Equation for the rotational instability due to convective turbulence and the Coriolis force

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The derivation of an averaged equation describing the production of large-scale hydrodynamic structures in an anisotropic turbulent medium in a Coriolis force field is discussed. The small-scale turbulence is assumed to be the result of the local convective instability resulting from an internal source of heat in the fluid. The calculation is performed by means of a statistical average over the small-scale turbulent eddies, assuming that the Reynolds number is low. The tensor structure of the equation for the average velocity on the large scale is found to be identical with that of the equation derived in a model for the production of large-scale hydrodynamic structures due to spiral turbulence.

1. INTRODUCTION

There have recently been a number of treatments of the processes by which large-scale hydrodynamic structures are produced under the influence of spiral turbulence.[1–6] The concept of spiral turbulence enables us to derive a system of averaged equations which describe the creation of large-scale vortices like tropical cyclones. This model is characterized by coupling between the toroidal and poloidal velocity field lines, which is responsible for the creation of large-scale vortices. The spiral property, i.e., the breaking of the mirror symmetry of the turbulence, implicitly assumes that some convection process is superposed on the Coriolis force.[7] This concept is so effective in producing a state in which structures are created that there is no need for explicitly invoking such obviously important factors as the Coriolis force and convection on the energy-containing scales. If we interpret the large-scale vortices obtained in this model as, e.g., the initial stage in the formation of tropical cyclones, it is necessary to assume that the concept of spiral turbulence contains within itself or parametrizes all of these factors.

Berezin and Zhukov[8] attempted to construct a model for the production of large-scale hydrodynamic structures directly through the action of the Coriolis force. They wound up concluding that it is impossible to construct such a model for an incompressible fluid, but in the case of a compressible fluid they derived equations for the large-scale motions superposed on a strong vertical variation, which they proposed to study numerically. Thus, the question of whether it is possible to construct a model for the generation of large-scale hydrodynamic structures in an incompressible fluid directly superposed on the small-scale convection in the Coriolis force field remains open.

The turbulence that develops in the region of a tropical depression is convective, and according to most models it is maintained by the release of the stored heat of vapor condensation. This process corresponds most closely to instability in the presence of an internal heat source in the fluid. The temperature profile under such conditions assumes the form of a quadratic parabola whose curvature is proportional to the rate at which the heating takes place. According to the basic idea of spiral turbulence,[7,9] the procedure for calculating the Reynolds stress by statistical averaging over the small-scale turbulent eddies including the Coriolis force and the curvature of the temperature profile in a convective cell should yield averaged equations analogous to those for the production of large-scale vortices under the action of spiral turbulence.[1–6]

2. FORMULATION OF THE PROBLEM

The process by which spiral turbulence develops can be understood most naturally as a rotation in one direction or the other of a fluid element floating in or immersed in a fluid with a convective cell due to the Coriolis force.[7] Thus it would seem that when we treat the problem of convection including the Coriolis force and averaging over the fine scales superposed on nonspiral turbulence we should be able to find the Reynolds stress due to large-scale rotational instability. It turns out, however, that including the vertical variation in the simplest problem of convection on the small scale with a constant vertical temperature gradient is inadequate to obtain the corresponding Reynolds stress. Atmospheric condensation, which proceeds on account of the internal release of the heat of vaporization, encourages us to take into account the curvature of the vertical temperature gradient. Hence the convective instability that takes place under these conditions is naturally regarded as the source of the turbulent motion. In this situation the curvature of the temperature profile should play a most important role.

When modeling the convective turbulence we assumed that its characteristic size is less than the dimensions of the convective cell. The local convection process is assumed to take place as a result of the local heat release in a layer of thickness λ, above and below which there is a slight stable stratification. We prescribe the temperature profile in the convective layer as a Taylor expansion in the vertical coordinate z, assuming that the curvature of the profile is weak:

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$$T_0(z)=\text{const}-Az-\frac{B}{2}z^2+\ldots,\quad A,B>0,\quad A\gg\frac{B}{2}\lambda. \qquad (1)$$

The question of convection under these conditions has been treated in Ref. 10. The critical Rayleigh number in such a system turns out to be much less than the Rayleigh number for convection in a layer of fluid with boundaries, and the cells are stretched out in the vertical direction due to convective penetration.

