}{R_1}+\frac{1}{R_2},\quad K_G=\frac{1}{R_1R_2}, \tag{2}$$

where R_1 and R_2 are the principal radii of curvature of the surface.

Expression (1) assumes symmetry under a change in the sign of the average curvature, so it has no linear component. This situation corresponds to the case of so-called bilayer or symmetric membranes, and the discussion below is limited to this case. It is expression (1) which elevates the physics of membranes to a position of fundamental importance. The membrane energy is determined by purely geometric characteristics of the surface, not by the particular molecular structure of the membrane. Only the numerical coefficients in the two relations depend on this structure.

In principle, one could consider an energy of more general form than that in (1), in which nonlinear terms of

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higher order in K_H or K_G are important for some reason or other. This is the situation which prevails for crystalline membranes,[10] for example. To find the shape of the membrane in this general case, we must minimize an integral of the form

$$\int du_1 du_2 E(K_H, K_G),$$

where u_1 and u_2 parametrize the surface, and E is some function of K_H and K_G. If we fix the topology of the membrane, then the integral

$$\int du_1 du_2 K_G$$

must be constant. In the general formulation of the problem of determining the equilibrium membrane shape, we must therefore minimize

$$E_{\text{eff}} = \int du_1 du_2 \{E(K_H, K_G) + \lambda K_G\},$$

where the Lagrange multiplier λ specifies the topology of the surface. The term λK_G in E_{eff} has the meaning of the Chern-Simon term[11] in field theory.

The range of the parameters u_1 and u_2 over which the integration is carried out is determined by the membrane topology under consideration. In the case of a torus, for example, the range is a square with identified opposite sides.

A complex liquid thus consists of membranes (surfaces) floating in a surrounding liquid. Membranes undergo fluctuations because of thermal motion. These fluctuations in turn induce motion in the liquid around the membrane, and the membrane relaxes to its equilibrium shape. A complete hydrodynamic description of a complex liquid thus consists of the solution of a system of equations which describes both the membrane and the surrounding liquid. In certain particular cases, such a description can indeed be found explicitly.[12] For membranes of arbitrary topology, however, it is hardly possible to find an explicit solution of this system of equations. At any rate, if we are not concerned with unstable situations, then fluctuations of the membrane shape relax to equilibrium, and the problem of solving the complex system of hydrodynamic equations and equations of motion of the membrane reduces simply to one of calculating a relaxation coefficient.

In this paper we adopt a phenomenological point of view. We consider the fluctuations in the shape of one membrane which is floating in a liquid surrounding it. If we eliminate all the hydrodynamic degrees of freedom other than those which describe the relaxation of the membrane shape, then the effective Langevin equation takes the following form for these degrees of freedom (for the time being, we denote these degrees of freedom by v, without specifying their physical meaning):

$$\gamma \partial_t v = \frac{\delta E}{\delta v} + y. \tag{3}$$

Here y is a random noise, and γ an effective relaxation coefficient. The meaning of the variable v will be clarified below; at this point we simply point out that the equilibrium membrane shape corresponds to a certain value v_0 of this variable, which satisfies the condition

$$\frac{\delta E}{\delta v}(v_0) = 0. \tag{4}$$

As justification for Eq. (3) we can also cite the following qualitative considerations, which are based on the analogy between the relaxation dynamics of any system and Brownian motion of particles. We assume that the membrane shape fluctuations can be described as a random Brownian motion of small regions which can be used to approximate the surface of the membrane. According to a Gaussian distribution, the fluctuations of these regions can be described by a correlation function of their displacements:

$$\langle \delta v(1) \delta v(2) \rangle = D\delta(1-2),$$

where the argument 1 or 2 specifies the position of the point on the surface and the time, and D is a diffusion constant. The probability density for a Gaussian distribution is given by the function

$$\exp\left(-\frac{y^2}{2D}\Delta\sigma\Delta t\right),$$

where y represents a random noise, $\Delta\sigma$ is the area of the region under consideration, and Δt is a time interval. Using this distribution we can find the mean value of any function Φ, which depends on the position of the point on the membrane surface:

$$\langle \Phi \rangle = \int \left(\prod_{i=1}^{N} dy_i \right) \Phi(y) \exp\left\{ - \sum_{i=1}^{N} \frac{y^2}{2D} \Delta\sigma\Delta t \right\}.$$

