Equilibrium hydromagnetic configurations in a gravitational field

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A class of solutions of the MHD equations is considered, describing plasma equilibrium in a gravitational field corresponding to complete splitting of the system into autonomous equations for the magnetic flux and for the gravitational potential. Models of the quasistatic structures on the surface of the photosphere are considered as applications, and the possibility that strong magnetic fields exist inside the Sun is investigated. In the quasi-one-dimensional approximation analytical formulas are found which supply a relation among the temperature and magnetic-field profiles and the profile of the Wilson depression in sunspots.

1. INTRODUCTION

The theory of axisymmetric equilibrium plasma configurations in a gravitational field is considered. It is shown that the problem simplifies substantially if instead of the conventional "equation of state" closure $p=p(\rho)$, which relates the gas kinetic pressure to the density, we use an equation $\Pi=\Pi(\psi)$ for the dependence of the magnetic pressure on the magnetic flux function. The class of solutions derived in this manner is applied to study equilibrium configurations in a uniform and spherically symmetric gravitational field.

As examples we consider in the case of a uniform gravitational field models of quasistatic structures on the surface of the Sun's photosphere, such as sunspots and filigrees. Hence the effective radiating surface of the photosphere is an isobaric interface between a dense and a diffuse plasma. In particular, we are able in the quasi-one-dimensional approximation to obtain analytical formulas which relate the radial temperature and magnetic-field profiles to the depth of the Wilson depression in sunspots. In the case of a spherically symmetric gravitational field we use existing exact solutions to study the possibility that strong magnetic fields can exist inside the Sun and to learn how they influence the distribution of thermodynamic quantities.

2. EQUILIBRIUM AXISYMMETRIC MHD CONFIGURATIONS IN A GRAVITATIONAL FIELD

The equilibrium equation for a MHD plasma configuration in a gravitational field takes the form

$$\nabla p-\frac{1}{c}[\mathbf{jB}]+\rho\nabla\Phi=0, \tag{1}$$

where p is the pressure, ρ is the density, Φ is the gravitational potential, $\mathbf{B}$ is the magnetic field, and $\mathbf{j}=(c/4\pi)\mathrm{rot}\,\mathbf{B}$ is the current. It is necessary to supplement Eq. (1) with equations determining $\mathbf{B}$ and Φ.

In the absence of a magnetic field it follows from (1) that

$$p=p(\rho). \tag{2}$$

Introducing the enthalpy $w=\int dp/\rho$ and adding the Poisson equation, we find a complete system of equations for the case $\mathbf{B}=0$:

$$w(\rho)+\Phi(\rho)=\mathrm{const},\quad \Delta\Phi=4\pi G\rho, \tag{3}$$

where G is the gravitational constant.

In the axisymmetric case, where in terms of cylindrical coordinates r, φ, and z there is no dependence on the azimuthal angle φ, the meridional components of $\mathbf{B}$ and $\mathbf{j}$ are expressed in terms of the magnetic flux function $\psi(r,z)$ and the electric current $I(r,z)=rB_\varphi$:

$$\mathbf{B}=\frac{1}{r}[\nabla\psi,\mathbf{e}_\varphi]+B_\varphi\mathbf{e}_\varphi,\quad \mathbf{j}=\frac{c}{4\pi r}[\nabla I,\mathbf{e}_\varphi]+j_\varphi\mathbf{e}_\varphi. \tag{4}$$

Since we have $\mathbf{B}\nabla\psi=0$ and $\mathbf{j}\nabla\psi=0$, the magnetic field lines lie on magnetic surfaces $\psi=\mathrm{const}$ and the current lines lie on the $I=\mathrm{const}$ surfaces. From Eq. (1) it follows that the azimuthal component of the Lorentz force $\mathbf{F}=c^{-1}[\mathbf{jB}]$, vanishes, which yields the functional form $I=I(\psi)$. Consequently, the currents flow on the magnetic surfaces. Then the equilibrium equation in conjunction with the equations for ψ and Φ can be written in the form

$$\begin{gathered}\nabla p-\frac{j^*}{cr}\nabla\psi+\rho\nabla\Phi=0,\\ \Delta^*\psi+II'(\psi)=-\frac{4\pi}{c}rj^*,\quad \Delta\Phi=4\pi G\rho,\end{gathered} \tag{5}$$

where

$$j^*=j_\varphi-\frac{c}{4\pi r}II'(\psi),\quad \Delta^*=r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

If there is no gravitation it follows from the first of Eqs. (5) that

$$p=p(\psi),\quad j^*/cr=p'(\psi). \tag{6}$$

In this case the pressure is constant on the magnetic surfaces ψ=const, the density $\rho(r,z)$ is an arbitrary function, and the equilibrium magnetic field is described by the well-known Grad–Shafranov equation:[1]

$$\Delta^*\psi + II'(\psi) + 4\pi r^2 p'(\psi) = 0, \qquad (7)$$

which contains two arbitrary functions $I(\psi)$ and $p(\psi)$.

When the gravitational field is switched on in the equilibrium equation a new variable ρ appears, and consequently it becomes necessary to introduce an additional closure equation.

a) When the usual "equation of state" closure equation $p=p(\rho)$ is used, Eq. (5) reduces to a system of two coupled equations for ψ and ρ. Specifically, in this case it follows from the first of Eqs. (5) that

$$\nabla(\Phi + w) = \frac{j^*}{c\rho r}\nabla\psi, \quad \frac{j^*}{c\rho r} = -K'(\psi). \qquad (8)$$

Accordingly, from (5) we obtain the system of equations

$$\Delta^*\psi + II'(\psi) = 4\pi\rho r^2 K'(\psi),$$
$$\Delta w(\rho) + 4\pi G\rho = \Delta K(\psi), \qquad (9)$$

containing three arbitrary functions $w(\rho)$, $I(\psi)$, and $K(\psi)$.