We will study the derivation of equations for large-scale instability in an incompressible fluid under the action of small-scale convective turbulence superposed on the Coriolis force using the following system of equations:[11]

$$\frac{\partial V_i}{\partial t}-\nu\Delta V_i+V_k\nabla_k V_i+\frac{\nabla_i P}{\rho}+ge_i+2\Omega\epsilon_{ijs}e_jV_s=0, \qquad (2)$$

$$\frac{\partial T}{\partial t}-\chi\Delta T+V_k\nabla_k T=Q, \qquad (3)$$

$$\nabla_k V_k=0,\quad \nabla_k=\frac{\partial}{\partial x_k}, \qquad (4)$$

$$\rho=\rho_0(1-\beta T). \qquad (5)$$

Here ν is the kinematic viscosity, χ is the thermal conductivity (the Prandtl number is assumed equal to unity, Pr $=\nu/\chi=1$), $\beta=-1/\rho[\partial\rho/\partial T]_P$ is the thermal expansion coefficient, T is the temperature, and $\mathbf{e}$ is a unit vector directed vertically upward.

We assume that at some time a layer with unstable stratification described by the temperature profile (1), superposed on the weak stable stratification, develops in the fluid under the influence of the internal heat source Q. We will analyze the instability for perturbations of the velocity $V(t,x)$, temperature $\theta(t,x)[T=T_0(z)+\theta(t,x)]$, and pressure $P_1(t,x)[P=P_0(z)+P_1(t,x)$, grad $P_0(z)=\rho_0 ge(1-\beta T_0(z))]$ superposed on the basic state of $T_0(z)$ and $P_0(z)$ resulting from the heating. The system of equations for the perturbations in the Boussinesq approximation takes the form

$$\frac{\partial V_i}{\partial t}-\nu\Delta V_i+V_k\nabla_k V_i+\frac{\nabla_i P_1}{\rho_0}-ge_i\beta\theta+2\Omega\epsilon_{isj}e_sV_j=F_i, \qquad (6)$$

$$\frac{\partial\theta}{\partial t}-\chi\Delta\theta-(V_ke_k)(A+Bz)+V_k\nabla_k\theta=0. \qquad (7)$$

The Navier–Stokes equation (6) contains an additional external random force $F_i(\langle F_i\rangle=0)$ due to the fine-scale turbulence in the fluid, whose parameters are assumed to be consistent with the parameters of the convection process. This means, in particular, that the homogeneous and steady turbulence should be regarded as anisotropic. As is well known, the correlation of the anisotropic random velocity field in the Fourier representation takes the form (see, e.g., Ref. 7)

$$T_{ij}(t_1-t_2,\mathbf{k})=G(t_1-t_2,\mathbf{k}_\perp,\mathbf{ek})\left\{\left(\delta_{ij}-(1-\mu)\frac{k_ik_j}{\mathbf{k}^2}\right)\right.$$

$$\left.+\mu\left(e_ie_j\frac{k^2}{(\mathbf{ek})^2}-\frac{e_ik_j+e_jk_i}{(\mathbf{ek})}\right)\right\}, \qquad (8)$$

where μ is a parameter which characterizes the degree of anisotropy $(0\leqslant\mu\leqslant1)$.

In this work we restrict ourselves to deriving the Reynolds stress corresponding to the large-scale instability in the presence of the Coriolis force and small-scale convective penetration superposed on the prescribed nonspiral anisotropic turbulence with the simplest properties. The Reynolds stress due to the turbulent correction to the viscosity coefficient was studied by Krauze and Rüdiger.[12] Consequently, the characteristic scale of the turbulence is on the order of the thickness λ of the stably stratified layer, and its strength u_T is determined by the power of the external heat source.