Here N is the number of regions into which the surface has been partitioned. In the limit $N \to \infty$ we obtain the standard expression for $\langle \Phi \rangle$, in the form of a functional integral:

$$\langle \Phi \rangle = \int Dy \Phi(y) \exp\left(- \int \frac{y^2}{2D} d\sigma dt \right).$$

Interestingly, this representation of the mean displacement of a Brownian particle, which includes all the essential ideas of functional integration, was derived back in 1910 by Einstein and Hopf.[13]

In general, the analogy between the fluctuations of membranes and Brownian motion is rather far-reaching, not purely formal. To illustrate this assertion, we note that a Brownian particle can be described by the Langevin equation

$$m \frac{d^2x}{dt^2} = -\gamma \frac{dx}{dt} + \frac{\partial \varphi}{\partial x} + y,$$

where m is the mass of a particle, φ is the potential of the external force, and y is a noise. For a sufficiently light particle or, equivalently, for a sufficiently viscous medium, we can ignore the inertial term. As a result we find

$$\frac{dx}{dt}=\frac{1}{\gamma}\frac{\partial\varphi}{\partial x}+y.$$

This, however, is Eq. (3), with x replacing the variable δv, which describes the motion of the membrane, and with φ replacing the membrane energy E. This entire discussion is of course only qualitative, since the liquid around the membrane is being taken into account only as a source of a random noise.

2. CHIRAL THEORY OF SURFACE DEFORMATIONS

We treat the membrane as a surface which is parametrized by certain coordinates u_1, u_2: $\mathbf{r}=\mathbf{r}(u_1,u_2)$. At any nonsingular point on the surface we can specify a local Frenet frame of reference, which is characterized by the tangent vectors

$$\tau_\mu=\frac{\partial\mathbf{r}}{\partial u_\mu}, \quad (\mu=1,2)$$

and by the unit normal vector

$$\mathbf{n}=[\tau_1\tau_2]/|[\tau_1\tau_2]|.$$

This frame of reference varies from point to point on the surface. The corresponding derivatives are found from the equations of differential geometry:[14]

$$\partial_\mu\tau_\nu=\Gamma^\xi_{\mu\nu}\tau_\xi+B_{\mu\nu}\mathbf{n}, \quad \partial_\mu\mathbf{n}=C^\xi_\mu\tau_\xi, \tag{5}$$

where $\Gamma^\xi_{\mu\nu}$ are the Christoffel symbols, given by

$$\Gamma^\xi_{\mu\nu}=\frac{g^{\xi\delta}}{2}\left(\frac{\partial g_{\delta\nu}}{\partial u_\mu}+\frac{\partial g_{\mu\delta}}{\partial u_\nu}-\frac{\partial g_{\mu\nu}}{\partial u_\delta}\right), \tag{6}$$

and $g_{\mu\nu}$ are coefficients which specify the Riemann metric of the surface. In addition, Eqs. (5) contain the so-called second quadratic form of the surface, $B_{\mu\nu}$. The tensor C^ξ_μ is related to $B_{\mu\nu}$ by

$$C^\xi_\mu=-g^{\xi\nu}B_{\mu\nu}. \tag{7}$$

We can thus describe the variation in the local frame of reference, which we denote by the matrix X, by the general formula

$$\partial_\mu X=j_\mu X, \quad \mu=1,2. \tag{8}$$

Here

$$X_{i\mu}=(\tau_\mu)_i, \quad i=1,2,3, \quad \mu=1,2,$$
$$X_{i3}=n_i, \quad j_\mu=\begin{pmatrix}\Gamma^\xi_{\mu\nu} & B_{\mu\nu}\\ C^\xi_\mu & 0\end{pmatrix}. \tag{9}$$