Equations (9) reduce to a single equation for ψ in the case of "force-free" configurations with $K'(\psi)=0$, which were first treated by Chandrasekhar[2] and for an "incompressible" plasma (ρ=const) by Prendergast.[3] Trehan and Uberoi[4] studied the full system (9) for linear functions $I=\alpha\psi$ and $K=k\psi/4\pi$ and a polytropic dependence $p \propto \rho^{1+1/n}$.

b) We will next treat the class of equilibrium configurations which instead of (2) satisfy the auxiliary condition

$$j^*/cr = -\Pi'(\psi), \qquad (10)$$

which, in contrast to the first of Eqs. (8), does not contain the density ρ. Then from (5) we have

$$\nabla P + \rho\nabla\Phi = 0, \quad \Delta\Phi = 4\pi G\rho, \qquad (11)$$

where $P(\rho)=p+\Pi(\psi)$ is the total pressure and accordingly $\Pi(\psi)$ is the "magnetic pressure." The magnetic pressure introduced here can be either positive or negative. From (11) it follows that $P=P(\rho)$ and $\rho=\rho(\Phi)$. The last relation was suggested as a closure condition by Parker.[5] Molodenskii and Solov'ev[6] constructed a model of sunspots using a different auxiliary condition.

Equations (11) for P and ρ are independent of ψ and are completely identical to the equilibrium equations in the absence of a magnetic field. The construction of the generalized enthalpy $W=\int dP/\rho$ yields

$$W(\rho) + \Phi = \text{const}, \quad \Delta\Phi = 4\pi G\rho, \qquad (12)$$

entirely analogous to (3). The flux function ψ satisfies the equation

$$\Delta^*\psi + II'(\psi) - 4\pi r^2 \Pi'(\psi) = 0, \qquad (13)$$

in which ρ does not appear.

Thus, in this class of equilibrium configurations the total pressure $P(\rho)=p+\Pi(\psi)$ is determined by the gravitational field, and the magnetic field profile is determined by the magnetic pressure, which is specified by the arbitrary function $\Pi(\psi)$.

These results permit generalization to the case of steady axisymmetric rotating plasma configurations. Specifically, if the velocity $\mathbf{v}$ has only the azimuthal component $v=v_\varphi$, then we have $\mathbf{v}\nabla S\equiv 0$, (where S is the entropy), and as in the analysis given above it becomes necessary to have an auxiliary closure condition. Thus, if $v_\varphi = v_\varphi(r)$ holds then the force balance equation can be written in the form

$$\nabla p + \Pi'\nabla\psi + \rho\nabla\Phi^* = 0,$$

where we have written $\Phi^* = \Phi - \int\Omega^2 r dr$, $\Omega(r) = v_\varphi/r$ is the angular velocity of rotation, and $\Pi' \equiv -j^*/cr$. Taking the curl of this equation we find

$$[\nabla\Pi'\nabla\psi] + [\nabla\rho\nabla\Phi^*] = 0.$$

As before, introducing the auxiliary equation $\Pi' = \Pi'(\psi)$ we have $\rho=\rho(\Phi^*)$ and $P=P(\rho)$.

Hence it follows that the temperature profile on an isobaric surface of a rotating plasma can vary only due to the magnetic fields, since for $\Pi\equiv 0$ we have $p=p(\rho)$.

3. AXISYMMETRIC EQUILIBRIUM MHD CONFIGURATIONS IN A UNIFORM MAGNETIC FIELD

The equation $p=p(\rho)$ is inconsistent with the concept of sunspots as depressions formed by the emitting surface Σ of the photosphere on which p_Σ=const holds, under the influence of the magnetic field. Specifically, from the equation of state $p=(k/m)\rho T$, the temperature T_Σ on this surface is constant, which contradicts the observed strong temperature decrease in a sunspot.

Application of the present equilibrium theory to the case of a uniform gravitational field $\Phi = gz$, where g=const is the gravitational acceleration, yields the relations

$$\rho=\rho(z), \quad p + \Pi(\psi) = P(\rho), \quad W(\rho) + gz = \text{const}, \qquad (14)$$

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + II'(\psi) = 4\pi r^2\Pi'(\psi). \qquad (15)$$

We will take the plasma density above the emitting surface Σ to be negligibly small, and the pressure p_Σ to be constant. If we set $\Pi(\psi)=0$ in the region where there are no currents, then a solution of the problem for isolated configurations can be expressed in terms of the unperturbed background thermodynamic functions $\rho_e(z)$ and $p_e(z)$ in the peripheral region:

$$\rho=\rho_e(z), \quad p = p_e(z) - \Pi(\psi),$$
$$kT/m = [p_e(z) - \Pi(\psi)]/\rho_e(z). \qquad (16)$$

Here only the function ψ, which describes the magnetic field, is unknown.

If we assume a polytropic dependence $p = p_0(\rho/\rho_0)^{\gamma_0}$ near the unperturbed planar surface $z=0$ of