This model presupposes that the conditions for the occurrence of small-scale convection are satisfied, which means that the time-independent part of the small-scale linear operator vanishes. The weak convective instability growth rate $\gamma\ll\nu/\lambda^2$ is assumed to be consistent with the comparatively large correlation time $\tau=1/\gamma$ of the turbulence. Practically speaking this means that the convective motion is regarded as turbulent. The presence of a variation in the temperature gradient in the problem under these conditions has a very strong effect. The role of even weak turbulence is therefore very important in determining the Reynolds stress. We can say that the process of free convection, which produces turbulent motion in the inhomogeneous temperature gradient and Coriolis force field is at the same time an effective mechanism for producing the spiral property of this turbulence. However, it can be studied comparatively simply using the ordinary technique for calculating the Reynolds stress only if the turbulent correlation time does not become too large:

$$1\gg\frac{\lambda^2}{\nu\tau}\gg\frac{g\beta B\lambda^5}{2\nu^2}=\frac{\text{Re}}{2}\frac{B\lambda}{A}\cong\frac{B\lambda}{A}. \qquad (9)$$

3. THE REYNOLDS STRESS

The Reynolds stress will be derived using Eqs. (3), (6), and (7), assuming that the nonlinear terms in Eqs. (6) and (7) are small, i.e., that the Reynolds number satisfies $\text{Re}=u_T\lambda/\nu\ll1$. It is convenient to represent Eqs. (6)–(7) as a single equation for the velocity, solving the perturbed temperature equation (7) iteratively:

$$L_{ij}DV_j-\beta gP_{im}e_m\frac{1}{D}(Bz)e_jV_j=F_i-P_{im}\nabla_k(V_kV_m)$$

$$-\beta Ag\frac{1}{D}P_{im}e_me_j\nabla_k\left(V_k\frac{1}{D}V_j\right). \qquad (10)$$

Here we have introduced the notation L_{ij} for the part of the linear operator which is independent of position:

$$L_{ij}=\delta_{ij}-qP_{im}e_me_j+bP_{im}\epsilon_{msj}e_s,$$

$$D=\frac{\partial}{\partial t}-\nu\Delta,$$

$$q=\frac{\beta g A}{D^2}, \quad b=\frac{2\Omega}{D}, \quad P_{im}=\delta_{im}-\frac{\nabla_i\nabla_m}{\Delta}. \tag{11}$$

The projection operator P_{im} excludes the potential part of the velocity field. The differential operators in the denominator are taken to mean integral operators with the corresponding Green's functions.

Note that the operator L_{ij}, generally speaking, depends on the large-scale velocity that develops as a result of the growth of the instability. In this case it describes the phenomenon of stimulated convection, and including it would enable us to describe the reciprocal effect of the instability on the development of the spiral property. In this model it is pointless to include the large-scale velocity in the operator L_{ij}, however, since we are neglecting the variation of the large-scale velocity fields on the small scale, and including the large-scale velocity in the operator L_{ij} will describe only the Galilean motion of the small-scale cells as a whole, without affecting the parameters of the convection process.

To perform the average over Eq. (10), we follow Ref. 12 and represent the velocity V_i in the form

$$V_i=V_i^T+V_i+\langle V_i\rangle. \tag{12}$$

The field V_i^T is a homogeneous random velocity field resulting from the direct action of the external force F_i:

$$L_{ij}DV_j^T-\beta g P_{im}e_m\frac{1}{D}(Bz)e_jV_j^T=F_i-P_{im}\nabla_k(V_k^TV_m^T)$$
$$-\beta A g\frac{1}{D}P_{im}e_me_j\nabla_k\left(V_k^T\frac{1}{D}V_j^T\right). \tag{13}$$

The quantity $\langle V_i\rangle$ is the large-scale average velocity field ($\langle V\rangle \ll V^T$). The equation describing it is found by ensemble-averaging over realizations of the turbulent equation (10):

$$L_{ij}D\langle V_j\rangle-\beta g P_{im}e_m\frac{1}{D}(Bz)e_j\langle V_j\rangle$$
$$=-P_{im}\nabla_k(\langle V_k^TV_m\rangle+\langle \tilde V_kV_m^T\rangle)$$
$$-\beta A g P_{im}\nabla_k e_me_j\frac{1}{D}\left(\left\langle \tilde V_k\frac{1}{D}V_j^T\right\rangle\right.$$
$$\left.+\left\langle V_k^T\frac{1}{D}\tilde V_j\right\rangle\right). \tag{14}$$