Equations (8), which are called the "Gauss–Weingarten" equations, are the foundation of the classical theory of surfaces. The matrices j_μ introduced above must satisfy integrability conditions, which are evidently

$$\partial_1 j_2-\partial_2 j_1+[j_2, j_1]=0. \tag{10}$$

Equation (10) can be written in a different form, as is used in the theory of gauge fields:

$$F_{\mu\nu}=[\nabla_\mu,\nabla_\nu]=0, \quad \mu,\nu=1,2.$$

According to this representation, the quantity $F_{\mu\nu}$ (the curvature of the gauge field), which is determined by covariant derivatives, must be zero. For arbitrary deformations of the surface the chiral currents j_μ must vary in such a way that this condition ($F_{\mu\nu}=0$) is conserved.

Deformations of a surface can be looked at from a slightly different standpoint. Let us assume that we vary the chiral current

$$j_\mu\to j_\mu+\delta j_\mu$$

in a way corresponding to a rightward translation or a gauge transformation of the local frame of reference:

$$X\to RX, \quad R=1+\delta v,$$

where the matrix R conserves the structure of the Frenet frame of reference. In other words, it sends the unit normal $\mathbf{n}$ into the unit normal $\mathbf{n}'$, and it sends the tangent vectors τ_μ, $\mu=1,2$ into the tangent vectors τ'_μ, $\mu=1,2$, respectively. The following equations hold:

$$\delta j_\mu=\nabla_\mu\delta v, \quad F'_{\mu\nu}=RF_{\mu\nu}R^{-1}.$$

It follows from these equations that the integrability condition $F_{\mu\nu}=0$ is conserved and that a fluctuation of the gauge field, δv, induces a fluctuation of the surface shape. The latter are apparently isometries, in agreement with the incompressibility of membranes.

The incompressibility of membranes is of course only approximate. The bulk modulus of a membrane is determined by the derivative of the surface tension with respect to the concentration of the molecules making up the membrane. Although the surface tension of a membrane is extremely small, as we mentioned above, the derivative of this quantity is by no means small. It is on the order of the "normal" magnitude for organic liquids.[1-5] A large value of the bulk modulus justifies our approximation that all deformations of the membrane are isometries.

On the other hand, the Gauss–Weingarten equations in (8) can be used to introduce a chiral field on a group of real, nondegenerate matrices $GL(3,R)$ such that Eq. (8), formally solved for j_μ, determines a chiral current:

$$j_\mu=\partial_\mu X\cdot X^{-1}, \quad \mu=1,2. \tag{11}$$

The approach which we are proposing here starts from the important fact that the mean curvature and the Gaussian curvature of a surface and thus a physical quantity such as the energy of a membrane can be expressed in terms of the same chiral currents:

$$K_G=\det C=\det(B\cdot g^{-1}), \quad K_H=\mathrm{Tr}(B\cdot g^{-1}). \tag{12}$$

The mean curvature can also be written explicitly in terms of the chiral currents:

$$K_H=j^1_{13}+j^2_{23}, \tag{13}$$

where $j^1_{\mu\nu}$ and $j^2_{\mu\nu}$ are matrix elements of j_1 and j_2. An explicit expression for K_G can also be written, but we omit it since we will not need it below.

The Helfrich energy (1) can be expressed in terms of chiral currents:

$$E_H=\int du_1 du_2 a^{\mu\nu}_{ijkl} j^{\mu}_{ij} j^{\nu}_{kl}. \tag{14}$$

We can therefore formulate a theory for membrane shape fluctuations as a chiral field theory. We also note that the energy in (14) is quadratic in the chiral currents. All the nonlinear terms in the expression (1) for the initial energy, are cancelled by the choice of the moving frame of reference. Nevertheless, this is of course not a free-field theory, since we must satisfy the integrability conditions (10). However, as we explained above, the gauge condition $F_{\mu\nu}=0$ holds automatically for small fluctuations if it holds for the equilibrium surface shape.