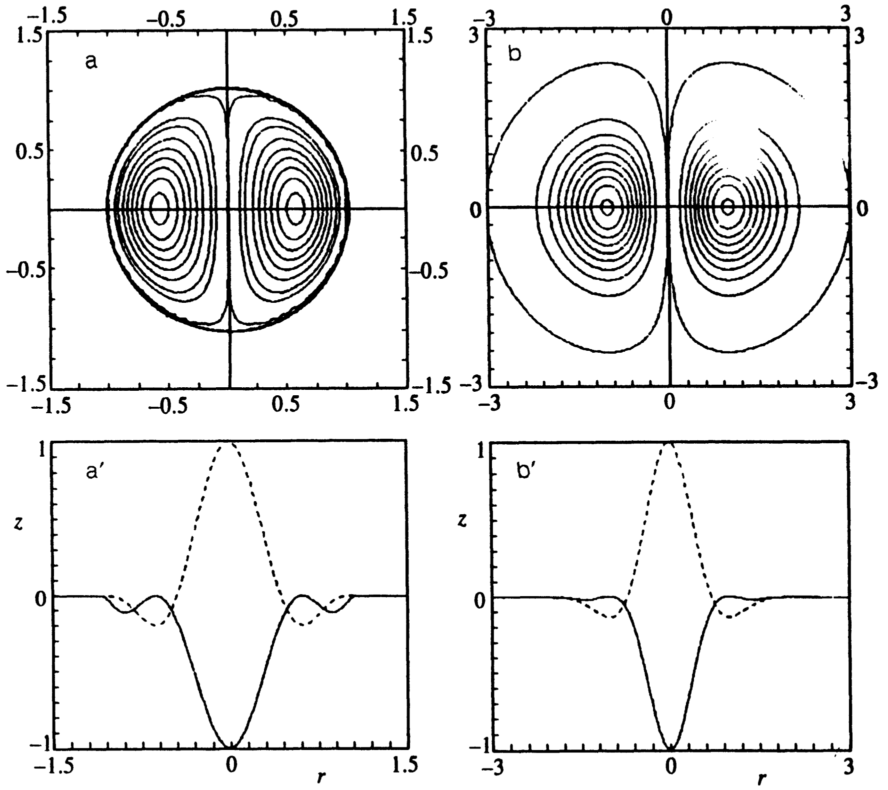

FIG. 1. Cross sections of the magnetic surfaces and the current surfaces that coincide with them: a) for the magnetic configurations (20) and (21) localized within the sphere $r^2+z^2=R^2$; b) the corresponding $z(r)$ profiles of the photosphere surface; a′) for case a according to Eqs. (18a) and (19); b′) for the continuous magnetic configurations (22) and (23) with magnetic fields that vanish in the limit $r^2+z^2\rightarrow\infty$, and also for case b according to Eqs. (18a) and (19). The solid traces correspond to filigrees and the broken traces to sunspots.

the photosphere, then the solution of the corresponding one-dimensional problem for equilibrium in the absence of currents yields

$$\rho_e(z)=\rho_0\left(\frac{T_e}{T_0}\right)^{1/(\gamma_0-1)},\quad p_e(z)=p_0\left(\frac{T_e}{T_0}\right)^{\gamma_0/(\gamma_0-1)},$$
$$\frac{T_e}{T_0}=1-\frac{\gamma_0-1}{\gamma_0}\frac{mgz}{kT_0}. \tag{17}$$

Assuming that the current density drops off rapidly in the limit $r\rightarrow\infty$, where the interface Σ goes over to the plane $z=0$, we find the following expressions for the temperature and the shape of the boundary surface $z=z(r)$:

$$T=T_0\left[1+\frac{\Pi(\psi)}{p_0}\right]^{-1/\gamma_0},$$
$$z(r)=-\frac{\gamma_0}{\gamma_0-1}\frac{p}{\rho_0 g}\left\{\left[1+\frac{\Pi(\psi)}{p_0}\right]^{(\gamma_0-1)/\gamma_0}-1\right\}. \tag{18}$$

In the incompressible plasma model ($\gamma_0\rightarrow\infty$) the expression for $z(r)$ simplifies considerably, but the effect of temperature variation on the interface drops out:

$$T=T_0,\quad z(r)=-\Pi(\psi)/\rho_0 g. \tag{18a}$$

The problem under consideration here, regarding the deformation of the surface of the photosphere, is one-dimensional for the density and gravitational potential and two-dimensional for the magnetic field. In order to determine the magnetic pressure $\Pi(r,z)$ we must solve Eq. (15) with the corresponding boundary conditions.

An approximate solution can be found in the quasi-one-dimensional model if we neglect the second derivative with respect to z in Eq. (15). Then we have

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}+II'(\psi)=4\pi r^2\Pi'(\psi). \tag{15a}$$

Multiplying (15a) by $(1/r^2)(\partial\psi/\partial r)$ and integrating yields

$$\Pi(r)=\frac{B_z^2}{8\pi}+\frac{1}{4\pi}\int_\infty^r\frac{B_\varphi}{r}d(rB_\varphi). \tag{19}$$

From this it follows that the sign of the magnetic pressure $\Pi(r)$ is determined by the ratio of the magnetic field components $B_\varphi(r)$ and $B_z(r)$, which can be regarded as prescribed functions of r. Equations (18) show that for $\Pi(r)>0$ we have $z(r)<0$, $T(r)<T_0$, while for $\Pi(r)<0$ we have $z(r)>0$, $T(r)>T_0$. Thus, in the first case the effective emitting surface of the photosphere has a depression, while in the second it has a bump with increased temperature. If the vertical magnetic field B_z dominates, so that $\Pi(r)>0$ holds, we obtain a sunspot, while if B_φ dominates we obtain filigrees.

Rough approximate models of bounded MHD equilibrium configurations can be constructed by describing the corresponding flux functions ψ and current I and calculating the magnetic pressure from the approximate one-dimensional formula (19):

1) A magnetic configuration with azimuthal currents and meridional magnetic field localized inside a sphere of radius R can be approximately described by

$$\psi=\frac{B_0 r^2}{2}\left(1-\frac{r^2+z^2}{R^2}\right),\quad \Pi=\frac{B_0}{8\pi}\left(1-\frac{r^2}{R^2}\right)^2\left(1-\frac{3r^2}{R^2}\right)^2; \tag{20}$$

2) An analogous configuration with poloidal currents and azimuthal magnetic field can be constructed by setting $I=\alpha\psi$:

$$I=\frac{\alpha B_0 r^2}{2}\left(1-\frac{r^2+z^2}{R^2}\right),$$
$$\Pi=-\frac{\alpha^2 R^2 B_0^2}{160\pi}\left(1-\frac{r^2}{R^2}\right)^4\left(1-\frac{6r^2}{R^2}\right). \tag{21}$$

Figure 1a,a′ displays the shape of the photospheric surface determined by Eqs. (20) and (21) for $\gamma_0=\infty$. The magnetic field outside the sphere $r^2+z^2=R^2$ vanishes, while the function Π together with its derivatives is continuous on the matching surface $r=R$, $z=0$.

Let us now consider similar configurations described by continuous analytical functions whose currents and fields go to zero in the limit $r^2+z^2\to\infty$.