Equation (14) describes the velocity field $\langle V_i\rangle$, in which the parameters A and B refer to the large-scale background temperature profile. Since the background profile in this model is assumed to be neutral on the average, the parameters A and B in this equation must be set equal to zero, so that the large-scale equation assumes the form

$$L_{ij}D\langle V_j\rangle=-P_{im}\nabla_k(\langle V_k^T\tilde V_m\rangle+\langle \tilde V_kV_m^T\rangle). \tag{15}$$

The average velocity field $\langle V_i\rangle$ superposed on the turbulent eddies V_i^T gives rise to a small inhomogeneous correction $\tilde V_i(\tilde V\ll V^T)$, which is therefore a functional of V^T and $\langle V\rangle$: $V_i=V_i\{V^T,\langle V\rangle\}$. Subtracting the equation for the turbulent part of the velocity V_i^T, Eq. (13), and the averaged equation (14) from the full equation (10), we find to lowest order an equation describing the inhomogeneous part of the turbulent velocity $\tilde V_i$:

$$L_{ij}D\tilde V_j-\beta g P_{im}e_m\frac{1}{D}(Bz)e_j\tilde V_j=-P_{im}\nabla_k(V_k^T\langle V_m\rangle$$
$$+\langle V_k\rangle V_m^T)-\beta A g P_{im}\nabla_k e_me_j\frac{1}{D}\left(V_k^T\frac{1}{D}\langle V_j\rangle\right.$$
$$\left.+\langle V_k\rangle\frac{1}{D}V_j^T\right). \tag{16}$$

The averages of quadratic combinations (the Reynolds stress) enter into Eq. (15) for the large-scale velocity. These can be expressed in terms of the average field $\langle V_i\rangle$ and the turbulent correlation by using the functional dependence of the field $\tilde V_i$ on the random field V_i^T, assumed to be Gaussian, by means of the Furutsu–Novikov formula:[13]

$$\left\langle V_k^T(t,\mathbf{x})\tilde V_m(t,x)\right\rangle=\lim_{\substack{t_1\to t\\ \mathbf{x}_1\to\mathbf{x}}}\int ds\int d\mathbf{y}\langle V_k^T(t,\mathbf{x})V_r^T(s,\mathbf{y})\rangle$$
$$\times\left\langle\frac{\delta\tilde V_m(t_1\mathbf{x}_1)}{\delta V_r^T(s,\mathbf{y})}\right\rangle, \tag{17}$$

similar to the way the production of large-scale structures superposed on spiral turbulence is treated.[2–4,6]

Let us determine how the inhomogeneous part of the turbulent velocity field $\tilde V_i$ depends on the turbulent field V_i^T. We represented as a sum of the main part $\tilde V_{0i}$, which depends on $\langle V\rangle$ and V^T, and a small correction $\tilde V_{1i}$:

$$\tilde V_i=\tilde V_{0i}+\tilde V_{1i}. \tag{18}$$

The correction $\tilde V_{1i}$, which is associated with the variation in the temperature gradient, in view of the inequality (11) can be determined by using $\tilde V_{0i}$:

$$\tilde V_{0j}=-\frac{1}{D}L_{ji}^{-1}P_{im}\left[\langle V_k\rangle\nabla_kV_m^T\right.$$
$$\left.-\beta A g e_me_n\frac{1}{D}\langle V_k\rangle\frac{1}{D}\nabla_kV_n^T\right], \tag{19}$$

$$\tilde V_{1j}=\frac{1}{D}L_{ji}^{-1}P_{im}e_me_j\beta g\frac{1}{D}(Bz)\tilde V_{0j}. \tag{20}$$