In this approximation the static binary correlation function can be calculated for an arbitrary membrane topology. For this purpose we consider the generating functional

$$Z(\chi)=\int Dv \exp\left\{-\frac{1}{T}\int du_1 du_2 [E(j_1\ j_2) + \mathrm{Tr}(\chi\delta v+\delta v\chi)]\right\},$$

where we have introduced a matrix field χ which is the conjugate of δv. Expanding the membrane energy in E in δv, retaining terms of up to second order, and using the integrability condition in the form $[\nabla_\mu,\nabla_\nu]=0$, we find the following expression for $Z(\chi)$:

$$Z(\chi)=\exp\left\{-\frac{1}{T}\mathrm{Tr}(\chi D\chi)\right\}.$$

Here the binary correlation function D is determined by the solution of the equation

$$\nabla_\mu\left\{\frac{\partial^2 E}{\partial j^{pq}_\mu \partial j^{rs}_\nu}(\nabla_\nu D^{rs})\right\}=\delta_{pq}\delta(1-2).$$

3. DYNAMICS OF FLUCTUATIONS

In the preceding section of this paper we showed that static fluctuations of a membrane shape can be described within the framework of chiral field theory. In the present section we show how the dynamics of fluctuations can be studied within the framework of this theory.

Any variation of a chiral field which is related to the rightward translation

$$X\to(1+\delta v)X, \tag{15}$$

where δv is a real 3×3 matrix, determines the following variation of the chiral current:

$$j_\mu\to j_\mu+\nabla_\mu\delta v,\quad j_0=\partial_t\delta v. \tag{16}$$

Here we have introduced the definition of a covariant derivative,

$$\nabla_\mu=\partial_\mu+[j_\mu,\ldots] \tag{17}$$

(the square brackets mean a commutator), and we have determined the component j_0 of the chiral current, which determines the time evolution of the chiral field (the local frame of reference),

$$j_0=\partial_t X\cdot X^{-1}. \tag{18}$$

These definitions allow us to write the Langevin equation (3) as an equation for a chiral field:

$$\gamma\partial_t\delta v=-\nabla_\mu(a^{\mu\nu}j_\nu)+y, \tag{19}$$

where y, a real 3×3 matrix, describes the thermal noise (the Brownian motion of the surface). If we consider small fluctuations near the equilibrium shape, given by the equation

$$\nabla_\mu j^{(0)}_\mu=0, \tag{20}$$

then we can linearize Eq. (19) around the equilibrium values of the chiral currents:

$$\gamma\partial_t\delta v+\{\partial_\mu(a^{\mu\nu}\nabla_\nu\delta v)+[\nabla_\mu\delta v,a^{\mu\nu}j^{(0)}_\nu] + [j^{(0)}_\mu,a^{\mu\nu}\nabla_\nu\delta v]\}=y. \tag{21}$$

As we mentioned above, one could in principle also discuss the general form of the membrane energy consisting of an arbitrary invariant function of chiral currents,

$$\int du_1 du_2 E(j_1,j_2). \tag{22}$$

The Langevin equation for this general form of the energy is

$$\gamma\partial_t\delta v=\frac{\delta E}{\delta v}+y.$$

A simple calculation of the functional derivatives yields

$$\gamma\partial_t\delta v+\nabla_\mu\left(\frac{\partial E}{\partial j_\mu}\right)=y.$$

Linearizing, we find in place of (21)

$$\gamma\partial_t\delta v+\nabla^{(0)}_\mu\left(\frac{\partial^2 E^{(0)}}{\partial j_\mu\partial j_\nu}\nabla^{(0)}_\nu\delta v\right)+\left[\nabla^{(0)}_\mu\delta v,\frac{\partial E^{(0)}}{\partial j_\mu}\right]=y$$

[the superscript (0) denotes the equilibrium value].