3) A magnetic configuration with azimuthal currents and poloidal magnetic field which falls off monotonically at infinity, is given by the functions

$$\psi=\frac{B_0 r^2}{2}\exp[-k^2(r^2+z^2)],$$
$$\Pi=\frac{B_0^2}{8\pi}(1-k^2r^2)\exp(-2k^2r^2); \tag{22}$$

4) If, on the other hand, the currents are poloidal, then the magnetic field is azimuthal. Setting $I=\alpha\psi$ we obtain

$$I=\frac{\alpha B_0 r^2}{2}\exp[-k^2(r^2+z^2)],$$
$$\Pi=-\frac{\alpha^2 B_0^2}{64\pi k^2}(1-2k^2r^2)\exp(-2k^2r^2). \tag{23}$$

The deformations of the photosphere surface described by Eqs. (22) and (23) for $\gamma_0=\infty$ are shown in Fig. 1b,b′.

5) We conclude by presenting a configuration in which the vertical and azimuthal magnetic fields and the plane of symmetry $z=0$ are described by Bessel functions:

$$B_z=B_0J_0(kr),\quad B_\varphi=b_\varphi J_1(kr). \tag{24}$$

Then from (19) the magnetic pressure is given by

$$\Pi=\frac{1}{8\pi}(B_0^2-b_\varphi^2)[J_0^2(kr)-J_0^2(kR)], \tag{25}$$

where the sign is determined by the difference between the squares of the vertical and azimuthal fields. Setting $\Pi(r)=0$ for $r>R$ we see that the function $\Pi(r)$ and its derivative are continuous if $\Pi'(R)=0$, i.e., $kR=x_{0n}$ or $kR=x_{1n}$, where x_{0n} and x_{1n} are the roots of the Bessel functions $J_0(kr)$ and $J_1(kr)$. In the case $kR=x_{1n}$ the total current $J(R)=cI(R)/2$ vanishes. The shape of the photosphere surface for a two-pump profile $\Pi(R)=0$ with $kR=x_{12}$, $b_\varphi^2>B_0^2$, $\gamma_0=\infty$ is shown in Fig. 2. In this case

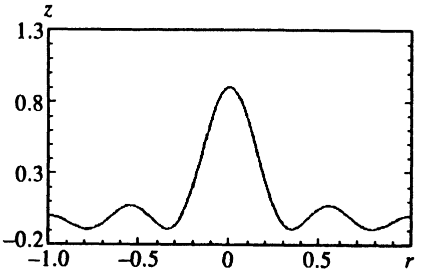

FIG. 2. Profile of the photosphere surface (24) described by Bessel functions in the case where the azimuthal field dominates according to Eqs. (25).

the entire current flowing out of the circle $0<r<a$ returns to the annulus $a<r<R$. The direction of the current can of course be reversed.

4. THE MAGNETIC FIELD DEPENDENCE OF THE SURFACE PROFILE AND THERMODYNAMIC PARAMETERS OF THE PHOTOSPHERE

The equilibrium configuration models described above contain the dimensionless parameter γ_0. The resulting formulas agree well with the familiar empirical behavior for the variables of sunspots when we take $\gamma_0=3.73$. A good approximate average value for γ_0 can also be found with the polytropic approximation $p=\rho^{\gamma_0}$, corresponding to the functional dependence $p=p(\rho)$ of the Harvard–Smithsonian standard model of the atmosphere of the quiet Sun.[7] The anomalously large value of the polytropic index is probably related to the presence of the magnetic field in the photosphere.

The affect of the magnetic field on the distribution of equilibrium thermodynamic variables of the photosphere can be characterized by the effective polytropic index $\gamma_0^* = d\ln p/d\ln\rho$. According to (16) the value of γ_0^* for autonomous equilibrium MHD configurations in a gravitational field is expressed by

$$\gamma_0^*(z)=\frac{\rho_e}{\rho_e'}\frac{p_e'}{p_e-\Pi}=\frac{\gamma_0}{1-\Pi/p_e}=\gamma_0\left(1+\frac{\Pi}{p}\right). \tag{26}$$

The magnetic field dependence is given here by the function $\Pi(\psi)$.

In the case $B_\varphi=0$, $\Pi=B_z^2/8\pi$ (a sunspot) the central temperature T_{max}, the depth $-z_{max}$ of the depression, and the polytropic index γ_0^* can be expressed according to (18), (19), and (26) in terms of the central magnetic field B_0 by

$$T_{max}=T_0\left(1+\frac{B_0^2}{8\pi p_0}\right)^{-1/\gamma_0},$$
$$z_{max}=-\frac{\gamma_0}{\gamma_0-1}\frac{p_0}{\rho_0 g}\left[\left(1+\frac{B_0^2}{8\pi p_0}\right)^{(\gamma_0-1)/\gamma_0}-1\right], \tag{27}$$
$$\gamma_0^*=\gamma_0\left(1+\frac{B_0^2}{8\pi p_0}\right).$$