In the expression for $\tilde V_{0j}$ we have omitted unimportant terms which contain derivatives with respect to the large-scale coordinates, since the averaged equation contains a single large-scale derivative on the right-hand side, while the Reynolds stress with a second derivative with respect to the large-scale coordinates describes the turbulent viscosity. The tensor L_{ji}^{-1} represents the inverse operator, whose explicit form can be determined through simple but rather lengthy calculations:

$$L_{ji}^{-1}=P[P^{-1}\delta_{ji}+(b^2+q)\Delta e_j e_i-(1-q)b^2\nabla_j\nabla_i$$
$$+(1-q)b^2 m_j m_i-bP^{-1}\epsilon_{jst}e_i-b^2(b^2+q)(\mathbf{e}\nabla)$$
$$\times e_j\nabla_i-(b^2+q)(\mathbf{e}\nabla)\nabla_j e_i+(b^2+q)b(\mathbf{e}\nabla)e_j m_i$$
$$-(b^2+q)b(\mathbf{e}\nabla)m_j e_i+b(1-q)\nabla_j m_i-b^3(1-q)$$
$$\times m_j\nabla_i],$$
$$P=[(1+b^2)[\Delta(1-q)+(\mathbf{e}\nabla)^2(b^2+q)]]^{-1},$$
$$m_j=\epsilon_{jst}\nabla_s e_t. \qquad (21)$$

Hence the variational derivative which enters into the Furutsu–Novikov formula (17) can be represented as

$$\left\langle\frac{\delta\widetilde{V}_m(t_1\mathbf{x}_1)}{\delta V_r^T(s,\mathbf{y})}\right\rangle=-\frac{1}{D}L_{ji}^{-1}P_{im}\langle V_k\rangle\Bigg[\nabla_k(\delta_{rm}+qe_me_r)$$
$$+\beta g e_m\frac{1}{D}(Bz)e_g\frac{1}{D}L_{gi}^{-1}P_{in}\nabla_k$$
$$\times(\delta_{nr}+qe_ne_r)\Bigg]\delta(\mathbf{x}-\mathbf{y})\delta(s-t). \qquad (22)$$

The calculations in Eq. (17) can be carried out conveniently in the Fourier coordinate representation:[14]

$$f(\mathbf{x})=\int\frac{dk}{(2\pi)^3}\hat{f}(\mathbf{k})e^{-i\mathbf{kx}},\quad \hat{f}(\mathbf{k})=\int d\mathbf{x}f(\mathbf{x})e^{i\mathbf{kx}}.$$

Then the Furutsu–Novikov formula (17) assumes the form

$$\langle V_k^T(t,\mathbf{x})\widetilde{V}_m(t,\mathbf{x})\rangle=\lim_{\substack{t_1\to t\\ \mathbf{x}_1\to\mathbf{x}}}\int ds\int d\mathbf{y}\int\frac{d\mathbf{k}}{(2\pi)^3}$$
$$\times e^{-i\mathbf{kx}}e^{i\mathbf{ky}}\hat{T}_{kr}^T(t-s,\mathbf{k})\int\frac{dk_1}{(2\pi)^3}$$
$$\times e^{-ik_1x_1}\left\langle\frac{\delta\widetilde{V}_m(t_1k_1)}{\delta V_r^T(s,y)}\right\rangle, \qquad (23)$$

where

$$\left\langle\frac{\delta\widetilde{V}_m(t_1\mathbf{k}_1)}{\delta V_r^T(s,\mathbf{y})}\right\rangle$$
$$=-\frac{1}{\hat{D}(\mathbf{k}_1)}\hat{L}_{mi}^{-1}(\mathbf{k}_1)\hat{P}_{ig}(\mathbf{k}_1)\langle V_a\rangle$$
$$\times\Bigg[(-ik_{1a})(\delta_{gr}+\hat{q}e_ge_r)+\beta gBe_g\frac{1}{\hat{D}(\mathbf{k}_1)}e_s\left(i\frac{\partial}{\partial k_{1s}}\right)$$
$$\times e_c\frac{1}{\hat{D}(\mathbf{k}_1)}\hat{L}_{cb}^{-1}(\mathbf{k}_1)\hat{P}_{bn}(k_1)(-ik_{1a})(\delta_{nr}+\hat{q}e_ne_r)\Bigg]$$
$$\times e^{ik_1y}\delta(s-t). \qquad (24)$$