We write this equation in the symbolic form

$$M\delta v=y,$$

where the structure of the operator M is to be found from the explicit form of the left side of this equation [or from Eq. (21) for the Helfrich energy of the membrane].

We have thus formulated phenomenological Langevin equation (1) as an equation for a chiral field which determines a movable frame of reference on the surface. A single energy characteristic of the surface, e.g., the Helfrich energy, also expressed in terms of chiral currents, describes an incompressible membrane, which is a good physical approximation.[1-5] Accordingly, all possible variations of the chiral field leave the metric of the surface unchanged; i.e., they are isometries.

We also note a useful relation for variations of the chiral current:

$$j_\mu=j^{(0)}_\mu+\nabla_\mu\delta v. \tag{23}$$

For the planar case the covariant derivative reduces to an ordinary derivative, and Eq. (23) simplifies.

4. CORRELATION FUNCTIONS

The expressions derived in the preceding section of this paper can be used to find various averages. We will take the standard approach, developed by Sourlas[15] and Zinn-Justin,[16] to describe the Langevin dynamics. We assume that the thermal noise is Gaussian:

$$\langle y_{ij}y_{kl}\rangle = D\delta(t_1-t_2)\delta_{ik}\delta_{jl} \tag{24}$$

(D is a kinetic coefficient on the order of $1/\gamma$). We take an average over the noise by integrating with the following measure:

$$d\rho(y) = dy\exp\left\{-\frac{1}{2D}\int du_1du_2dty_{ij}y_{kl}\right\}. \tag{25}$$

Using the measure (25), we can calculate the average of any function $F(X)$ of the local frame of reference:

$$\langle F\rangle = \int dyF(X)\exp\left\{-\frac{1}{2D}\int du_1du_2dty_{ij}y_{ji}\right\}. \tag{26}$$

The next step is to formally solve Eq. (19) for the noise. We can then write the following expression:

$$\langle F\rangle = \int DyD\delta vF(\delta v)\delta(y-M\delta v)$$
$$\times\exp\left\{-\frac{1}{2D}\int du_1du_2dty_{ij}y_{ji}\right\}. \tag{27}$$

Changing variables in (27), i.e., switching to an integration over the chiral currents j, we find the formal average of any function of the chiral currents:

$$\langle F(j)\rangle = \int DyD\delta vF(\delta v)\delta(y-M\delta v)\det\frac{\delta(M\delta v)}{\delta(\delta v)}$$
$$\times\exp\left\{-\frac{1}{2D}\int du_1du_2dty_{ij}y_{ji}\right\}. \tag{28}$$

Recalling that we have

$$\delta j_\mu = \nabla_\mu\delta v \quad\text{and}\quad \det\frac{\delta(M\delta v)}{\delta(\delta v)} = \det M,$$

by definition, we obtain the following result for the Helfrich energy in (14):

$$\langle F(j)\rangle = \int D\delta vF(j)\det M\exp\left\{-\frac{1}{2D}\int du_1du_2dt\right.$$
$$\left.\times\mathrm{Tr}([j_0-\nabla_\mu(a^{\mu\nu}j_\nu)][j_0-\nabla_\mu(a^{\mu\nu}j_\nu)^{(t)}])\right\}, \tag{29}$$

where $(a^{\mu\nu}j_\nu)^{(t)}$ means the transposed matrix.