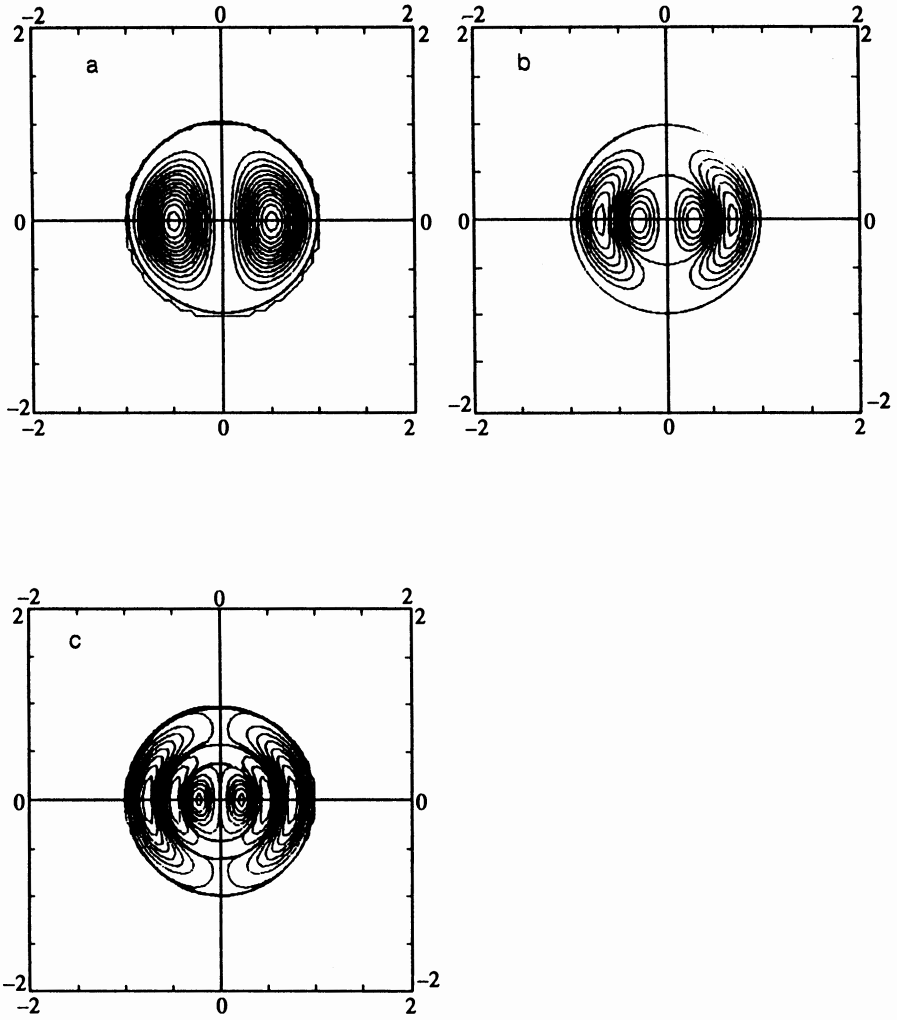

FIG. 3. The cross sections of the magnetic surfaces and the current surfaces that coincide with them for the equilibrium configuration (36) with external magnetic field equal to zero for the first three radial modes $z_{2k}=5.764$ (a), 9.095 (b), 12.32 (c).

In the opposite limit, i.e, $B_z=0$ (filigree), bounded by the configuration (25), we find for the magnetic pressure at

$$\Pi_0=-\frac{B_\varphi^2(a)}{8\pi}\frac{1-J_0^2(kR)}{J_1^2(ka)}\equiv-\lambda\frac{B_\varphi^2(a)}{8\pi}.$$

For the "two-pump" configuration shown in Fig. 3 we have $ka=x_{01}\approx 2.41$, $kR=x_{12}\approx 7.01$. For the values of $T_{\max}$, $z_{\max}$, and γ_0^* at the center we have

$$T_{\max}=T_0\left[1-\frac{\lambda B_\varphi^2(a)}{8\pi p_0}\right]^{-1/\gamma_0},$$

$$z_{\max}=-\frac{\gamma_0}{\gamma_0-1}\frac{p_0}{\rho_0 g}\left\{\left[1-\frac{\lambda B_\varphi^2(a)}{8\pi p_0}\right]^{(\gamma_0-1)/\gamma_0}-1\right\}, \tag{27a}$$

$$\gamma_0^*=\gamma_0\left[1-\frac{\lambda B_\varphi^2(a)}{8\pi p_0}\right].$$

Here ρ_0, p_0, T_0, and γ_0 are the parameters of the unperturbed photosphere near the effective radiating layer. In Tables I and II the values of $T_{\max}$, $z_{\max}$, and γ_0^* are given as functions of the magnetic field for the cases $B_\varphi=0$ and $B_z=0$, evaluated according to Eqs. (27) and (27a). We used as the unperturbed parameters of the Sun's photosphere $T_0=5920$ K, $p_0=1.023\cdot 10^5$ dyn/cm^2, $\rho_0=2.884\cdot 10^{-7}$ g/cm^3, $m=2.29\cdot 10^{-24}$ g, $g=2.74\cdot 10^4$ cm/s^2, $\gamma_0=3.73$.

Tables I and II show that the value of the polytropic index γ_0^* increases as a function of the magnetic field B_z, so that the plasma becomes increasingly incompressible, and decreases as a function of B_φ.

5. EQUILIBRIUM MHD CONFIGURATIONS IN A SPHERICALLY SYMMETRIC GRAVITATIONAL FIELD

The class of axisymmetric equilibria with $\Pi=\Pi(\psi)$ in Sec. 1 admits MHD configurations maintained in equilibrium by a spherically symmetric gravitational field. The quiet Sun, which is very close to having spherical shape, probably belongs to this class.

In spherical coordinates r, θ, φ equilibrium configurations of this class are described by the equations

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}+II'(\psi)=4\pi\Pi'(\psi)r^2\sin^2\theta, \tag{28}$$

TABLE I.

B_{z0}, kG	T_{max}, kK	$-z_{max}$, km	γ_0^*
0	5.92	0	3.73
0.5	5.77	12.4	4.09
1.0	5.42	48	5.18
1.5	5.00	103	6.99
2.0	4.60	175	9.53
2.5	4.25	259	12.8
3.0	3.96	355	16.8
3.5	3.70	460	21.5
4.0	3.48	575	26.9
4.5	3.30	697	33.1
5.0	3.13	827	40.0
5.5	2.99	964	47.6
6.0	2.86	1110	56.0
6.5	2.75	1260	65.0
7.0	2.65	1410	74.8

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\Phi}{dr}=4\pi G\rho,\quad W(\rho)+\Phi=\text{const},$$
$$W(\rho)=\int\frac{dP}{\rho},\quad P(\rho)=p+\Pi(\psi). \tag{29}$$

It is evident that Eq. (28), which determines the magnetic field, is independent of Eqs. (29) for the profile of the density ρ, the gravitational potential $\Phi(\rho)$, and the total pressure $P(\rho)$. Eliminating the gravitational potential from (29) we find

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dW(\rho)}{dr}+4\pi G\rho=0. \tag{30}$$

In particular, for a polytropic dependence $P\propto\rho^{\gamma_0}$, $\gamma_0=1+1/n=\text{const}$ it follows from (30) that we have an equation for a generalized Emden equilibrium in which the total pressure $P(\rho)=p+\Pi(\psi)$ appears in place of the gas pressure p. The spherical components of the magnetic field are expressed in terms of ψ and $I(\psi)$ by

$$B_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\quad B_\theta=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},\quad B_\varphi=\frac{I(\psi)}{r\sin\theta}. \tag{31}$$

TABLE II.