The first term in square brackets in Eq. (24) can be omitted, since it vanishes when integrated over the angles of the wave vector. In performing the differentiation with respect to $\mathbf{k}_1$ we should keep in mind that the main contribution to the integral over the magnitude of the wave vector comes from the vicinity of the pole in the inverse operator responsible for producing small-scale turbulence. To integrate over the wave vector we must explicitly give the correlation function $\hat{G}(t_1-t_2,\mathbf{k}_\perp,\mathbf{ek})$, which must describe both the way the spectrum falls off as the scale of the turbulence decreases and the shape of a convective cell. We prescribe the correlation function as follows:

$$\hat{G}(t_1-t_2,\mathbf{k}_\perp,\mathbf{ek})=G_0[\delta(k_z-\eta|\mathbf{k}_\perp|)$$
$$+\delta(k_z+\eta|\mathbf{k}_\perp|)]$$
$$\times\delta(\mathbf{k}_\perp^2-1/\lambda^2)e^{-|t-s|/\tau}. \qquad (25)$$

The quantity η in Eq. (25) describes the ratio of the vertical to the horizontal scale. It should be regarded as small, $\eta\ll 1$, since we are using a model with convective cells that are elongated vertically. In order to simplify the integration we assume that the turbulent spectrum has a trivial form, since in this model it is assumed that the turbulent spectrum has no wave numbers less than the convective wave numbers. The main contribution of the integral comes from the vicinity of the pole, and the manner in which the turbulent spectrum falls off has no great significance. If we neglect the variation of the large-scale velocity on the turbulent scale, and also assuming that the ratio of the square of the vertical wave number to that of the horizontal wave number is much less than unity, we can transform the expression for the quadratic combination $\langle V_k^T(t,\mathbf{x})\widetilde{V}_m(t,\mathbf{x})\rangle>\kappa$ as follows:

$$\langle V_k^T(t,\mathbf{x})\widetilde{V}_m(t,\mathbf{x})\rangle$$
$$=\lim_{t_1\to t}\int ds\int\frac{dk}{(2\pi)^3}\hat{G}(t_1-t_2,\mathbf{k}_\perp,\mathbf{ek})\langle V_a(t_1,x)\rangle$$
$$\times\delta(s-t_1)\frac{-4(\mathbf{ek})\hat{q}^3(B/A)\hat{D}^5k_a}{k^6(\hat{D}^2\mathbf{k}^2-\mathbf{k}_\perp^2\beta Ag)^3}[\mathbf{k}^2e_m-(\mathbf{ek})k_m$$
$$-\hat{b}(\mathbf{ek})\hat{m}_m]\Bigg[\hat{b}(\mathbf{ek})\hat{m}_k+\Bigg[(\mathbf{k}^2(1+\hat{q})e_k-(\mathbf{ek})k_k)$$
$$-\hat{q}(1-\mu)k_k(ek)+\mu\left(\frac{1}{(\mathbf{ek})^2}k^4(1+\hat{q})e_k\right.$$
$$\left.-\frac{1}{(\mathbf{ek})}(\hat{q}\mathbf{k}^2e_k(\mathbf{ek})+\mathbf{k}^2(1+\hat{q})k_k)\right)\Bigg]\Bigg].$$

When integrating near the pole we neglect the shift due to the Coriolis parameter. As a result we find the following expression for the Reynolds stress:

$$\langle V_k^T(t,\mathbf{x})\tilde{V}_m(t,\mathbf{x})\rangle+\langle V_m^T(t,\mathbf{x})\tilde{V}_k(t,\mathbf{x})\rangle$$

$$=\frac{G_0}{(2\pi)^2}\left(\frac{\nu T}{\lambda^2}\right)^3\frac{B\lambda}{A}\frac{\lambda}{\nu}\mu[(\langle V_k^\perp\rangle e_m+\langle V_m^\perp\rangle e_k)$$

$$-2e_ke_m(\mathbf{e}\langle\mathbf{V}\rangle)-\tfrac{1}{2}b^*(e_k\epsilon_{mra}+e_m\epsilon_{kra})e_r\langle V_a\rangle],\tag{26}$$