The last step is to write $\det M$ in exponential form with the help of the Grassmann variables ψ, $\bar\psi$. We can then find any averages through functional integration with the effective action S given by

$$S = \int d^2udt\left\{-\frac{1}{2D}\mathrm{Tr}([j_0+\nabla_\mu(a^{\mu\nu}j_\nu)]\right.$$
$$\times[j_0+\nabla_\mu(a^{\mu\nu}j_\nu)^{(t)}]) + \bar\psi(\nabla_0\psi - a^{\mu\nu}\partial_\mu\nabla_\nu\psi$$
$$\left.-[\nabla_\mu\psi,a^{\mu\nu}j_\nu]-[j_\mu,a^{\mu\nu}\nabla_\nu\psi])\right\}. \tag{30}$$

When this action is used, the averages are calculated in the following way:

$$\langle F\rangle = \int djd\psi d\bar\psi\exp S. \tag{31}$$

A direct check verifies that the effective action in (30) is supersymmetric, i.e., invariant under the transformations

$$\delta v = (\bar\epsilon\psi+\bar\psi\epsilon),\quad \delta\psi = \epsilon(\partial_t-M)\delta v,$$
$$\delta\bar\psi = (\partial_t-M)\delta v\epsilon,$$

where ϵ and $\bar\epsilon$ are infinitesimal anticommuting fields.

Action (30) generates a diagram technique which involves seed correlation functions determined by the quadratic part of the action,

$$S^{(2)} = \int d^2udt\left\{-\frac{1}{2D}\mathrm{Tr}(j_0\ j_0^{(t)})+\bar\psi\nabla_0\psi\right\},$$

while the structure of the interaction vertices is specified by nonlinear terms,

$$S^{(\mathrm{int})} = S-S^{(2)}. \tag{32}$$

Equation (32) contains vertices of two types: vertices which are unrelated to the nontrivial geometry of the surface (i.e., vertices which exist even in the case of a planar membrane) and vertices which arise from terms in (32) which contain covariant derivatives (i.e., which arise directly from the nontrivial geometry of the surface).

The action (30) can also be put in a different form by introducing a Bose field p, which is the conjugate of δv:

$$S = \int d^2udt\left\{-p\left(\partial_t\delta v - M\delta v+\frac{1}{2D}p\right)+\bar\psi(\partial_t-M)\psi\right\}. \tag{33}$$

The averages calculated with action (30) and action (33) are the same, but in the latter case it is also necessary to integrate over the Bose variable p. The effective action in (33) is convenient in that it contains only first time derivatives, like the original Langevin equations.

Using the effective action in the form in (33), we can prove a fluctuation dissipation theorem in general form. For this purpose we introduce a special notation D for the correlation function:

$$D = \langle\delta v,\delta v\rangle.$$

The correlation function $\langle\delta v,p\rangle$, which is a generalized susceptibility or response function,[12] is denoted by G. We change variables in the action (33):

$$p \to p' + \frac{i}{2T}\frac{\partial \delta v}{\partial t}. \tag{34}$$

When this change is made, as can be seen directly and easily from the definitions of the averages, we have

$$G(t) - \frac{i}{2T}\partial_t D(t) = G(-t) + \frac{i}{2T}\partial_t D(t),$$

i.e.,

$$\frac{i}{T}\partial_t D(t) = G(t) - G(-t). \tag{35}$$

Relation (35) is the expression which we wanted for the fluctuation-dissipation theorem. The fact that it has been possible to prove this theorem in a general form is not a consequence of the specific expressions for the vertices and correlation functions. It is instead a consequence of the purely relaxational nature of the dynamics or, in other words, the linear dependence of the effective action (33) on the derivative $\partial_t \delta v$. This situation makes it possible to use the transformation of variables in (34).

The method proposed in this section of the paper allows one to calculate not only correlation functions of local chiral fields but also such global characteristics of a membrane as the correlation functions of the average curvature or Gaussian curvature. According to Eqs. (12), the latter are also expressed in terms of chiral fields. In the zeroth approximation in the curvature, for example, we have

$$\langle \delta K_H(\omega,q)\delta K_H(0,0)\rangle = \frac{T\gamma q}{\omega^2 + \kappa^2\gamma^2 q^2}. \tag{36}$$

This formula agrees with that found in Ref. 12, by a different method, through a suitable choice of the phenomenological effective coefficient γ.