$B_{\varphi a}$, kG	T_{max}, kK	$-z_{max}$, km	γ_0^*
0	5.92	0	3.73
0.1	5.92	1.71	3.68
0.2	6.01	6.88	3.53
0.3	6.13	15.6	3.29
0.4	6.31	28.2	2.94
0.5	6.58	44.9	2.50
0.6	7.03	66.5	1.96
0.7	7.82	94.2	1.32
0.8	9.74	131.5	0.58
0.85	13.4	158	0.18
0.874	∞	175	0

For linear functions $I(\psi)=\alpha\psi$ and $I(\psi)=\alpha\psi$ and $\Pi(\psi)=\Pi_0+\Pi_0'\psi$, Eq. (28) becomes a linear equation with a given right-hand side:

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}+\alpha^2\psi=4\pi\Pi_0'r^2\sin^2\theta. \tag{32}$$

The solution of this equation which is regular at $r=0$ can be expressed in terms of a series[2,3]

$$\psi=\frac{4\pi\Pi_0'}{\alpha^2}\left\{r^2+\sum_n a_nf_n(\alpha r)P_n'(\cos\theta)\right\}\sin^2\theta, \tag{33}$$

where we have written $f_n(z)=\sqrt{z}J_{n+1/2}(z)$, $J_k(z)$ are Bessel functions, and $P_n'(x)$ are derivatives of the Legendre polynomials. If there are no currents in the external region, the solution of Eq. (32) which is regular for $r\to\infty$ takes the form

$$\psi_e=\sum_n\frac{b_n}{r^n}P_n'(\cos\theta)\sin^2\theta. \tag{34}$$

The solution of Eq. (32) obtained by matching the harmonics (33) and (34) at the sphere $r=R$ is expressed by the equations

$$\psi=\left\{K\left[r^2-\frac{3Rf_1(\alpha r)}{\alpha f_0}\right]+a_2f_2(\alpha r)P_2'(\cos\theta)+\ldots+a_nf_n(\alpha r)P_n'(\cos\theta)+\ldots\right\}\sin^2\theta,$$
$$\psi_e=\left\{-K\left[\frac{R^3f_2}{f_0r}\right]+\frac{a_2R^2}{r^2}f_2P_2'(\cos\theta)+\ldots+\frac{a_nR^n}{r^n}f_nP_n'(\cos\theta)+\ldots\right\}\sin^2\theta. \tag{35}$$

Here $K=4\pi\Pi_0'/\alpha^2$, $f_n\equiv f_n(\alpha R)$. The first harmonic $n=1$ is continuous at $r=R$ together with its normal derivative, while from the requirement of continuity of the normal derivatives for $n\geqslant2$ it follows that for the case of a nonvanishing external field we have $\alpha_n\neq0$ only for $f_{n-1}(\alpha R)=0$. The constant K is related to the central magnetic field B_0 by

$$K=\frac{B_0}{2(1-\alpha R/\sin\alpha R)}.$$

The terms of the series (35) for ψ and ψ_e are expressible in terms of elementary functions.

If we restrict ourselves to the fundamental mode $n=1$, we have

$$\psi=K\left[r^2-\frac{3Rf_1(\alpha r)}{\alpha\sin\alpha R}\right]\sin^2\theta,\quad\psi_e=-K\frac{R^3f_2(\alpha R)}{r\cdot\sin\alpha R}\sin^2\theta. \tag{36}$$

A notable property of this solution is that the external magnetic field vanishes when αR coincides with one of the roots z_{2k} of the function $f_2(z)$. Then the sphere $r=R$ becomes a magnetic surface $\psi=0$, on which $\mathbf{B}=0$ holds. The solution (36) adequately describes the quiet Sun, which is an almost perfect sphere with a negligible external mag-