where

$$b^*=\frac{2\Omega\lambda^2}{\nu}.$$

The anisotropic turbulence is characterized by different values of the turbulent energy in the vertical and horizontal directions:

$$E=\frac{1}{2}\langle\mathbf{V}^2\rangle=\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}\hat{T}_{ii}^T(t-s,\mathbf{k})$$

$$=\frac{G_0}{(2\pi)^2}\left(2-\mu+\frac{\mu k^2}{(\mathbf{ek})^2}\right),$$

$$E_z=\frac{1}{2}\langle V_z^2\rangle=\frac{1}{2}\int\frac{dk}{(2\pi)^3}e_ie_j\hat{T}_{ij}^T(t-s,\mathbf{k})$$

$$=\frac{G_0}{(2\pi)^2}\left(1-2\mu-\frac{\mu(ek)^2}{k^2}+\frac{\mu k^2}{(\mathbf{ek})^2}\right),$$

$$E_\perp=E-E_z\left(1+\mu+\frac{\mu(\mathbf{ek})^2}{\mathbf{k}^2}\right).$$

For strong anisotropy ($\mu\to 1$) most of the energy is concentrated in the vertical motion:

$$E\cong E_z\cong\frac{G_0}{(2\pi)^2}\frac{\mu\mathbf{k}^2}{(\mathbf{ek})^2}.$$

The coefficient G_0 can thus be expressed in terms of the turbulent energy density E:

$$G_0=(2\pi)^2E\eta^2.$$

Then the equation for the average velocity $\langle V_i\rangle$ can be represented in the form

$$\left(\frac{\partial}{\partial t}-\nu_T\Delta\right)\langle V_i\rangle+2\Omega P_{im}\epsilon_{msj}e_s\langle V_j\rangle$$

$$=CP_{im}\nabla_k\Bigg[-(\langle V_k^\perp\rangle e_m+\langle V_m^\perp\rangle e_k)-2e_ke_m(\mathbf{e}\langle\mathbf{V}\rangle)$$

$$+\frac{1}{2}b^*(e_k\epsilon_{mra}+e_m\epsilon_{kra})e_r\langle V_a\rangle\Bigg],\tag{27}$$

where

$$C=E\eta^2\left(\frac{\nu T}{\lambda^2}\right)^3\frac{B\lambda}{A}\frac{\lambda}{\nu},$$

and ν_T is the turbulent viscosity coefficient on the large scale, whose magnitude will be assumed much larger than that of the corresponding small-scale coefficient $\nu(\nu_T\gg\nu)$.

4. LARGE-SCALE MOTION

Equation (27) describes the large-scale instability in an incompressible fluid. This instability can conveniently be studied in terms of the poloidal and toroidal parts of the velocity $\langle V\rangle$,

$$\langle V\rangle=\mathrm{rot}(e\psi)+\mathrm{rot}\,\mathrm{rot}(e\varphi).$$

The system of equations for the poloidal potential φ and the toroidal potential ψ of the fields takes the form

$$\left(\frac{\partial}{\partial t}-\nu_T\Delta\right)\Delta\varphi+2\Omega\mathbf{e}\nabla\psi=C\Bigg[(3\Delta_\perp-(\mathbf{e}\nabla)^2)\mathbf{e}\nabla\varphi$$

$$+\frac{b^*}{2}((\mathbf{e}\nabla)^2-\Delta_\perp)\psi\Bigg],$$

$$\left(\frac{\partial}{\partial t}-\nu_T\Delta\right)\psi-2\Omega\mathbf{e}\nabla\varphi=-C\left[\mathbf{e}\nabla\psi+\frac{b^*}{2}(\mathbf{e}\nabla)^2\varphi\right].\tag{28}$$