5. PERTURBATION THEORY

In this section of the paper we describe an iterative procedure for solving the Gauss–Weingarten equations. This procedure makes it possible to construct a perturbation theory in the characteristic curvature of the equilibrium surface. For this purpose we write the Gauss–Weingarten equations in integral form

$$X = \int_{P_1}^{P_2} du^\mu j_\mu. \tag{37}$$

Using (37), we find the following result for small fluctuations of the local frame of reference:

$$\delta X = \int_{P_1}^{P_2} du^\mu (\delta j_\mu X + j_\mu \delta X + \delta j_\mu \delta X). \tag{38}$$

We recall that (37) and (38) do not depend on the positions of the points P_1 and P_2 on the surface—only on the distance between these points [this situation is a consequence of the integrability conditions (10)].

Using Eq. (38), we can formulate an iterative procedure, making a systematic transformation from a point on the surface to a nearby point, and choosing the integration path in such a way that δj_μ and δX remain small in each step. We can thus express δX in terms of δj as a sum of terms:

$$\int_{P_1}^{P_2} du^\mu(P') \int_{P_1}^{P'} du^\nu(P'') j_\mu(P'')\delta j_\nu(P'')X(P'').$$

The expression found in this manner is extremely cumbersome and thus of little use. However, it can be utilized to construct a perturbation theory in the curvature of the surface. We will illustrate the procedure using a spherical surface of radius R as an example. In this case Eq. (38) can be rewritten as

$$\delta X = \int_{P_1}^{P_2} du^\mu(P') \left[\delta j_\mu(P') X^{(\mathrm{sph})}(P') + \frac{1}{R} j_\mu^{(\mathrm{sph})}\delta X\right]. \tag{39}$$

Here $X^{(\mathrm{sph})}$ and $j_\mu^{(\mathrm{sph})}$ are a local Frenet frame of reference and the chiral current for a sphere. Simple calculations lead to

$$X^{(\mathrm{sph})} = \begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix},$$

$$j_1^{(\mathrm{sph})} = \frac{1}{R}\begin{pmatrix} 0 & 0 & -1 \\ 0 & \mathrm{ctg}\,\theta & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{40}$$

$$j_2^{(\mathrm{sph})} = \frac{1}{R}\begin{pmatrix} 0 & -\sin\theta\cos\theta & 0 \\ \mathrm{ctg}\,\theta & 0 & -\sin^2\theta \\ 0 & 1 & 0 \end{pmatrix}.$$

There is an important circumstance to be noted here: The chiral currents $j_\mu^{(\mathrm{sph})}$ are on the order of $1/R$, and a current variation δj_μ makes contributions of zeroth and first orders in $1/R$. Equation (39), which is an integral equation of the Volterra type, can thus be used to construct a perturbation theory in $1/R$. Specifically, working in the standard way, we can write a solution of Eq. (39) in series form:

$$\delta X = \int du^\mu(P')\delta j_\mu(P')X^{(\mathrm{sph})}(P') + \frac{1}{R}\int du^\mu(P')\int du^\nu(P'')j_\mu^{(\mathrm{sph})} \times (P'')\delta j_\nu(P'')X^{(\mathrm{sph})}(P'') + \dots . \tag{41}$$

Solution (41) expresses $\delta X(P)$ in terms of known functions for the equilibrium surface ($X^{(\mathrm{sph})}$ and $j_\mu^{(\mathrm{sph})}$ for the case of a sphere) and in terms of the variation $\delta j_\mu = \nabla_\mu \delta v$. According to (41), any correlation function $\langle \delta X(P)\delta X(P')\rangle$, which is directly related to fluctuations in the surface shape can therefore be expressed in terms of the correlation functions $\langle \delta j_\mu \delta j_\nu\rangle$ calculated from an energy which depends on them [e.g., the Helfrich energy in (14)].