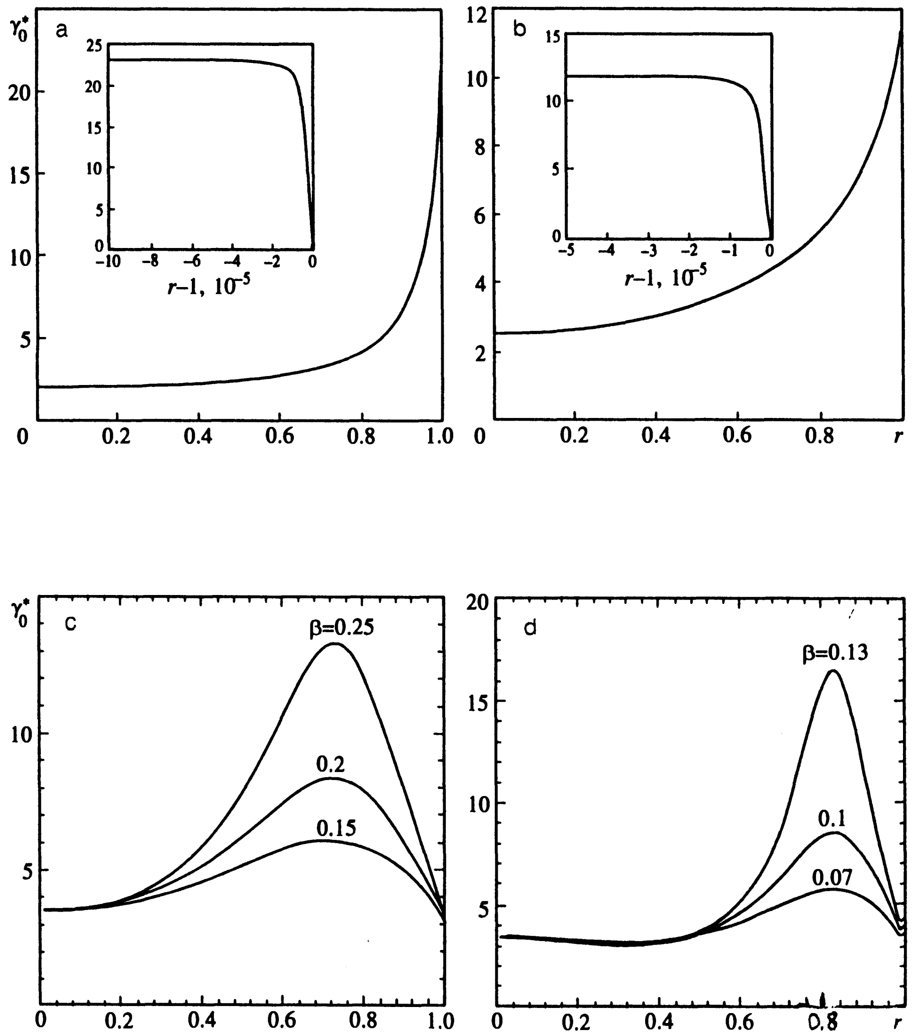

FIG. 4. Radial dependence of the polytropic index $\gamma_0^*(r)$ in an equilibrium gravitating sphere in the presence of an internal magnetic field for density profiles $N=3$ (a), 2 (b), and 4 (c,d) from Table III, constructed according to Eq. (43). The magnetic field profiles correspond to the first $z_{21}=5.764$ (a–c) and second $z_{22}=9.095$ (d) radial modes (36). In cases c and d as β increases toward β_{cr} the curvature of the function $\gamma_0^*(r)$ increases. The inserts in Figs. a and b show the behavior of the functions near the boundary $R=1$.

netic field. The meridional cross sections of the magnetic surface $\psi(r,\theta)=\text{const}$ for the configurations determined by the first few roots $z_{2k}=5.7635$, 9.0950, 12.3224 (Ref. 3) are shown in Fig. 3a–c.

We can represent the functions ψ and $\Pi=\Pi'\psi$ in the form

$$\psi=K\Psi(r)\sin^2\theta,\quad \Pi=(\alpha^2K^2/4\pi)\Psi(r)\sin^2\theta,$$
$$\Psi(r)=r^2-\frac{3R}{\alpha\sin\alpha R}\left(\frac{\sin\alpha r}{\alpha r}-\cos\alpha r\right). \tag{37}$$

The density ρ and total pressure $P(\rho)$ of the equilibrium configuration are determined according to (29) by the gravitational field, while the gas pressure is equal to

$$p=P(\rho)-\beta p_0\Psi(r)\sin^2\theta, \tag{38}$$

where $\beta=(\alpha RK)^2/4\pi p_0$ is the characteristic ratio of the magnetic pressure to the total pressure.

The magnetic pressure increases as a function of the magnetic field, while the gas kinetic pressure decreases. At a critical value of the dimensionless parameter $\beta=\beta_{cr}$ the gas kinetic pressure vanishes at some point $r=r_{cr}$, $\theta=\pi/2$. Consequently, the condition for the existence of the corresponding equilibrium configuration is that the inequality $\beta\leqslant\beta_{cr}$ hold.

On the other hand, Fermi and Chandrasekhar[8] derived a stability criterion which imposes a limit $\eta=\mathfrak{M}/|\Omega|<1$, on the ratio of the magnetic energy $\mathfrak{M}=1/8\pi\int B^2d\tau$, to the gravitational energy Ω, $\Omega=1/2\int\rho\Phi d\tau$. For the equilibrium configuration (37) with $\alpha R=z_{2k}$ these energies are given by[3]

$$\mathfrak{M}=\frac{2}{5}K^2\alpha^2R^5. \tag{39}$$

The gravitational energy for Emden equilibria ($P\propto\rho^{\gamma_0}$) is expressed in terms of the mass M and radius R of a star by the formula

$$\Omega=-\frac{\gamma_0-1}{5\gamma_0-6}\frac{3GM^2}{R}. \tag{40}$$

It follows that the Fermi–Chandrasekhar limit for an Emden equilibrium configuration confined by the magnetic field (37) can be written in the form

$$\eta=\frac{2}{15}\frac{5\gamma_0-6}{\gamma_0-1}\frac{K^2\alpha^2R^6}{GM^2}<1. \tag{41}$$

In addition to the criterion $\beta<\beta_{cr}$, there is another restriction on the equilibrium for the magnetic field (37): $\gamma_0\geqslant2$. Specifically, the magnetic pressure $\Pi(r,0)$ near

TABLE III.

N	ρ	P	λ	μ	ν	β/ν	$\beta_{cr}^{(1)}$	$\beta_{cr}^{(2)}$
1	ρ_0	$p_0\left(1-\frac{r^2}{R^2}\right)$	$\frac{3}{8\pi}$	$\frac{2}{3}$	$\frac{2}{3}$	1	0.5	0.35
2	$\rho_0\left(1-\frac{r^2}{R^2}\right)$	$p_0\left(1-\frac{r^2}{R^2}\right)^2\left(1-\frac{r^2}{2R^2}\right)$	$\frac{15}{16\pi}$	$\frac{14}{25}$	$\frac{4}{15}$	0.48	0.12	0.049
3	$\rho_0\frac{\sin(\pi r/R)}{\pi r/R}$	$p_0\left(\frac{\sin(\pi r/R)}{\pi r/R}\right)^2$	$\frac{\pi}{8}$	$\frac{8}{15}$	$\frac{2}{\pi^2}$	0.38	0.062	0.024
4	$\rho_0\sqrt{1-r^2/R^2}$ $\left(\frac{r}{R}=\sin\varphi\right)$	$\frac{8p_0}{16-\pi^2}\left[\frac{\varphi^2}{2}+\varphi\,\mathrm{ctg}\,\varphi+\frac{\cos^2\varphi}{2}(1+\cos^2\varphi)-\pi^2/8\right]$	$\frac{16-\pi^2}{\pi^3}$	0.61	$\frac{\pi^2/4}{16-\pi^2}$	0.66	0.33	0.16

$\alpha R = z_{2k}$ approaches zero for $\delta r = R - r \to 0$ like $(\delta r)^2$: $\Pi \propto (\delta r)^2$. Hence it follows that equilibrium is possible only under the condition $P \propto (\delta r)^m$, where $m \leqslant 2$. If we have $\rho \propto (\delta r)^l$, then $P \propto (\delta r)^{l+1}$ holds and if P and ρ are related near $r=R$ according to the polytropic law $P \propto \rho^{\gamma_0}$, we find $\gamma_0 = 1 + 1/l \geqslant 2$. The Emden equilibrium ($\gamma_0 = \text{const}$) can therefore support the magnetic field (37) only if $\gamma_0 \geqslant 2$ holds.