The large-scale instability described by Eqs. (28) is most clearly manifested for structures whose horizontal dimensions are large in comparison to the vertical, $K_z\gg K_\perp$, where $\mathbf{K}=(\mathbf{K}_\perp,K_z)$ is the wave vector of the large-scale instability, whose growth rate is given by

$$\gamma=\pm C\frac{b^*}{2}K_z-\nu_TK_z^2.\tag{29}$$

When the instability develops, structures form with typical size $K_{zm}^{-1}=4\nu_T/Cb^*$ corresponding to the maximum value of the growth rate (29), with driven oscillations of the medium at frequencies $\omega=-CK_z\pm2\Omega$. It is reasonable to expect that the exponential growth of the amplitude in the initial stage is slowed down by the possible nonlinearity. However, if the amplitude in the process of amplification nevertheless approaches the phase velocity $V_{\mathrm{ph}}=\omega K_{zm}^{-1}$, the solutions become multivalued and the motion acquires a turbulent character. This system can be regarded as a source of large-scale turbulence with the energy contained on the scale $4\nu_T/Cb^*$ and the characteristic amplitude V_{ph} of the turbulent eddies. This can be treated as an inverse cascade of energy in the turbulent spectrum, which in general causes its shape to change. The study of the shape of the turbulent spectrum lies outside the scope of the present model.

If we treat a medium which is bounded in the vertical direction, then the thickness h of this layer may turn out to be less than the characteristic vertical scale of the instability, $h<4\nu_T/Cb^*$. In this case a single structure will occupy the entire volume of the layer, and the instability will be characterized by a threshold determined by a boundary-value problem.

Consider the limiting case of large values of the parameter $b^*-2\Omega\lambda^2/\nu\gg1$, realized, e.g., when the small-scale viscosity coefficient ν is sufficiently small. Equation (27) with lengths scaled by the layer thickness h and times scaled by h^2/ν_T assumes the form

$$\left(\frac{\partial}{\partial t}-\Delta\right)\langle V_i\rangle+\frac{2\Omega h^2}{\nu_T}P_{im}\epsilon_{msj}e_s\langle V_j\rangle$$
$$=E\frac{\lambda h}{\nu\nu_T}\eta^2\left(\frac{\nu\tau}{\lambda^2}\right)^3$$
$$\times\frac{B\lambda}{A}\frac{2\Omega\lambda^2}{\nu}P_{im}\nabla_k(e_k\epsilon_{mra}+e_m\epsilon_{kra})e_r\langle V_a\rangle. \qquad (30)$$

Equation (30) differs from the equation of the large-scale instability derived using the idea of spiral turbulence by Lupyan *et al.*[6] only in having a term with the Taylor number, which arises naturally in this model. The role of the spiral coefficient S is consequently played by the product

$$S=2\Omega\tau\eta^2\frac{E\tau^2}{\lambda^2}\frac{h}{\lambda}\frac{\nu}{\nu_T}\frac{B\lambda}{A}.$$

Thus, in this model the procedure for performing a statistical average over the small-scale turbulent eddies including the Coriolis force and curvature of the temperature profile in the convective cell yields large-scale equations analogous to the equations in the model for the production of large-scale vortices under the action of spiral turbulence.[1–6,15] Consequently, the concept of spiral turbulence parametrizes the combined effects of the variation in the temperature gradient and the Coriolis force, while the spiral coefficient is proportional to a product of the variation in the temperature gradient, the intensity of the turbulence, and the Coriolis force.

Note that both Eqs. (28) and Eqs. (30) lead to a large-scale instability due to the interaction of the poloidal and toroidal velocity fields. Just as in the case of spiral turbulence, its growth rate does not depend on the sign of the spiral parameters $Cb^*/2$ or S. The direction of the spiral characterizes the direction of the velocity fields; for example, with a prescribed direction of the poloidal field (say, the fluid rises in the central part of the cell and falls at the periphery) the sign of the spiral parameter determines the sign of the toroidal field, i.e., the direction in which the large-scale vortex rotates (when the spiral has a positive sign seen from above, the vortex rotates counterclockwise, while with a negative sign it rotates clockwise).

Equation (30) for the values of the spiral coefficient S which exceeds some threshold value describes creation of a large-scale vortex like a tropical cyclone, whose horizontal dimensions are substantially greater than the layer thickness h. Under certain conditions the toroidal velocity in such a vortex is considerably greater than the poloidal field. Including the term with the Taylor number does not significantly change the instability process. According to Klyatskin,[13] the effect of this term reduces to raising the instability threshold somewhat and changing the shape of the neutral curves.

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