6. CONCLUSION

The main advantage of the method proposed in this paper for describing fluctuating surfaces (membranes) is that chiral field theory allows one to distinguish effects stemming from the nontrivial topology of a membrane and those stemming from the nonlinear interaction of fluctuations. The latter simplify significantly, since the choice of a movable frame of reference is in a sense equivalent to a diagonalization of the energy [the Helfrich energy in (14) is thus quadratic in the chiral currents].

In order to study physical phenomena, of course, an analysis of only those degrees of freedom which are associated with the membrane shape relaxation must be supplemented with a consideration of some other, harder degrees of freedom, which describe the physical phenomena of interest. However, again in this regard, chiral field theory can be useful for reaching an understanding of certain qualitative effects.

For example, a previous study[12] by Lebedev and one of the present authors established a $\omega^{5/3}$ law for the absorption of sound in a solution of nearly planar membranes. Deriving that law requires consideration of the interaction of sound waves with shape fluctuations of the membrane. In terms of diagrams, the incorporation of that interaction reduces to a summation of some ladder diagrams in which loops are governed by correlation functions of fluctuations in the membrane shape, while vertices are determined by the interaction of these fluctuations with the sound. If the membrane has a complicated geometry, then it is natural to assume (at least for long waves) that the vertices determined by the nonlinear terms in the hydrodynamic equations are insensitive to the geometry of the membrane. The entire dependence on the geometry is thus in the correlation functions, which are calculated in chiral field theory. This discussion leads to an $A\omega^{5/3}$ law for the fluctuation-related absorption of sound in solutions of membranes of arbitrary geometry. An important point is that the entire dependence on the geometry is in the coefficient A.

Universal expressions describing the dependence on the geometry of the membrane also demonstrate an analogy with the Brownian motion of a particle. To make this analogy more transparent, we need to introduce a quantity equivalent to the mean square displacement Δ of a particle in the course of a Brownian motion. For membranes it is natural to use as Δ^2 the integral characteristic

$$\Delta^2=\int du_1 du_2\,\mathrm{Tr}(\delta v\cdot\delta v).$$

Using the Langevin equation for δv we find the time evolution of Δ^2:

$$\tfrac{1}{2}\partial_t\langle\Delta^2\rangle=\gamma\langle E-E^{(0)}\rangle.$$

In deriving this relation we made use of the value $\langle\delta v y\rangle=0$, and we denoted the equilibrium value of the membrane energy by $E^{(0)}$.

In the quadratic approximation in the deviations of the membrane shape from equilibrium, we have, according to the definition of the average,

$$\langle E-E^{(0)}\rangle=\frac{1}{Z}\int D\delta v\,\mathrm{Tr}\left\{\frac{\partial^2 E^{(0)}}{\partial j_\mu\partial j_\nu}\nabla_\mu\delta v\nabla_\gamma\delta v\right\}\times\exp\left(-\frac{E-E^{(0)}}{T}\right),$$

where Z is a normalization factor. Using the change of variables

$$\delta v=T^{1/2}\delta\tilde{v},$$

we can relate $E-E^{(0)}$ to the reduced energy E_{red}:

$$E-E^{(0)}=T\int du_1 du_2\frac{\partial^2 E^{(0)}}{\partial j_\mu\partial j_\nu}\nabla_\mu\delta\tilde{v}\nabla_\nu\delta\tilde{v}=TE_{red}.$$

Correspondingly, we find

$$Z=T^{1/2}Z_{red},$$

and thus

$$\langle E-E^{(0)}\rangle=T\langle E-E^{(0)}\rangle_{red}/Z_{red}=TI,$$

where I has the meaning (and dimensionality) of a moment of inertia. The equation for Δ^2 thus becomes

$$\partial_t\Delta^2=2I\gamma T.$$

The solution of this equation is obvious:

$$\frac{\Delta^2}{t}=2I\gamma T.$$

This equation is the Einstein relation for rotational Brownian diffusion. All the dependence on the membrane geometry is in the moment of inertia I.

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