The central gas kinetic pressure p_0 and the dimensionless parameters η and β can be expressed in terms of the mass M and the radius R of a star by the formulas

$$p_0 = \lambda GM^2/R^4, \quad \eta = \mu K^2\alpha^2 R^6/GM^2, \quad \beta = \nu K^2 \alpha^2 R^6/GM^2, \tag{42}$$

where λ, μ, and ν are new dimensionless parameters. The results of calculating the parameters λ, μ, and ν, along with the critical parameters $\beta_{cr}^{(1)}$ and $\beta_{cr}^{(2)}$ for the first and second modes $\alpha R = 5.764$ and $\alpha R = 9.094$ are plotted respectively in Figs. 3a and 3b, and are shown in Table III for different equilibrium profiles of the density ρ and the total pressure $P(\rho)$.

The data for $N=1$ shown in Table III correspond to the "incompressible" equilibrium configuration treated by Prendergast,[3] where $P \propto \delta r$ for $r \to R$. In the $N=2$, 3, configurations the total pressure behaves as $P \propto (\delta r)^2$, similar to the magnetic pressure $\Pi \propto (\delta r)^2$ in the limit $r \to R$. The $N=1$, 3 configurations belong to the class of Emden equilibria with $\gamma_0 = \infty$ and $\gamma_0 = 2$. In the $N=2$, 4 configurations the value of γ_0 is variable and changes from $\gamma_0(0) = 2.5$ to $\gamma_0(R) = 2$ for $N=2$ and from $\gamma_0(0) = 3.48$ to $\gamma_0(R) = 3$ for $N=4$. The β/η column shows that the restrictions on equilibrium are more severe than the Fermi–Chandrasekhar stability limit.

The equilibrium restrictions on the central magnetic field for a uniform ($N=1$) and Emden ($N=3$) gravitating gaseous spheres with the solar mass and radius are $B_0 \leqslant 4 \cdot 10^8$ G and $B_0 \leqslant 2.5 \cdot 10^8$ G respectively for the first radial mode and $B_0 \leqslant 4.5 \cdot 10^8$ G and $B_0 \leqslant 2.1 \cdot 10^8$ G for the second mode. These inequalities show that it is possible for strong magnetic fields to exist in the internal regions of the Sun, even though there is almost no magnetic field on the surface of the quiet Sun.

The models of equilibrium self-gravitating plasma spheres which we have considered allow us to at least qualitatively explain the anomalous behavior of the polytropic index $\gamma_0 = d \ln p / d \ln \rho$ as a function of distance into the solar photosphere. As shown by the analysis of the Harvard–Smithsonian model of the quiet-Sun atmosphere,[7] the quantity γ_0 increases considerably as one passes to a depth of order ~ 100 km below the photosphere boundary, which in all probability is a consequence of the internal magnetic field.

Specifically, when an internal magnetic field is present the distribution of the equilibrium thermodynamic parameters characterized by an adiabatic index which from (38) is equal to

$$\gamma_0^*(r,\theta) = \frac{P'(r) - \beta p_0 \Psi'(r) \sin^2\theta}{P(r) - \beta p_0 \Psi(r) \sin^2\theta} \frac{\rho(r)}{\rho'(r)}. \tag{43}$$

From this it can be seen that for these equilibria the effect of the magnetic field is greatest in the equatorial plane $\theta = \pi/2$ and vanishes at the poles $\theta = 0$. At $\beta = \beta_{cr}$ the gas kinetic pressure which appears in the denominator vanishes at some point $r = r_{cr}$, and consequently we have $\gamma_0^*(r_{cr}, \pi/2) \to \infty$. For $\beta \leqslant \beta_{cr}$ the quantity γ_0^* also increases as r approaches r_{cr} from the direction of $r=R$ and then drops off in the limit $r \to 0$.

For configurations in which $P(r)$ approaches zero in the limit $r \to R$ as $(\delta r)^2$, i.e., in the same way as $\Psi(r)$, the point $r = r_{cr}$ is near the boundary $r=R$, and consequently $\gamma_0^*(r)$ increases rapidly with increasing $\delta r = R - r$. But if $P(r)$ increases faster than $(\delta r)^2$, then the critical radius moves away into the sphere. Figure 4 shows plots of the function $\gamma_0^* = \gamma_0^*(r)$ for $\theta = \pi/2$ in equilibrium configurations with the different $\rho(r)$ and $P(r)$ profiles given in Table III. It can be seen that the rapid increase of $\gamma_0^*(r)$ as $R - r$ increases may be at least qualitatively explained by the effect of the internal magnetic field of the equilibrium gravitating sphere.

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[1] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, Oxford (1960).
[2] S. Chandrasekhar, Proc. Natl. Acad Sci. **42**, 1 (1956).
[3] K. H. Prendergast, Astrophys. J. **123**, 498 (1956).
[4] S. K. Trehan and M. S. Uberoi, Astrophys. J. **175**, 161 (1972).
[5] E. N. Parker, *Cosmical Magnetic Fields*, Clarendon Press, Oxford (1979).
[6] M. M. Molodenskii and L. S. Solov'ev, Astrofiz. Zh. **70**, 156 (1993) [Astron. Repts. **37**, 83 (1993)].
[7] C. W. Allen, *Astrophysical Quantities*, 3rd ed., Athlone, London (1976).
[8] S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 116 (1953).

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